A NONLOCAL VECTOR CALCULUS WITH APPLICATION TO NONLOCAL BOUNDARY-VALUE PROBLEMS

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• The elliptic equation

\[-\nabla \cdot (D(x) \cdot \nabla w(x)) = b(x) \quad \text{in } \Omega \subset \mathbb{R}^d\]

models (steady state) diffusion

• The “nonlocal” equation

\[2 \int_{\Omega} \left( u(x') - u(x) \right) \mu(x, x') \, dx' = b(x) \quad \text{in } \Omega \subset \mathbb{R}^d\]

models (steady state) nonlocal diffusion
to see this, consider the nonlocal diffusion equation
\[ u_t = \int_{\Omega} (u(x') - u(x)) \mu(x, x') \, dx' \]

suppose that
\[ \int_{\Omega} \int_{\Omega} \mu(x, x') \, dx' \, dx = 1 \]
and
\[ \mu(x, x') = \mu(x', x) \geq 0 \]
so that \( \mu(x, x') \) can be interpreted as the joint probability density of moving between \( x \) and \( x' \)

then
\[ \int_{\Omega} (u(x') - u(x)) \mu(x, x') \, dx' = \int_{\Omega} u(x') \mu(x, x') \, dx' - u(x) \int_{\Omega} \mu(x, x') \, dx' \]
is the rate at which \( u \) “enters” \( x \) less the rate at which \( u \) “departs” \( x \)
the nonlocal diffusion equation can be derived from a “nonlocal” random walk

any skew-symmetry of $\mu$ represents drift

the nonlocal diffusion equation is an example of a differential Chapman-Kolmogorov equation
Motivations and goals

\[ \int_{\Omega} (u(x') - u(x)) \mu(x, x') \, dx' \quad \text{vs.} \quad \nabla \cdot (D(x) \cdot \nabla w(x)) \]

• The nonlocal operator always contains length scales
  it is a multiscale operator

  – the local operator contains length scales
    only when the diffusion tensor \( D \) does

• The nonlocal operator has lower regularity requirements
  \( u \) may be discontinuous

• The nonlocal diffusion equation does not necessarily
  smooth discontinuous initial conditions
• Extension to the vector case is (formally) straightforward

  – our ultimate goal – the analysis and numerical analysis of Silling’s nonlocal, continuum peridynamic model for materials – then follows

  – note that the peridynamic model is a mechanical model so that we have $u_{tt}$ and not $u_t$

• Here we consider the steady-state case

• Nonlocal models are able to model phenomena at smaller length and time scales at which the underlying assumption of locality associated with the classical diffusion equation or the classical balance of linear momentum lead to less accurate modeling
A NONLOCAL GAUSS’S THEOREM

• For any mapping \( r(x, x') : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \), it is easily seen that

\[
\int_{\hat{\Omega}} \int_{\hat{\Omega}} r(x, x') \, dx' \, dx = \int_{\hat{\Omega}} \int_{\hat{\Omega}} r(x', x) \, dx' \, dx \quad \forall \hat{\Omega} \subseteq \mathbb{R}^d
\]

• If \( p(x', x) \) denotes an anti-symmetric mapping, i.e.,

\[
p(x', x) = -p(x, x') \text{ for all } x, x' \in \mathbb{R}^d
\]

then

\[
\int_{\hat{\Omega}} \int_{\hat{\Omega}} p(x, x') \, dx' \, dx = 0 \quad \forall \hat{\Omega} \subseteq \mathbb{R}^d
\]

• Let \( \Omega \) denote an open bounded subset of \( \mathbb{R}^d \). Obviously, if \( \Gamma \subseteq \mathbb{R}^d \setminus \Omega \) and \( p(x', x) \) is anti-symmetric, by setting \( \hat{\Omega} = \Omega \cup \Gamma \), we have

\[
\int_{\Omega} \int_{\Omega \cup \Gamma} p(x, x') \, dx' \, dx = -\int_{\Gamma} \int_{\Omega \cup \Gamma} p(x, x') \, dx' \, dx
\]
Let \( \alpha(x, x') : \Omega \cup \Gamma \times \Omega \cup \Gamma \rightarrow \mathbb{R} \) denote a symmetric function
\[ \Rightarrow \alpha(x', x) = \alpha(x, x') \]

