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Global Existence of Weak Solutions for Viscous Incompressible Flows around a Moving Rigid Body in Three Dimensions

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Abstract. We study the motion of a rigid body of arbitrary shape immersed in a viscous incompressible fluid in a bounded, three-dimensional domain. The motion of the rigid body is caused by the action of given forces exerted on the fluid and on the rigid body. For this problem, we prove the global existence of weak solutions.

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1. Introduction

We consider a coupled system of nonlinear partial and ordinary differential equations modeling the motion of a rigid body inside of a fluid flow. The governing equations for the fluid flow are given by the classical Navier–Stokes system whereas the motion of the solid is governed by the equations for the balance of linear and angular momentum. The motion of the rigid body is caused by the action of given forces exerted on the fluid and the rigid body.

Suppose that a rigid body B moves inside of a domain D_0 filled up with a viscous fluid. At the initial moment of time, the body occupies a set $\overline{S_0} \subset D_0$, where $\overline{S_0}$ is the closure of a simply connected domain S_0. This is the *reference configuration.* The Cartesian coordinates $\mathbf{y}$ of points of the body at the initial moment are called material or Lagrangian coordinates. We assume that the center of mass is located at $\mathbf{y} = \mathbf{0}$. The trajectory of any material point $\mathbf{y}$ is completely described by two functions:

$$t \mapsto Q(t) \in SO(3) \qquad \text{and} \qquad t \mapsto \overline{\mathbf{x}}(t) \in \mathbb{R}^3,$$

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where the second function is just the trajectory of the center of mass of the rigid body. The equations of the trajectory of any material point have the form

$$\mathbf{x}(\mathbf{y},t) = Q(t)\mathbf{y} + \overline{\mathbf{x}}(t), \quad \mathbf{y} \in \overline{S_0}. \tag{1.1}$$

Thus, at the time t, the body occupies the set $\overline{S(t)}$ where

$$S(t) = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = Q(t)\mathbf{y} + \overline{\mathbf{x}}(t),\ \mathbf{y} \in S_0\right\}. \tag{1.2}$$

As a matter of convenience, let us recall some notation from tensor algebra which we are going to use. We shall use only fixed Cartesian coordinates. For that reason, we will not distinguish between tensors and their coordinate matrices with respect to the chosen Cartesian coordinates. Thus, algebraic operations with tensors can be expressed via algebraic operations with matrices. Denoting by $M^{3\times3}$ the space of all real 3×3 matrices and adopting the convention on summation over repeated lower Latin indices running from 1 to 3, we shall use the following notations:

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \qquad |\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}},$$
$$\mathbf{u}\wedge\mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \in \mathbb{R}^3,$$
$$\mathbf{u}\otimes\mathbf{v} = (u_iv_j) \in M^{3\times3}, \qquad A : B = tr(A^*B) = A_{ij}B_{ij}, \qquad |A| = \sqrt{A : A},$$
$$A^* = (A_{ji}) \in M^{3\times3}, \qquad trA = A_{ii}, \qquad \text{and} \qquad A\mathbf{u} = (A_{ij}u_j)$$

for $\forall\,\mathbf{u} = (u_j) \in \mathbb{R}^3$, $\forall\,\mathbf{v} = (v_j) \in \mathbb{R}^3$, $\forall\,A = (A_{ij}) \in M^{3\times3}$, and $\forall\,B = (B_{ij}) \in M^{3\times3}$.

The motion of the fluid is described by the velocity field $\mathbf{v}^f$. The vector $\mathbf{v}^f(\mathbf{x},t)$ is interpreted as the velocity of the fluid material point which has Cartesian coordinates $\mathbf{x}$ at the time t. This provides the *Euler description* of the fluid motion. With (1.1), one can give an Eulerian description of the motion of the solid body B:

$$\mathbf{v}^s(\mathbf{x},t) = \dot{Q}(t)Q^*(t)(\mathbf{x} - \overline{\mathbf{x}}(t)) + \overline{\mathbf{w}}(t), \qquad \mathbf{x} \in S(t), \tag{1.3}$$

where

$$\overline{\mathbf{w}}(t) = \dot{\overline{\mathbf{x}}}(t) = \frac{d\overline{\mathbf{x}}(t)}{dt} \tag{1.4}$$

and we use the notation for the derivative in time

$$\dot{b}(t) = \frac{db(t)}{dt},$$

if a function b depends on time only.

