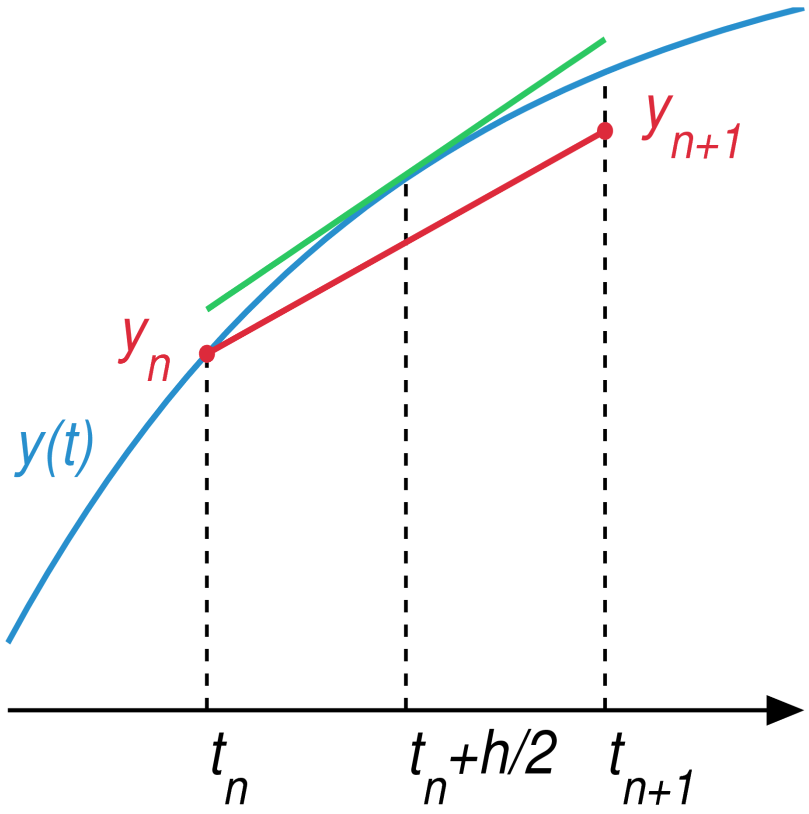
Bulirsch-Stoer Method

Midpoint Method

Recall “modified” Euler

$$y_{n-1/2} = y_{n-1} + \frac{h}{2} f(t_{n-1}, y_{n-1})$$
$$y_n = y_{n-1} + hf(t_{n-1/2}, y_{n-1/2})$$



This method is 2^{nd} order consistent

Also called the leap-frog formula.

Let's modify it a little bit.

"Modified" Midpoint Method

Take *n* small steps of size *h* to cover the interval t_0 to t_0+H.

$$\tilde{y}_0 = y(t_0)$$

First Step: Euler

$$\tilde{y}_1 = \tilde{y}(t_1) = \tilde{y}_0 + h\, f(\tilde{y}_0, t_0)$$

Modified Midpoint

i = 1... n-1

$$\tilde{y}_2 = \tilde{y}(t_2) = \tilde{y}_0 + 2h\, f(\tilde{y}_1, t_1)$$

$$\tilde{y}_3 = \tilde{y}(t_3) = \tilde{y}_1 + 2h\, f(\tilde{y}_2, t_2)$$

$$\tilde{y}_{i+1} = \tilde{y}(t_{i+1}) = \tilde{y}_{i-1} + 2h\, f(\tilde{y}_i, t_i)$$

Combination

$$y(t_0+H) = \frac{1}{2}\left[\tilde{y}_n + \tilde{y}_{n-1} + h\, f(\tilde{y}_n, t_n)\right]$$

n+1 function evaluations required

“Modified” Midpoint Method

Useful because:

It has an error series that consists of only the even powers of h

$$y(t_0+H)-y(t_0)=\sum k_i h^{2i}$$

Reminiscent of Romberg integration with trapezoidal rule for quadrature

Can play the same trick of combining steps with different values of h to get higher order accuracy – Bulirsch Stoer method

Both based on Richardson's extrapolation idea.