Let \( \mathcal{D} \) denote the linear operator mapping functions \( f \) into functions defined over \( \Omega \) given by
\[
\mathcal{D}(f)(x) = \int_{\Omega \cup \Gamma} \left( f(x, x') - f(x', x) \right) \alpha(x, x') \, dx' \quad \text{for } x \in \Omega
\]
- \( \mathcal{D} \) is a nonlocal “divergence” operator

Similarly, let \( \mathcal{N} \) denote the linear operator mapping functions \( f \) into functions defined over \( \Gamma \) given by
\[
\mathcal{N}(f)(x) = -\int_{\Omega \cup \Gamma} \left( f(x, x') - f(x', x) \right) \alpha(x, x') \, dx' \quad \text{for } x \in \Gamma
\]
- \( \mathcal{N} \) is a nonlocal “normal flux” operator
• Note that the operators $\mathcal{D}$ and $\mathcal{N}$ differ only in their domains and in their signs

• Then, setting $\rho(x, x') = \left( f(x, x') - f(x', x) \right) \alpha(x, x')$
results in the nonlocal Gauss’s theorem

$$\int_{\Omega} \mathcal{D}(f) \, dx = \int_{\Gamma} \mathcal{N}(f) \, dx$$
We apply two remarkable Lemmas due to Walter Noll


See also W. Noll, Derivation of the fundamental equations of continuum thermodynamics from statistical mechanics; Translation with corrections by R. Lehoucq and O. von Lilienfeld, to appear in J. Elasticity, 2009

Given a mapping $f(x, x')$, let

$$p(x, x') = (f(x, x') - f(x', x)) \alpha(x, x')$$

and, with $z = x' - x$

$$\varphi(x, z) = \int_0^1 p(x + \lambda z, x - (1 - \lambda)z) \, d\lambda$$

$$q(x) = -\int_{\mathbb{R}^d} (x' - x) \varphi(x, x' - x) \, dx'$$
• Then, a formal application of Lemma I of Noll implies
\[ \nabla \cdot q(x) = \int_{\mathbb{R}^d} (f(x, x') - f(x', x)) \alpha(x, x') \, dx' \quad \text{for } x \in \Omega \]

using the definition of the operator \( D(\cdot) \), we then have
\[ \nabla \cdot q = Df \quad \text{for } x \in \Omega \]

• Lemma II of Noll implies
\[ \int_{\partial\Omega} q(x) \cdot n \, dA = - \int_{\Gamma} \int_{\Omega \cup \Gamma} (f(x, x') - f(x', x)) \alpha(x, x') \, dx' \, dx \]

where
\[ \partial\Omega = \text{boundary of } \Omega \]
\[ dA = \text{surface element on } \partial\Omega \]
\[ n = \text{outward pointing unit normal vector along } \partial\Omega \]

using the definition of the operator \( \mathcal{N}(\cdot) \), we then have
\[ \int_{\partial\Omega} q(x) \cdot n \, dA = \int_{\Gamma} \mathcal{N}(f) \, dx \]
• Summarizing, given a function \( f(\cdot, \cdot) \), if \( q(\cdot) \) is determined from \( f(\cdot, \cdot) \) according to

\[
p(x, x') = (f(x, x') - f(x', x)) \alpha(x, x')
\]

\[
\varphi(x, z) = \int_0^1 p(x + \lambda z, x - (1 - \lambda)z) \, d\lambda \quad \text{with } z = x' - x
\]

\[
q(x) = -\int_{\mathbb{R}^d} (x' - x) \varphi(x, x' - x) \, dx'
\]

then, the no-local Gauss’s theorem for \( f \)

\[
\int_{\Omega} D(f) \, dx = \int_{\Gamma} N(f) \, dx
\]

implies the classical Gauss’s theorem for \( q \)

\[
\int_{\Omega} \nabla \cdot q \, dx = \int_{\partial\Omega} q \cdot n \, dx
\]
• Evidently, Gauss's theorem can be given a meaning without the notions of a divergence operator, a unit normal vector, or a surface.

• Lehoucq and Silling have shown that the vector-valued function \( q \) satisfies a constrained energy principle.