Since $Q(t) \in SO(3)$, the matrix $\dot{Q}(t)Q^*(t)$ is skew symmetric. Thus, one can determine the rotation vector $\overline{\boldsymbol{\varpi}}(t)$ by the identity

$$\dot{Q}(t)Q^*(t)\mathbf{z} = \overline{\boldsymbol{\varpi}}(t) \wedge \mathbf{z}, \qquad \mathbf{z} \in \mathbb{R}^3. \tag{1.5}$$

Then, (1.3) can be written in the form

$$\mathbf{v}^s(\mathbf{x},t) = \overline{\boldsymbol{\varpi}}(t) \wedge (\mathbf{x} - \overline{\mathbf{x}}(t)) + \overline{\mathbf{w}}(t), \qquad \mathbf{x} \in S(t). \tag{1.6}$$

The dynamics of the coupled fluid-rigid body system is described by the following system of partial and ordinary differential equations:

$$\begin{cases} \rho^f \partial_t \mathbf{v}^f(\mathbf{x},t) + \rho^f \operatorname{div}\left(\mathbf{v}^f(\mathbf{x},t)\otimes \mathbf{v}^f(\mathbf{x},t)\right) - \operatorname{div}\sigma^f(\mathbf{x},t) = \rho^f\,\mathbf{f}(\mathbf{x},t) \\ \operatorname{div}\mathbf{v}^f(\mathbf{x},t) = 0 \\ \sigma^f(\mathbf{x},t) = 2\nu\varepsilon\left(\mathbf{v}^f(\mathbf{x},t)\right) - p^f(\mathbf{x},t)I \end{cases} \tag{1.7}$$

for all $\mathbf{x}\in D_0\backslash\overline{S(t)}$, $t\in[0,T]$,

$$\begin{cases} m_s\,\dot{\overline{\mathbf{w}}}(t) = \int_{\partial S(t)} \sigma^f(\mathbf{x},t)\,\mathbf{n}(\mathbf{x},t)\,ds + \rho^s\int_{S(t)}\mathbf{f}(\mathbf{x},t)\,d\mathbf{x} \\ \overline{J}\dot{\overline{\boldsymbol{\omega}}}(t) = \overline{J}\overline{\boldsymbol{\omega}}(t)\wedge\overline{\boldsymbol{\omega}}(t) + \int_{\partial S(t)}(\mathbf{x}-\overline{\mathbf{x}}(t))\wedge\left(\sigma^f(\mathbf{x},t)\,\mathbf{n}(\mathbf{x},t)\right)\,ds \\ \qquad\qquad + \rho^s\int_{S(t)}(\mathbf{x}-\overline{\mathbf{x}}(t))\wedge\mathbf{f}(\mathbf{x},t)\,d\mathbf{x} \end{cases} \tag{1.8}$$

for all $t\in[0,T]$, and

$$\begin{aligned} &\mathbf{v}^f(\mathbf{x},t) = \mathbf{v}^s(\mathbf{x},t), \qquad \mathbf{x}\in\partial S(t), \quad t\in[0,T], \\ &\mathbf{v}^f(\mathbf{x},t) = \mathbf{0}, \qquad \mathbf{x}\in\partial D_0 \quad t\in[0,T]. \end{aligned} \tag{1.9}$$

Here, $\rho^f > 0$ and $\rho^s > 0$ are the constant densities of the fluid and the rigid body, respectively, $m_s = \rho^s|S_0|$, $\mathbf{f}$ is a given external force per unit mass, σ^f is the Cauchy stress tensor in the fluid, p^f is the pressure field in the fluid, ν is the constant viscosity of the fluid, $\varepsilon(\mathbf{v}) \equiv \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^*)$, I is the identity matrix, $\mathbf{n}(\cdot,t)$ is the unit outer normal vector to the surface $\partial S(t)$ of the set $S(t)$, and the inertial matrix $\overline{J}$ of the body is defined by the following identity:

$$(\overline{J}\mathbf{a})\cdot\mathbf{b} = \rho^s\int_{S(t)}(\mathbf{a}\wedge(\mathbf{x}-\overline{\mathbf{x}}(t)))\cdot(\mathbf{b}\wedge(\mathbf{x}-\overline{\mathbf{x}}(t)))\,d\mathbf{x} \tag{1.10}$$

for any $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$.

