2 DRAFT February 10, 2006

2.1 Orthogonal vectors and matrices

A pair of vectors $\mathbf{x}$ and $\mathbf{y}$ are *orthogonal* if $\mathbf{x}^T\mathbf{y} = 0$. If $\mathbf{x}$ and $\mathbf{y}$ are real then the lie at right angles to each other. Two sets of vectors $\mathbf{X}$ and $\mathbf{Y}$ are orthogonal if every $\mathbf{x} \in \mathbf{X}$ is orthogonal to every $\mathbf{y} \in \mathbf{Y}$. A set on nonzero vectors in $\mathbf{S}$ is orthogonal if its elements are pairwise orthogonal, i. e.,

$$\text{if for } \mathbf{x}, \mathbf{y} \in \mathbf{S}, \mathbf{x} \neq \mathbf{y} \quad \Longrightarrow \quad \mathbf{x}^T\mathbf{y} = 0.$$

so we have, for example

$$\left[\begin{bmatrix} \\ \mathbf{x}_1 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \mathbf{x}_2 \\ \\ \end{bmatrix} \quad \begin{bmatrix} \\ \mathbf{x}_3 \\ \\ \end{bmatrix} \right] \Longrightarrow \begin{array}{l} \mathbf{x}_1^T\mathbf{x}_2 = 0 \\ \mathbf{x}_1^T\mathbf{x}_3 = 0 \\ \mathbf{x}_2^T\mathbf{x}_3 = 0 \end{array}$$

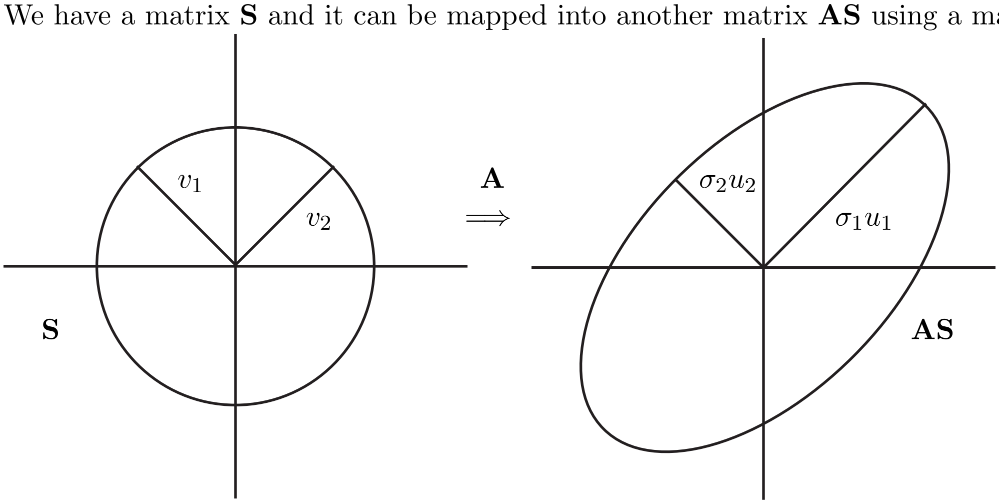
A set of vectors is *orthonormal* if it is orthogonal and, in addition, every $\mathbf{x} \in \mathbf{S}$ has $||\mathbf{x}|| = 1$. [definitions from Trefethen 1997]

2.2 Singular value decomposition

The singular value decomposition (SVD) is a matrix factorization.

2.2.1 A geometric explanation

We have a matrix $\mathbf{S}$ and it can be mapped into another matrix $\mathbf{AS}$ using a mapping $\mathbf{A}$.



Trefethen: "We define n *singular values* of $\mathbf{A}$. These are the lengths of the n principal semiaxes of $\mathbf{AS}$, written $\sigma_1, \sigma_2, ..., \sigma_n$." Conventionally, they are orders from the largest to the smallest. We also define the *n left singular vectors of* $\mathbf{A}$. These are the unit vectors $u_1, u_2, ..., u_n$. $\sigma_1 u_1$ is therefore the largest semiaxes of $\mathbf{AS}$. The vectors $v_1, v_2, ..., v_n$ are the *right singular vectors*. So we can say that