• Noll’s Lemma also holds for vector-valued functions in which case \( q \) is tensor-valued.
In the recent paper


the nonlocal divergence theorem

\[ \int_{\Omega} (Df)(x) \, dx = 0 \]

is given

- this is a the special case of our nonlocal Gauss’s theorem for \( \Gamma = \emptyset \)

- Gilboa and Osher also provide the germs of a nonlocal calculus but do not provide Green’s identities or consider nonlocal BVPs
• Important related papers by Rossi and co-workers that deal with nonlocal diffusion problems


+ several others
Notational simplification

- In the sequel, we frequently let

\[ u = u(x) \quad u' = u(x') \]
\[ v = v(x) \quad v' = v(x') \]
\[ f = f(x, x') \quad f' = f(x', x) \]
\[ \alpha = \alpha(x, x') \quad \alpha' = \alpha(x', x) \]

and similarly for functions still to be introduced
An application of the nonlocal Gauss’s theorem

• For functions $v(x)$ and $s(x, x')$, set $f = sv$ so that

$$f - f' = sv - s'v' = (s - s')v + s'(v - v')$$

• Using the definitions of the operators $\mathcal{D}$ and $\mathcal{N}$, one obtains

$$\int_{\Omega} v\mathcal{D}(s) \, dx + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} s(v' - v)\alpha \, dx' \, dx = \int_{\Gamma} v\mathcal{N}(s) \, dx$$

• Let $\mathcal{G}$ denote the linear operator mapping functions $v : \Omega \cup \Gamma \rightarrow \mathbb{R}$ into functions defined over $\Omega \cup \Gamma \times \Omega \cup \Gamma$ given by

$$\mathcal{G}(v) = (v' - v)\alpha \quad \text{for} \ x, x' \in \Omega \cup \Gamma$$
• Then, we obtain

\[
\int_{\Omega} v D(s) \, dx + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} s G(v) \, dx' \, dx = \int_{\Gamma} v N(s) \, dx
\]

– this is the nonlocal analog of the classical result

\[
\int_{\Omega} v \nabla \cdot q \, dx + \int_{\Omega} \nabla v \cdot q \, dx = \int_{\partial \Omega} v q \cdot n \, dA
\]

– the particular choice \( v = \text{constant} \) results in

\[
\int_{\Omega} D(f) \, dx = \int_{\Gamma} N(f) \, dx
\]

i.e., the nonlocal Gauss’s theorem
• Let

$$U(\Omega \cup \Gamma) \text{ and } V(\Omega \cup \Gamma)$$
denote Banach spaces

$$\Gamma = \Gamma_e + \Gamma_n \quad \text{with} \quad \Gamma_e \cap \Gamma_n = \emptyset$$

$$V_0(\Omega \cup \Gamma) = \{v \in V(\Omega \cup \Gamma) : v = 0 \text{ for } x \in \Gamma_e\}$$

• Define the data functions

$$b : \Omega \rightarrow \mathbb{R}$$

$$h_e : \Gamma_e \rightarrow \mathbb{R}$$

$$h_n : \Gamma_n \rightarrow \mathbb{R}$$
• For \( u \in U(\Omega \cup \Gamma) \), let
\[
s(x, x') = A(u)
\]
for a possibly nonlinear operator \( A \) which also may depend explicitly on \( x \) and \( x' \)

• Consider the variational problem

\[
\text{seek } u \in U(\Omega \cup \Gamma) \text{ such that } \\
\quad u = h_e \quad \text{for } x \in \Gamma_e
\]

and
\[
\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} A(u) G(v) \, dx' dx = \int_{\Omega} vb \, dx + \int_{\Gamma_n} vh_n \, dx \quad \forall v \in V_0(\Omega \cup \Gamma)
\]
The nonlocal Green’s formula shows that the variational formulation can be viewed as a weak formulation of the nonlocal “boundary-value” problem

\[-\mathcal{D}(A(u)) = b \quad \text{for } x \in \Omega\]
\[u = h_e \quad \text{for } x \in \Gamma_e\]
\[\mathcal{N}(A(u)) = h_n \quad \text{for } x \in \Gamma_n\]

- the second equation is a “Dirichlet boundary” conditions that is essential for the variational principle
- the third equation is a “Neumann boundary” condition that is natural for the variational principle
if $\Gamma_e = \emptyset$, then