The system (1.1)–(1.10) must be complemented by the initial conditions

$$\begin{cases} \mathbf{v}^f(\mathbf{x},0) = \mathbf{v}_0^f(\mathbf{x}), \qquad \mathbf{x}\in D_0\backslash\overline{S_0} \\ \mathbf{v}^s(\mathbf{x},0) = \overline{\boldsymbol{\omega}}_0\wedge\mathbf{x} + \overline{\mathbf{w}}_0, \qquad \mathbf{x}\in S_0 \end{cases} \tag{1.11}$$

and

$$\begin{cases} \overline{\boldsymbol{\omega}}(0) = \boldsymbol{\omega}_0, \qquad \overline{\mathbf{w}}(0) = \mathbf{w}_0 \\ Q(0) = I, \qquad \overline{\mathbf{x}}(0) = \mathbf{0}. \end{cases} \tag{1.12}$$

We always seek solutions $\mathbf{v}^f$, Q, $\overline{\mathbf{x}}$, $\overline{\boldsymbol{\omega}}$, and $\overline{\mathbf{w}}$ to the initial-boundary value problem (1.1)–(1.12) which satisfy the relation

$$\operatorname{dist}\{\partial D_0, S(t)\} > 0 \quad \text{for all } t\in[0,T], \tag{1.13}$$

i.e., the rigid body does not touch the confining boundary of the fluid domain.

In (1.7), we have used the following differential operators:

$$\nabla p = (p_{,j}), \qquad \nabla \mathbf{v} = (v_{i,j}), \qquad \operatorname{div} \mathbf{v} = (v_{i,i}),$$
$$\operatorname{div} \sigma = (\sigma_{ij,j}), \qquad \text{and} \qquad \partial_t \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t},$$

where the comma in the lower indices indicates differentiation with respect to the corresponding Cartesian coordinate; for example, $p_{,i} \stackrel{\text{def}}{=} \partial p / \partial x_i$.

The goal of this paper is to prove the global existence of weak solutions to the initial-boundary value problem (1.1)–(1.12). In particular, the problem we consider corresponds to the motion of a rigid body of arbitrary shape immersed in a fluid; the fluid and the rigid body are subject to prescribed external forces and the fluid is subject to square integrable initial data. Of course, one should expect some restrictions on T but they are connected with condition (1.13) only. This problem, in different settings, was studied by other authors. The case $D_0 = \mathbb{R}^3$ was treated in [8] and [9], essentially for the *steady* case. For a bounded domain D_0 in $\mathbb{R}^3$, the existence of at least one local strong solution, even for many bodies inside of D_0, was proved in [2]. The authors of that paper assumed that the initial data have square integrable derivatives. In [1], the one body problem was considered in the case of a bounded domain D_0. Those authors assumed that the body is a ball, the external force $\mathbf{f}$ is equal to zero, and the initial data are square integrable. Their definition of a weak solution is similar to the one given here and their main theorem states the existence of at least one such a weak solution.

We also note the papers [3] and [5] on the closely related topic of the existence of weak solutions to initial-boundary value problems for the Navier–Stokes equations in domains with *prescribed* time-varying boundaries. The paper [4] is related in a different way; in it, the *steady* self-propelled motion of a body in a viscous incompressible fluid is studied.

We next discuss a consequence of (1.7) and (1.8), assuming that a solution to (1.1)–(1.12) exists and is smooth. We first introduce

$$\overline{\rho}\,(\mathbf{x},t) = \begin{cases} \rho^s, & \mathbf{x} \in S(t) \\ \rho^f, & \mathbf{x} \in D_0 \backslash \overline{S(t)}, \end{cases} \qquad \mathbf{v}(\mathbf{x},t) = \begin{cases} \mathbf{v}^s(\mathbf{x},t), & \mathbf{x} \in S(t) \\ \mathbf{v}^f(\mathbf{x},t), & \mathbf{x} \in D_0 \backslash \overline{S(t)}. \end{cases} \tag{1.14}$$