$$Av_j = \sigma_j u_j, \qquad 1 \leq j \leq n. \tag{1}$$

2.2.2 Reduced SVD

The formula 1 clooks like this

$$\left[\begin{array}{c} \\ \\ \mathbf{A} \\ \\ \\ \end{array}\right] \left[\left[\begin{array}{c} \\ v_1 \\ \\ \end{array}\right] \left[\begin{array}{c} \\ v_2 \\ \\ \end{array}\right] \cdots \left[\begin{array}{c} \\ v_n \\ \\ \end{array}\right]\right] = \left[\left[\begin{array}{c} \\ \\ u_1 \\ \\ \\ \end{array}\right] \left[\begin{array}{c} \\ \\ u_2 \\ \\ \\ \end{array}\right] \cdots \left[\begin{array}{c} \\ \\ u_n \\ \\ \\ \end{array}\right]\right] \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$$

or more compact

$$\mathbf{AV} = \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}$$
$$\mathbf{A} = \hat{\mathbf{U}}\hat{\mathbf{\Sigma}}\mathbf{V}^{-1}$$

This factorisation is called *reduced singular value decomposition* or *reduced SVD* and works when there are fewer columns than rows ($m \geq n$).

2.2.3 Full SVD

There exists a general version that allows the SVD of any matrix.

2.3 Matrix properties via the SVD

The SVD allows us the transform any matrix into a diagonal matrix with appropriate change of base (or coordinate system). Trefethen: "Any $b \in \mathbb{C}^m$ can be expanded into the base of left singluar vectors of $\mathbf{A}$ (columns of $\mathbf{U}$), and any $x \in \mathbb{C}^n$ can be expanded in the base of right singular vectors of mtA (columns of $\mathbf{V}$). The coordinate vectors for these expansions are

$$b' = \mathbf{U}^{-1} b$$
$$x' = \mathbf{V}^{-1} x.$$

We can now express $\mathbf{b} = \mathbf{A}\mathbf{x}$ in terms of $\mathbf{b}'$ and $\mathbf{x}'$

$$\mathbf{b} = \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \mathbf{U}^{-1}\mathbf{b} = \mathbf{U}^{-1}\mathbf{A}\mathbf{x} = \mathbf{U}^{-1}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{-1}\mathbf{x} \quad \Longleftrightarrow \quad \mathbf{b}' = \boldsymbol{\Sigma}\mathbf{x}'$$

2.4 Eigenvalue decomposition

Eigenvalue decomposition is a close cousing to SVD. *Eigenvalue decomposition* factorizes a matrix into the diagonal matrix $\boldsymbol{\Lambda}$ and the eigenvectors $\mathbf{X}$ so that

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$$

$$\mathbf{A}\mathbf{X} = \mathbf{X}\boldsymbol{\Lambda}$$

If $\mathbf{b} = \mathbf{A}\mathbf{x}$ and $\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$ then we can define

$$b' = \mathbf{X}^{-1}b$$

$$x' = \mathbf{X}^{-1}x.$$

then these vectors satisfy $\mathbf{b}' = \boldsymbol{\Lambda}\mathbf{x}'$.

2.5 SVD versus Eigenvalue decomposition

(1) SVD use two different bases, the left and right singular vectors, whereas the ED uses only one, the eigenvectors. (2) SVD uses orthonormal bases, but ED most often does not even use orthogonal bases. Although in many applications it seems that the eigenvectors are orthogonal. (3) All matrices have a SVD but not all have an ED. ED is most often used when one workds with iterated forms of matrices, and SVd is most often used in problems that involve the behavior of the matrix itself.

2.6 Matrix exponentiation

By definition the exponentiation of a variable x is

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

replacing x with a matrix $\mathbf{A}$ then we have

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

We can express now

$$e^{\mathbf{A}} = e^{\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}} = \sum_{k=0}^{\infty} \frac{(\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1})^k}{k!} \tag{2}$$

We can factors this because $\mathbf{X}\mathbf{X}^{-1} = \mathbf{I}, \mathbf{X}^k (X^{-1})^k = \mathbf{I}$ this way:

$$e^{\mathbf{A}} = \mathbf{X} \left(\sum_{k=0}^{\infty} \frac{\mathbf{\Lambda}^k}{k!} \right) \mathbf{X}^{-1} = \mathbf{X} e^{\mathbf{\Lambda}} \mathbf{X}^{-1} \tag{3}$$

2.7 Sources and Additional Reading

Golub, G. H., and C. F van Loan 1996. *Matrix computations.* 3rd edition. John Hopkins University Press, Baltimore and London.

Trefethen, L.N. and D. Bau, III. 1997. *Numerical Linear Algebra.* Philadelphia, PA, Society for Industrial and Applied Mathematics.