- the space of test functions $V_0(\Omega)$ is replaced by $V(\Omega \cup \Gamma)/\mathbb{R}$

- the compatibility condition

$$\int_{\Omega} b d\mathbf{x} + \int_{\Gamma} h_n d\mathbf{x} = 0$$

must hold

- this is entirely analogous to the classical case
We now specialize to the case of \( U(\Omega \cup \Gamma) = V(\Omega \cup \Gamma) \) and to linear operators

For a mapping \( \beta(x, x') \), let

\[
s = A(u) = \beta G(u) = (u' - u)\alpha\beta
\]

- this is a constitutive relation

From the nonlocal Gauss’s theorem, we then obtain the nonlocal Green’s first identity

\[
\int_{\Omega} v D(\beta G(u)) \, dx + \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v) G(u) \, dx' \, dx = \int_{\Gamma} v N(\beta G(u)) \, dx
\]

- this is the nonlocal analog of the classical Green’s first identity

\[
\int_{\Omega} v \Delta u \, dx + \int_{\Omega} \nabla v \cdot \nabla u \, dx = \int_{\partial \Omega} v n \cdot \nabla u \, dA
\]
• One can immediately obtain a nonlocal Green’s second identity

\[ \int_{\Omega} v D(\beta G(u)) \, dx - \int_{\Omega} u D(\beta G(v)) \, dx = \int_{\Gamma} \left( v N(\beta G(u)) - u N(\beta G(v)) \right) \, dx \]

– this is the nonlocal analog of the classical Green’s second identity

\[ \int_{\Omega} v \Delta u \, dx - \int_{\Omega} u \Delta v \, dx = \int_{\partial \Omega} v n \cdot \nabla u \, dA - \int_{\partial \Omega} u n \cdot \nabla v \, dA \]

• A nonlocal Green’s third identity will come later
Constitutive relation

- The relation

\[ s = \mathcal{A}(u) = \beta \mathcal{G}(u) = (u' - u)\alpha \beta \]

is a “constitutive” relation

- We need to say something about \( \beta \)
  
  - we want something analogous to the diffusion tensor \( \mathbf{D} \) appearing in second-order elliptic partial differential equations

- Let \( \mathbf{K}(\mathbf{x}, \mathbf{x}') \) denote a tensor
  
  - then, a general nonlocal constitutive function \( \beta \) is given by

\[
\beta = (\mathbf{x}' - \mathbf{x}) \cdot \mathbf{K} \cdot (\mathbf{x}' - \mathbf{x})
\]
LINEAR, NONLOCAL BOUNDARY-VALUE PROBLEMS

• For \( s = \mathcal{A}(u) = \beta G(u) = (u' - u)\beta\alpha \), the variational problem reduces to

\[
\text{seek } u \in V(\Omega \cup \Gamma) \text{ such that } \\
\quad u = h_e \text{ for } x \in \Gamma_e \\
\text{and } \\
\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v) G(u) \, dx' \, dx = \int_{\Omega} vb \, dx + \int_{\Gamma_n} vh_n \, dx \quad \forall \, v \in V_0(\Omega \cup \Gamma)
\]

• The corresponding “boundary-value” problem reduces to the linear problem

\[
-\mathcal{D}(\beta G(u)) = b \quad \text{for } x \in \Omega \\
u = h_e \quad \text{for } x \in \Gamma_e \\
\mathcal{N}(\beta G(u)) = h_n \quad \text{for } x \in \Gamma_n
\]
We have the explicit relations

\[
\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v)G(u) \, dx' dx = \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v' - v)(u' - u)\alpha^2 \beta \, dx' dx
\]

\[
\mathcal{D}(\beta G(u)) = 2 \int_{\Omega \cup \Gamma} (u' - u)\alpha^2 \beta \, dx' \quad \text{for } x \in \Omega
\]

\[
\mathcal{N}(\beta G(u)) = -2 \int_{\Omega \cup \Gamma} (u' - u)\alpha^2 \beta \, dx' \quad \text{for } x \in \Gamma_n
\]
We now assume that the constitutive function $\beta$ is positive