Next, let $\mathbf{V}$ be a sufficiently smooth solenoidal function in $D_0 \times]0,T[$ such that

$$\mathbf{V}(\mathbf{x},t) = \mathbf{0}, \qquad \mathbf{x} \in \partial D_0, \quad t \in [0,T],$$

and

$$\mathbf{V}(\mathbf{x},t) = \overline{\boldsymbol{\Omega}}(t) \wedge (\mathbf{x} - \overline{\mathbf{x}}(t)) + \overline{\mathbf{W}}(t), \qquad \mathbf{x} \in S(t), \quad t \in [0,T],$$

for some $t \mapsto \overline{\boldsymbol{\Omega}}(t) \in \mathbb{R}^3$ and $t \mapsto \overline{\mathbf{W}}(t) \in \mathbb{R}^3$. Then, using (1.10) and integrating by parts, we obtain

$$\int_{D_0} \Big[\overline{\rho}(\mathbf{x},t)\left(\partial_t \mathbf{v}(\mathbf{x},t)\cdot \mathbf{V}(\mathbf{x},t) + \nabla \mathbf{v}(\mathbf{x},t) : (\mathbf{V}(\mathbf{x},t)\otimes \mathbf{v}(\mathbf{x},t))\right)$$
$$+ 2\nu\varepsilon(\mathbf{v}(\mathbf{x},t)) : \varepsilon(\mathbf{V}(\mathbf{x},t)) - \overline{\rho}(\mathbf{x},t)\,\mathbf{f}(\mathbf{x},t)\cdot \mathbf{V}(\mathbf{x},t)\Big]\,d\mathbf{x} = 0 \quad (1.15)$$

for any $t \in [0,T]$.

It turns out to be more convenient to interchange the roles of the reference and current configurations. To this end, we make the following change of variables and functions:

$$\overline{\rho}(\mathbf{x},t) = \rho(\mathbf{y}), \quad \mathbf{v}(\mathbf{x},t) = Q(t)\mathbf{u}(\mathbf{y},t), \qquad \mathbf{x}\in D_0, \quad \mathbf{y}\in D(t), \tag{1.16}$$

where

$$D(t) = \left\{\mathbf{y}\in\mathbb{R}^3 \mid \mathbf{y} = Q^*(t)(\mathbf{x}-\overline{\mathbf{x}}(t)), \quad \mathbf{x}\in D_0\right\}, \tag{1.17}$$

and

$$\overline{\boldsymbol{\omega}}(t) = Q(t)\boldsymbol{\omega}(t), \qquad \overline{\mathbf{w}}(t) = Q(t)\mathbf{w}(t), \qquad \overline{J} = Q(t)JQ^*(t). \tag{1.18}$$

Now, (1.4)–(1.6) and (1.9)–(1.14) take the form

$$\dot{\overline{\mathbf{x}}} = Q(t)\mathbf{w}(t), \tag{1.19}$$

$$Q^*(t)\dot{Q}(t)\mathbf{z} = \boldsymbol{\omega}(t)\wedge\mathbf{z} \quad \forall\,\mathbf{z}\in\mathbb{R}^3, \tag{1.20}$$

$$\mathbf{u}^s(\mathbf{y},t) = \boldsymbol{\omega}(t)\wedge\mathbf{y} + \mathbf{w}(t), \quad \mathbf{y}\in S_0, \tag{1.21}$$

$$\begin{cases} \mathbf{u}^f(\mathbf{y},t) = \mathbf{u}^s(\mathbf{y},t), & \mathbf{y}\in\partial S_0, \quad t\in[0,T],\\ \mathbf{u}^f(\mathbf{y},t) = \mathbf{0}, & \mathbf{y}\in\partial D(t), \quad t\in[0,T], \end{cases} \tag{1.22}$$

$$(J\mathbf{a})\cdot\mathbf{b} = \rho^s\int_{S_0}(\mathbf{a}\wedge\mathbf{y})\cdot(\mathbf{b}\wedge\mathbf{y})\,d\mathbf{y}, \quad \mathbf{a},\mathbf{b}\in\mathbb{R}^3, \tag{1.23}$$