Richardson Extrapolation and Bulirsch-Stoer Method

Take a "large" step size H

Consider the answer as an analytic function f(h) of h=H/n.

Fit the function by polynomial or rational function interpolation.

Choose a method (e.g., midpoint) such that f(h) is even in h. And finally extrapolate to h=0.

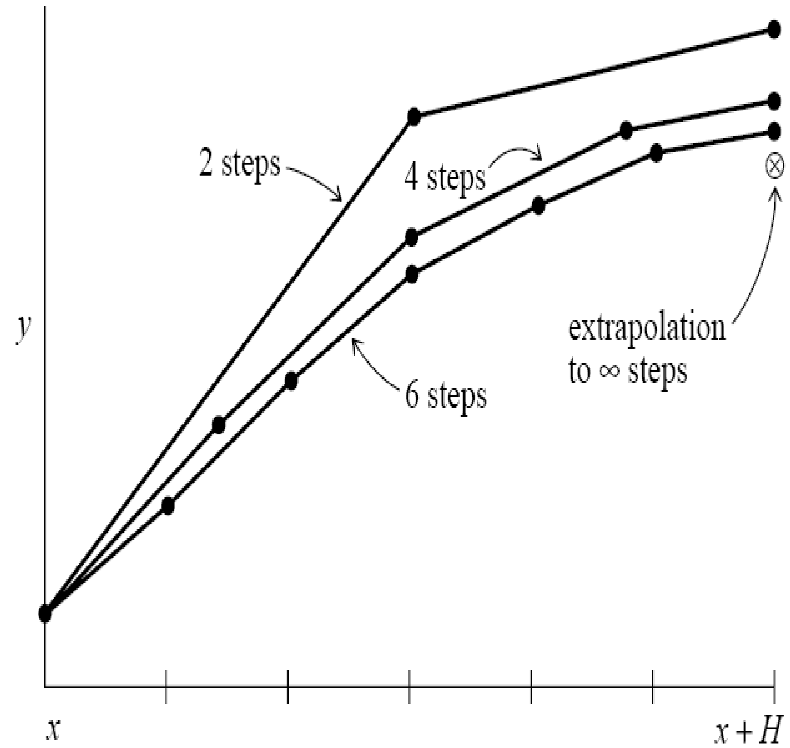


Figure 16.4.1. Richardson extrapolation as used in the Bulirsch-Stoer method. A large interval H is spanned by different sequences of finer and finer substeps. Their results are extrapolated to an answer that is supposed to correspond to infinitely fine substeps. In the Bulirsch-Stoer method, the integrations are done by the modified midpoint method, and the extrapolation technique is rational function or polynomial extrapolation.

Polynomial Extrapolation

Get two estimates for $y(t_0+H)$ using n and $2n$ steps.

$$y(t_0+H)=\frac{4\,y_{2\mathrm{n}}-y_n}{3}$$

This estimate 4^{th} order accurate, same as 4^{th} order Runge-Kutta

Can use exactly the same idea of "successive refinement" used in Romberg integration to get higher order estimates.

Polynomial Extrapolation

If $Y_n^{(k)}$ represents the k^{th} order estimate of $y(t_0+H)$, then

$$Y_j^{(k+1)} = Y_j^{(k)} + \frac{Y_j^{(k)} - Y_{j-1}^{(k)}}{\left(\frac{n_j}{n_{j-k}}\right)^2 - 1}$$

This is exactly the same as what we did for Romberg Integration by building the table.

Bulirsch Stoer

A commonly used sequence of “n”s is:

$n = \{2, 4, 6, 8, 10, \ldots\}$ $n_j = 2j$

After each n_j, extrapolate and obtain error estimate

This technique (and extrapolation in general) works best for smooth functions.

If not very smooth, use adaptive RK, since it does does a better job of negotiating abruptly changing regions of the domain.