- $\mathbf{K}$ symmetric, positive definite is sufficient for $\beta > 0$

For all $u, v \in V(\Omega \cup \Gamma)$, define the symmetric bilinear form

$$B(u, v) = \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v) G(u) \, dx'dx$$

$$= \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} (v' - v)(u' - u) \beta \alpha^2 \, dx'dx$$

- note that $B(u, u) \geq 0$
Let

\[(u, v) = B(u, v) \quad \text{and} \quad |||u||| = (B(u, u))^{1/2}\]

and

\[V(\Omega \cup \Gamma) = \{u : |||u||| < \infty\}\]

We have, for both \(V_0(\Omega \cup \Gamma)\) and \(V(\Omega \cup \Gamma) \setminus \mathbb{R}\) that

\[(\cdot, \cdot) \quad \text{defines an inner product} \quad \text{and} \quad ||| \cdot ||| \quad \text{defines and norm}\]
• We restrict attention to the linear nonlocal variational formulation with homogeneous “Dirichlet boundary” condition:

\[
\text{given } b, \text{ seek } u \in V_0(\Omega \cup \Gamma) \text{ such that }
\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v) G(u) \, dx' \, dx = \int_{\Omega \cup \Gamma} vb \, dx \, dx \quad \forall \, v \in V_0(\Omega \cup \Gamma)
\]

or

\[
\text{given } b, \text{ seek } u \in V_0(\Omega \cup \Gamma) \text{ such that }
B(u, v) = F(v) \quad \forall \, v \in V_0(\Omega \cup \Gamma)
\]

where

\[
F(v) = \int_{\Omega \cup \Gamma} vb \, dx
\]

• The Lax-Milgram theorem can be applied to obtain the well-posedness of the variational problem
• Related papers that give more extensive analyses, including making connections between the function space $V(\Omega \cup \Gamma)$ and fractional Sobolev spaces with index $0 \leq s \leq 1$

Q. DU AND K. ZHOU, Mathematical analysis for the peridynamic nonlocal continuum theory; submitted.

and a paper in preparation (Du, G., Lehoucq, Zhou)
Decomposition of the solution space

- Let the space $S(\Omega \cup \Gamma)$ consist of functions $u \in V(\Omega \cup \Gamma)$ that satisfy

\[
\begin{align*}
\mathcal{D}(\beta \mathcal{G}(u)) &= 2 \int_{\Omega \cup \Gamma} (u' - u) \beta \alpha^2 \, dx' = 0 \quad \forall x \in \Omega \\
\mathcal{N}(\beta \mathcal{G}(u)) &= -2 \int_{\Omega \cup \Gamma} (u' - u) \beta \alpha^2 \, dx' = 0 \quad \forall x \in \Gamma.
\end{align*}
\]

- Then, we have that, for all $u \in S(\Omega \cup \Gamma)$ and $v \in V_0(\Omega \cup \Gamma)$,

\[
((u, v)) = \int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \mathcal{G}(v) \mathcal{G}(u) \beta \, dx' \, dx = 0
\]

- Thus, we conclude that

\[
V(\Omega \cup \Gamma) = V_0(\Omega \cup \Gamma) \oplus S(\Omega \cup \Gamma)
\]
thus, any function in $V(\Omega \cup \Gamma)$ can be written as a sum of two functions that are orthogonal with respect to the inner product $(\cdot, \cdot)$

- the first a function that vanishes on $\Gamma_e$
- the second a function belonging to $S(\Omega \cup \Gamma)$

this is entirely analogous to the decomposition of the Sobolev space $H^1(\Omega)$ into functions belonging to $H^1_0(\Omega)$ and harmonic functions
• For each $y \in \mathbb{R}^d$, let $g(x; y)$ denote a solution of

$$
\mathcal{D}(\beta G(g(x; y))) = \delta(|x - y|) \quad \forall x \in \mathbb{R}^d
$$

$g(x; y)$ is then a nonlocal fundamental solution or nonlocal free-space Green’s function for the operator $\mathcal{D}(\beta G(\cdot))$