$$\begin{cases} \mathbf{u}^f(\mathbf{y},0) = \mathbf{u}_0^f = \mathbf{v}_0^f, & \mathbf{y}\in D_0\backslash\overline{S_0},\\ \mathbf{u}^s(\mathbf{y},0) = \boldsymbol{\omega}_0\wedge\mathbf{y} + \mathbf{w}_0, & \mathbf{y}\in S_0, \end{cases} \tag{1.24}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \quad \mathbf{w}(0) = \mathbf{w}_0, \quad Q(0) = I, \quad \overline{\mathbf{x}}(0) = \mathbf{0}, \tag{1.25}$$

$$\operatorname{dist}\{\partial D(t), S_0\} > 0, \quad t\in[0,T], \tag{1.26}$$

and

$$\rho(\mathbf{y}) = \begin{cases} \rho^s, & \mathbf{y}\in S_0,\\ \rho^f, & \mathbf{y}\in D(t)\backslash\overline{S_0}, \end{cases} \qquad \mathbf{u}(\mathbf{y},t) = \begin{cases} \mathbf{u}^s(\mathbf{y},t), & \mathbf{y}\in S_0,\\ \mathbf{u}^f(\mathbf{y},t), & \mathbf{y}\in D(t)\backslash\overline{S_0}. \end{cases} \tag{1.27}$$

If we choose a test function $\mathbf{U}$ such that

$$\mathbf{V}(\mathbf{x},t) = Q(t)\mathbf{U}(\mathbf{y},t), \qquad \mathbf{x}\in D_0,, \quad \mathbf{y}\in D(t),$$
$$\mathbf{U}(\mathbf{y},t) = \boldsymbol{\Omega}(t)\wedge\mathbf{y} + \mathbf{W}(t), \qquad \mathbf{y}\in S_0,$$

where $\mathbf{\Omega}(t) = Q^*(t)\overline{\mathbf{\Omega}}(t)$ and $\mathbf{W}(t) = Q^*(t)\overline{\mathbf{W}}(t)$, then (1.15) yields the identity

$$\begin{aligned}\int_{D(t)} \Big[&\rho(\mathbf{y}) \big((\partial_t \mathbf{u}(\mathbf{y},t) + \boldsymbol{\omega}(t) \wedge \mathbf{u}(\mathbf{y},t)) \cdot \mathbf{U}(\mathbf{y},t) \\ &+ \nabla \mathbf{u}(\mathbf{y},t) : (\mathbf{U}(\mathbf{y},t) \otimes (\mathbf{u}(\mathbf{y},t) - \boldsymbol{\omega}(t) \wedge \mathbf{y} - \mathbf{w}(t))\big) \\ &+ 2\nu\, \varepsilon(\mathbf{u}(\mathbf{y},t)) : \varepsilon(\mathbf{U}(\mathbf{y},t)) - \rho(\mathbf{y})\, Q^*(t)\mathbf{f}(Q(t)\mathbf{y} \\ &+ \overline{\mathbf{x}}(t), t) \cdot \mathbf{U}(\mathbf{y},t)\Big]\, d\mathbf{y} = 0\end{aligned} \tag{1.28}$$

for any $t \in [0,T]$.

2. The notion of weak solutions and main results

In order to define the notion of weak solutions to the initial-boundary value problem (1.1)–(1.12), or, equivalently, to the initial-boundary value problem (1.19)–(1.28), we need to introduce some sets and function spaces.

First, we introduce the set of admissible configurations

$$\mathcal{R} \overset{\text{def}}{=} \big\{ D \subset \mathbb{R}^3 \mid \overline{S_0} \subset D = \{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = Q^*(\mathbf{x} - \overline{\mathbf{x}}),\, \mathbf{x} \in D_0\} \\ \text{for some } Q \in SO(3), \text{ and for some } \overline{\mathbf{x}} \in \mathbb{R}^3 \big\}.$$

Next, let us choose a ball D_+ with the center at the origin such that $D \subset D_+$ and $\operatorname{dist}\{\partial D_+, D\} \geq 1$ for all $D \subset \mathcal{R}$.

