2 DRAFT February 13, 2006

2.1 Factorizations

We want to solve $\mathbf{Ax} = \mathbf{b}$. If $\mathbf{A}$ has a triangular form this very easy. *Backsubstitution* will solve the equation, for example

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

we find the solution for $\mathbf{x}$ starting with the last row:

$$\begin{aligned} & & 0x_1 + 0x_2 + 6x_3 = 9 \quad &\rightarrow \quad x_3 = \frac{9}{6} \\ 0x_1 + 4x_2 + 5x_3 = 8 \quad &\rightarrow & 0x_1 + 4x_2 + 5\frac{9}{6} = 8 \quad &\rightarrow \quad x_2 = \frac{1}{4} \\ 1x_1 + 2x_2 + 3x_3 = 7 \quad &\rightarrow & 1x_1 + 2\frac{1}{4} + 5\frac{9}{6} = 7 \quad &\rightarrow \quad x_2 = \frac{1}{4} \end{aligned}$$

Solving the above equation is easy if the matrix is in triangular form, therefore the problem reduces to find the triangular matrix.

2.1.1 QR-factorization

We want to put the matrix $\mathbf{A}$ into the form

$$\mathbf{A} = \mathbf{QR}$$

where $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{Q}$ is a $m \times n$ orthnormal matrix, and $\mathbf{R}$ is a triangular $n \times n$ matrix. Having $\mathbf{Q}$ and $\mathbf{R}$ we can substitute,

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{QRx} &= \mathbf{b} \\ \mathbf{Q^{-1}QRx} &= \mathbf{Q^{-1}R} \\ \mathbf{Rx} &= \mathbf{Q^{-1}R} \end{aligned}$$

This boils down to a little rough algorithm **??**. Of course, we still need to know how step 1 in our algorithm is done.

2.1.2 Gram-Schmidt factorization

One of the first factorization of a matrix into triangular matrix is the Gram-Schmidt factorization (Algorithm **??**).

Algorithm 1 QR factorization

Compute factorization $\mathbf{A} = \mathbf{QR}$

Compute $\mathbf{y} = \mathbf{Qb}$

Solve $\mathbf{Rx} = \mathbf{y}$ for $\mathbf{x}$

Algorithm 2 Gram-Schmidt factorization

for $j = 1$ to n **do**

  $\mathbf{v}_j = \mathbf{a_j}$

  for $j = 1$ to $j - 1$ **do**

    $r_{ij} = \mathbf{q_i a_j}$

    $\mathbf{v}_j = \mathbf{v}_j - r_{ij}\mathbf{q_i}$

  end for

  $r_{ij} = ||\mathbf{v}_i||_2$

  $\mathbf{q}_j = \mathbf{v}_j / r_{ij}$

end for

2.1.3 Householder factorization, Householder reflection

2.2 Uses of these factorizations