• We assume that

- $\alpha$ and $\beta$ are radial functions of $x$ and $x'$, e.g., $\alpha(x, x') = \alpha(x' - x)$
- $g(x; y) = g(x - y)$
- $\int_{\mathbb{R}^d} \alpha^2 \beta \, dx = 1$, where the scaling 1 is without loss of generality
Let \( \hat{\mu} \) denote the Fourier transform of \( \mu = \beta \alpha^2 \)

then, it can be shown that a fundamental solution is given by

\[
g(x; y) = \frac{(2\pi)^{-d}}{2} \int_{\mathbb{R}^d} e^{ik \cdot (x-y)} \frac{1}{(2\pi)^{d/2} \hat{\mu} - 1} \, dk
\]

note that the special choice

\[
\mu(x' - x) = \delta(x' - x) + \frac{d^2}{dx^2} \delta(x' - x)
\]

leads to the same fundamental solution as that for the Laplace operator:

\[
-(2\pi)^{-d} \int_{\mathbb{R}^d} e^{ik \cdot (x-y)} |k|^{-2} \, dk = \begin{cases} 
\frac{1}{2} |x - y| & \text{for } d = 1 \\
\frac{1}{2\pi} \ln |x - y| & \text{for } d = 2 \\
\frac{1}{2\omega_d} \frac{|x - y|^{2-d}}{2 - d} & \text{for } d \geq 3
\end{cases}
\]

where \( \omega_d \) denotes the volume of the unit ball in \( \mathbb{R}^d \)
Nonlocal Green’s third identity

• For any \( y \in \Omega \cup \Gamma \), let \( G(x; y) : \Omega \cup \Gamma \rightarrow \mathbb{R} \) denote any function satisfying

\[
\mathcal{D}(\beta G(G(x; y))) = \delta(|x - y|) \quad \forall x \in \Omega
\]

• Then, from the nonlocal Green’s second identity we obtain the nonlocal Green’s third identity

\[
u(y) = \int_{\Omega} G(x; y) \mathcal{D}(\beta G(u(x))) \, dx \\
- \int_{\Gamma} \left( G(x; y) \mathcal{N}(\beta G(u(x))) - u(x) \mathcal{N}(\beta G(G(x; y))) \right) \, dx \quad \forall y \in \Omega
\]

– this the analog of the classical Green’s third identity

\[
u(y) = \int_{\Omega} G(x; y) \Delta u(x) \, dx - \int_{\partial \Omega} \left( G(x; y) \frac{\partial u(x)}{\partial n} - u(x) \frac{\partial G(x; y)}{\partial n} \right) \, dA
\]
Suppose that the constitutive function $\beta = 1$ so that
\[ D(G(u)) = 2 \int_{\Omega \cup \Gamma} (u' - u) \alpha^2 \, dx' = 0 \quad \forall \, x \in \Omega \]
then, the solution $u(x)$ represents the nonlocal “harmonic” function
\[ u(y) = \int_{\Gamma} \left( u(x)N(G(G(x; y))) - G(x; y)N(G(u(x))) \right) \, dx, \]
i.e., “harmonic” functions are determined by their “boundary” values on $\Gamma$

Nonlocal versions of the Poisson integral formula and Gauss’s law of the arithmetic mean can also be derived
Nonlocal Green’s functions

• Let $g(x; y)$ denote a nonlocal fundamental solution

• For each $y \in \Omega \cup \Gamma$, define the nonlocal Green’s function $G(x; y) : \Omega \cup \Gamma \to \mathbb{R}$ as

$$G(x; y) = g(x; y) - \tilde{g}(x; y)$$

where $\tilde{g}(\cdot; \cdot)$ is a solution of

\[
\begin{align*}
\mathcal{D}(\beta G(\tilde{g})) &= 0 \quad \text{for } x \in \Omega \\
\mathcal{N}(\beta G(\tilde{g})) &= \mathcal{N}(\beta G(g)) \quad \text{for } x \in \Gamma_n \\
\tilde{g}(x; y) &= g(x; y) \quad \text{for } x \in \Gamma_e
\end{align*}
\]

– then, $G(\cdot; \cdot)$ satisfies the homogeneous “boundary” conditions

$$G(x; y) = 0 \quad \text{for } x \in \Gamma_e \quad \text{and} \quad \mathcal{N}(\beta G(G)) = 0 \quad \text{for } x \in \Gamma_n$$
Then, from the nonlocal Green’s third identity, we have that the solution of the nonlocal “boundary-value” problem is given by

\[ u(y) = -\int_{\Omega} G(x; y) b(x) \, dx + \int_{\Gamma_e} h_e(x) \mathcal{N}(\beta G(G(x; y))) \, dx - \int_{\Gamma_n} G(x; y) h_n(x) \, dx \quad \forall y \in \Omega \cup \Gamma_n \]