We shall use the usual Lebesque and Sobolev spaces $L_r(D;\mathbb{R}^3)$ and $W^l_r(D;\mathbb{R}^3)$, respectively. Now, for $D \subset \mathcal{R}$, we define the set

$$C_\infty(D) \overset{\text{def}}{=} \big\{ \mathbf{v} \in W^1_\infty(D_+;\mathbb{R}^3) \mid \operatorname{spt} \mathbf{v} \subset D,\, \operatorname{div} \mathbf{v} = 0 \text{ in } D_+, \\ \mathbf{v}(\mathbf{y}) = \boldsymbol{\omega} \wedge \mathbf{y} + \mathbf{w} \text{ for any } \mathbf{y} \in S_0, \text{ for some } \boldsymbol{\omega} \in \mathbb{R}^3, \text{ and for some } \mathbf{w} \in \mathbb{R}^3 \big\}.$$

We denote by $H(D)$ and $H^1(D)$ the closure of the set $C_\infty(D)$ in $L_2(D_+;\mathbb{R}^3)$ and $W^1_2(D_+;\mathbb{R}^3)$, respectively. We assume that

$$D_0 \text{ and } S_0 \text{ are of class } C^2. \tag{2.1}$$

Then, one can claim that for any $D \in \mathcal{R}$, the following statements are valid:

$$\begin{cases} (1) \text{ the embedding } H^1(D) \text{ into } H(D) \text{ is dense and compact;} \\ (2)\ H(D) = \big\{ \mathbf{v} \in L_2(D_+;\mathbb{R}^3) \mid \mathbf{v} = \mathbf{0} \text{ in } D_+ \backslash D,\ \operatorname{div} \mathbf{v} = 0 \text{ in } D_+, \\ \qquad \mathbf{v}(\mathbf{y}) = \boldsymbol{\omega} \wedge \mathbf{y} + \mathbf{w} \text{ for any } \mathbf{y} \in S_0, \text{ for some } \boldsymbol{\omega} \in \mathbb{R}^3, \\ \qquad \text{and for some } \mathbf{w} \in \mathbb{R}^3 \big\}; \\ (3)\ H^1(D) = H(D) \cap W^1_2(D_+;\mathbb{R}^3). \end{cases} \tag{2.2}$$

We also assume that

$$(J\mathbf{a})\cdot\mathbf{a} \geq \gamma_0|\mathbf{a}|^2, \quad \mathbf{a}\in\mathbb{R}^3, \tag{2.3}$$

for some positive number γ_0.

We extend the function ρ to the whole ball D_+ by setting

$$\rho \stackrel{\text{def}}{=} \begin{cases} \rho^f & \text{in } D_+\backslash\overline{S_0}\,, \\ \rho^s & \text{in } S_0. \end{cases}$$

Then, for any $D\subset\mathcal{R}$ and for any $\mathbf{u}\in H(D)$, we let

$$E(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2}\int_{D_+}\rho\,|\mathbf{u}|^2\,d\mathbf{y} = \frac{1}{2}\left(\int_{D\backslash S_0}\rho^f\,|\mathbf{u}|^2\,d\mathbf{y} + m_s|\mathbf{w}|^2 + (J\boldsymbol{\omega})\cdot\boldsymbol{\omega}\right).$$

Concerning the external force, we assume that

$$A_{\mathbf{f}}^2 = \max\{\rho^f,\rho^s\}\sup_{0\leq t<+\infty}\int_{D_0}|\mathbf{f}(\mathbf{x},t)|^2\,d\mathbf{x} < +\infty \quad \text{and} \quad \mathbf{f}\in C(\overline{D}\times[0,T];\mathbb{R}^3) \tag{2.4}$$

for any bounded domain $D\subset\mathbb{R}^3$ and for any $T>0$. Concerning initial data, it is assumed that

$$\mathbf{u}_0\in H(D_0) \tag{2.5}$$

and

$$d \stackrel{\text{def}}{=} \operatorname{dist}\{\partial D_0, S_0\} > 0. \tag{2.6}$$