– the classical analog is

\[ u(y) = -\int_{\Omega} G(x; y) b(x) \, dx + \int_{\Gamma_e} h_e \frac{\partial G}{\partial n} \, dA - \int_{\Gamma_n} h_n G \, dA \quad \forall y \in \Omega \]
LOCAL SMOOTH LIMITS

- We now see what happens are a result of the assumptions:
  1. solutions of the nonlocal “boundary-value”-problems are smooth
  2. the nonlocal operators are asymptotically local

- we emphasize that these assumptions are made only to make the connection to classical problems for partial differential equations and are not required for the well posedness of the nonlocal “boundary-value”-problems

- in addition, the nonlocal “boundary-value”-problems admit solutions that are not solutions, even in the usual sense of weak solutions, of the partial differential equations

- thus, one can view solutions of the nonlocal “boundary-value”-problems as further generalizations of solutions of the partial differential equations, generalized in two ways: they are nonlocal and they lack the smoothness needed for them to be standard weak solutions
• **Smoothness assumption:** assume that the solution \( u(x) \) of the nonlocal “boundary-value”-problem satisfies

\[
u(x') = u(x) + \nabla u(x) \cdot (x' - x) + o(\varepsilon^2) \quad \text{if} \quad |x' - x| \leq \varepsilon
\]

where \( \varepsilon^{-2}o(\varepsilon^2) \to 0 \) as \( \varepsilon \to 0 \)

– actually, we only need this to hold weakly

• **Locality assumption:** assume that the constitutive function \( \beta(x, x') \) is integrable and

is a positive, symmetric function such that

\[
\beta(x, x') = 0 \quad \text{whenever} \quad |x' - x| > \varepsilon
\]
Next, set

\[ \Gamma = \Gamma(\varepsilon) = \bigcup_{x \in \Omega} \text{supp}(\beta) \setminus \Omega \]

\[ \partial \Omega_e = \partial \Omega \cap \partial \Gamma_e \quad \text{and} \quad \partial \Omega_n = \partial \Omega \cap \partial \Gamma_n \]

and assume that

\[ \partial \Omega_e \neq \emptyset \quad \text{and} \quad \partial \Omega_n \neq \emptyset \]
• Then

\[\varepsilon \to 0 \quad \begin{cases}
|\Gamma| = O(\varepsilon) \\
\partial\Gamma_e \to \partial\Omega_e \\
\partial\Gamma_n \to \partial\Omega_n
\end{cases}\]

• We also set

\[\alpha(x, x') = \frac{1}{|x - x'|}\]

and assume a scaling such that, for some positive constants \(\underline{\beta}\) and \(\overline{\beta}\),

\[\underline{\beta} < \int_{S_\varepsilon(x)} \beta(x, x') \, dx' < \overline{\beta}\]

uniformly for \(x \in \Omega\),

where \(S_\varepsilon(x) := \{x' \in \mathbb{R}^d \mid |x' - x| < \varepsilon\}\)
Then, using the assumptions we have made, we are led to

\[ B(u, v) = \int_{\Omega \cup \Gamma(\varepsilon)} \nabla v \cdot \left( D_\varepsilon + o(\varepsilon^0) \right) \cdot \nabla u \, dx \]

where

\[ D_\varepsilon(x) = \int_{S_\varepsilon(x) \cap (\Omega \cup \Gamma(\varepsilon))} \frac{(x' - x) \otimes (x' - x) K(x' - x) \otimes (x' - x)}{|x' - x|^2} \, dx' \]

so that

\[ \lim_{\varepsilon \to 0} B(u, v) = \int_{\Omega} \nabla v \cdot (D \cdot \nabla u) \, dx \]

where

\[ D(x) = \lim_{\varepsilon \to 0} D_\varepsilon(x) \]
• Then, using the local and nonlocal Green’s identities, it can be shown that, as $\varepsilon \to 0$, the nonlocal variational problem