Definition 2.1. *We say that a function* $\mathbf{u}$ *is a weak solution to the initial-boundary value problem* (1.19)–(1.28) *if it has the following properties:*

$$\mathbf{u}\in L_\infty\left(0,T;L_2(D_+;\mathbb{R}^3)\right)\cap L_2\left(0,T;W_2^1(D_+;\mathbb{R}^3)\right), \tag{2.7}$$

$$\mathbf{u}(\cdot,t)\in H^1(D(t)) \quad \textit{for a.a. } t\in[0,T], \tag{2.8}$$

where

$$D(t) \stackrel{\text{def}}{=} \left\{\mathbf{y}\in\mathbb{R}^3 \mid \mathbf{y} = Q^*(t)(\mathbf{x}-\overline{\mathbf{x}}(t)), \mathbf{x}\in D_0\right\}\in\mathcal{R} \quad \textit{for all } t\in[0,T], \tag{2.9}$$

and the functions

$$Q\in W_\infty^1(0,T;M^{3\times3}) \qquad \textit{and} \qquad \overline{\mathbf{x}}\in W_\infty^1(0,T;\mathbb{R}^3) \tag{2.10}$$

satisfy the relations

$$\overline{\mathbf{x}}(t) = \int_0^t Q(\tau)\mathbf{w}(\tau)\,d\tau \quad \textit{for all } t\in[0,T], \tag{2.11}$$

$$\begin{cases} Q^*(t)\dot{Q}(t)\mathbf{z} = \boldsymbol{\omega}(t)\wedge\mathbf{z} & \textit{for all } \mathbf{z}\in\mathbb{R}^3 \textit{ and for a.a. } t\in[0,T] \\ Q(0) = I, \quad Q(t)\in SO(3) & \textit{for all } t\in[0,T], \end{cases} \tag{2.12}$$

$$\mathbf{u}(\mathbf{y},t) = \boldsymbol{\omega}(t)\wedge\mathbf{y} + \mathbf{w}(t) \quad \textit{for all } \mathbf{y}\in S_0 \textit{ and for a.a. } t\in[0,T], \tag{2.13}$$

and, for almost all $t \in [0,T]$, the function $\mathbf{u}$ satisfies the variational identity

$$\begin{aligned} &\int_{D_+} \rho(\mathbf{y}) \big(\mathbf{u}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t) - \mathbf{u}_0(\mathbf{y})\cdot\mathbf{U}(\mathbf{y},0)\big)\,d\mathbf{y} - \int_0^t \int_{D_+} \rho\,\mathbf{u}\cdot\partial_\tau\mathbf{U}\,d\mathbf{y}d\tau \\ &\quad + \int_0^t \int_{D_+} \Big\{\rho\,\big((\boldsymbol{\omega}\wedge\mathbf{u})\cdot\mathbf{U} + \nabla\mathbf{u} : (\mathbf{U}\otimes(\mathbf{u}-\boldsymbol{\omega}\wedge\mathbf{y}-\mathbf{w}))\big) \\ &\quad + 2\nu\big(\varepsilon(\mathbf{u}):\varepsilon(\mathbf{U})\big) - \rho\,\mathbf{F}\cdot\mathbf{U}\Big\}\,d\mathbf{y}d\tau = 0 \end{aligned} \tag{2.14}$$

for all $\mathbf{U} \in W^1_\infty(D_+\times]0,T[;\mathbb{R}^3)$ such that $\mathbf{U}(\cdot,\tau) \in C_\infty(D(\tau))$ for all $\tau \in [0,T]$, where

$$\mathbf{F}(\mathbf{y},t) \stackrel{\text{def}}{=} Q^*(t)\mathbf{f}(Q(t)\mathbf{y} + \overline{\mathbf{x}}(t),t),$$

and $\mathbf{u}$ also satisfies, for almost all $t \in [0,T]$, the energy inequality

$$E(\mathbf{u}(\cdot,t)) + 2\nu\int_0^t\int_{D_+} |\varepsilon(\mathbf{u})|^2\,d\mathbf{y}d\tau \le E(\mathbf{u}_0) + \int_0^t\int_{D_+}\rho\,\mathbf{F}\cdot\mathbf{u}\,d\mathbf{y}d\tau. \tag{2.15}$$

The initial data are fulfilled in the following sense:

$$\begin{cases} \textit{the function } \ h_{\mathbf{U}}(t) = \displaystyle\int_{D_+} \rho(\mathbf{y})\,\mathbf{u}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t)\,d\mathbf{y} \textit{ is continuous} \\ \textit{ on } [0,T] \textit{ for all } \mathbf{U} \in C([0,T];L_2(D_+;\mathbb{R}^3)) \textit{ such that} \\ \mathbf{U}(\cdot,t) \in H(D(t)),\ t\in[0,T], \textit{ and } \ h_{\mathbf{U}}(0) = \displaystyle\int_{D_+} \rho(\mathbf{y})\,\mathbf{u}_0(\mathbf{y})\cdot\mathbf{U}(\mathbf{y},0)\,d\mathbf{y}. \end{cases} \tag{2.16}$$

Our main results are as follows.