$$\begin{align*}
\text{seek } u \in V(\Omega \cup \Gamma) \text{ such that } \\
u = h_e \quad \text{for } x \in \Gamma_e
\end{align*}$$

and

$$\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v)G(u) \, dx' \, dx = \int_{\Omega} vb \, dx + \int_{\Gamma_n} vh_n \, dx \quad \forall v \in V_0(\Omega \cup \Gamma)$$

reduces to the classical local variational problem

$$\begin{align*}
\left\{ \begin{array}{l}
\int_{\Omega} \nabla v \cdot (D \cdot \nabla u) = \int_{\Omega} vb \, dx + \int_{\partial \Omega_n} \tilde{v} h_n \, dx \quad \text{in } \Omega \\
u = \tilde{h}_e \quad \text{on } \partial \Omega_e
\end{array} \right.
\end{align*}$$

– $\tilde{h}_e$ and $\tilde{h}_n$ denote traces of the nonlocal data $h_e$ and $h_n$, respectively
Also,
as \( \varepsilon \to 0 \), the corresponding nonlocal “boundary-value” problem

\[
-\mathcal{D}(\beta \mathcal{G}(u)) = b \quad \text{for } x \in \Omega \\
u = h_e \quad \text{for } x \in \Gamma_e \\
\mathcal{N}(\beta \mathcal{G}(u)) = h_n \quad \text{for } x \in \Gamma_n
\]

reduces to classical boundary-value problem

\[
\begin{cases}
-\nabla \cdot (D \cdot \nabla u) = b & \text{in } \Omega \\
u = \tilde{h}_e & \text{on } \partial\Omega_e \\
(D \cdot \nabla u) \cdot n = \tilde{h}_n & \text{on } \partial\Omega_n
\end{cases}
\]
NONLOCAL LINEAR
CONVECTION-DIFFUSION-REACTION PROBLEMS

- Let \( \sigma(x, x') \) and \( \omega(x, x') \) denote anti-symmetric and symmetric functions, respectively.

- Consider the nonlocal variational principle

\[
\text{seek } u \in V(\Omega \cup \Gamma) \text{ such that } \quad u = h_e \text{ for } x \in \Gamma
\]

and

\[
\int_{\Omega \cup \Gamma} \int_{\Omega \cup \Gamma} \beta G(v)G(u) \, dx' \, dx + \int_{\Omega \cup \Gamma} v \int_{\Omega \cup \Gamma} \sigma G(u) \, dx' \, dx \\
+ \int_{\Omega \cup \Gamma} v \int_{\Omega \cup \Gamma} \omega(u' + u) \, dx' \, dx = \int_{\Omega} vb \, dx \quad \forall v \in V_0(\Omega \cup \Gamma)
\]

- here, we only examine nonlocal “Dirichlet” problems; clearly, their “Neumann” counterparts can be defined in a similar manner.
• The corresponding nonlocal “Dirichlet boundary-value” problem is given by

\[-\mathcal{D}(\beta \mathcal{G}(u)) + \sigma \mathcal{G}(u) + \omega(u' + u) = b \quad \text{for } x \in \Omega\]
\[u = h_e \quad \text{for } x \in \Gamma\]

• General problems may be defined by choosing the constitutive relations

\[\beta = (x' - x) \cdot K \cdot (x' - x)\]
\[\sigma = a \cdot (x' - x)\]
\[\omega = r\]

where
\[a(x, x')\text{ denotes a symmetric vector-valued function}\]
\[r(x, x')\text{ denotes a symmetric function}\]
Then, as before, we now have that the general nonlocal “Dirichlet boundary-value” problem reduces to the general linear convection-diffusion-reaction problem

\[-\nabla \cdot (D \cdot \nabla u) + w \cdot \nabla u + cu = b\]

along with a Dirichlet boundary condition, where \(w\) and \(c\) are related to \(a\) and \(r\) in a manner similar to the relation between \(D\) and \(K\).
CURRENT AND FUTURE WORK

• With Du, Lehoucq, and Zhou
  – fusing the nonlocal calculus to the connections made by Du and Zhou to Sobolev spaces
  – extension to the vector-valued case
  – extension to material interface problems
  – application to the peridynamic model of materials
  – extension to nonlinear problems

• With M. Parks and P. Seleson
  – extension to material interface problems