Theorem 2.1. *Assume that conditions* (2.1) *and* (2.3)–(2.6) *hold. Then, there exists a positive number $T_0 = T_0(\mathbf{u}_0,\mathbf{f},d,S_0,D_0)$ such that for any $T \in]0,T_0[$ there exists at least one weak solution to the initial-boundary value problem* (1.19)–(1.28). *Weak solutions satisfy all the relations* (2.7)–(2.16). *Moreover, only one of the following cases can be realized.*

1. $T_0 = +\infty$.
2. *Let $\{T_k > 0\}_{k=1}^\infty$ be any sequence such that $T_k \uparrow T_0$. For given T_k, choose an arbitrary weak solution to* (1.19)–(1.28) *satisfying Definition 2.1. Let the function $t \in [0,T] \mapsto D_k(t) \in \mathcal{R}$ correspond to the chosen weak solution. Then,*

$$\operatorname{dist}\{\partial D_k(T_k),S_0\} = \operatorname{dist}\{\partial D_0, S_k(T_k)\} \to 0 \quad \textit{as} \quad k \to +\infty.$$

In fact, all the statements of Theorem 2.1 are simple consequences of the following theorem.

Theorem 2.2. *Let*

$$\lambda_+ \stackrel{\text{def}}{=} \inf\Big\{ \int_{D_+} |\varepsilon(\mathbf{v})|^2 \, d\mathbf{y} \mid \mathbf{v} \in \overset{o}{W}{}_2^1 (D_+;\mathbb{R}^3), \operatorname{div}\mathbf{v} = 0 \text{ in } D_+,$$

$$\mathbf{v}(\mathbf{y}) = \boldsymbol{\Omega} \wedge \mathbf{y} + \mathbf{W}, \ \mathbf{y} \in S_0, \ \text{for some } \boldsymbol{\Omega} \in \mathbb{R}^3 \tag{2.17}$$

$$\text{and for some } \mathbf{W} \in \mathbb{R}^3, \ \int_{D_+} \rho\, |\mathbf{v}|^2 \, d\mathbf{y} = 1 \Big\}$$

and

$$T_*(E_0, A_{\mathbf{f}}, d) \stackrel{\text{def}}{=} \min\left\{ 1, \frac{d}{2\sqrt{2}} \left(\left(\frac{\operatorname{diam} S_0}{\sqrt{\gamma_0}} + \frac{1}{\sqrt{m_s}} \right) \sqrt{E_0 + E_{\mathbf{f}}} \right)^{-1} \right\}, \tag{2.18}$$

where

$$E_0 \stackrel{\text{def}}{=} E(\mathbf{u}_0) \quad \text{and} \quad E_{\mathbf{f}} \stackrel{\text{def}}{=} \frac{A_{\mathbf{f}}^2}{4\nu\lambda_+}.$$

Then, for any weak solution to the initial-boundary value problem (1.19)–(1.28) *satisfying Definition 2.1 with $T = T_*$, we have the estimate*

$$\operatorname{dist}\{\partial D(t), S_0\} \geq \frac{d}{2} \quad \text{for all } t \in [0, T_*]. \tag{2.19}$$

Moreover, there is at least one weak solution to (1.19)–(1.28) *satisfying Definition 2.1 with $T = T_*$.*

Proof of Theorem 2.1. Suppose that we are given an arbitrary collection $\{\mathbf{f}, \mathbf{u}_0, d\}$ satisfying (2.4)–(2.6). Assume that a positive number T exists such that there is a weak solution to (1.19)–(1.28) satisfying Definition 2.1. Let us denote by T_0 the supremum of such T. According to the second statement of Theorem 2.2 we have $T_0 > 0$. If $T_0 = +\infty$, then we take another collection $\{\mathbf{f}, \mathbf{u}_0, d\}$, satisfying conditions (2.4)–(2.6). Thus, we assume that $T_0 < +\infty$.

Suppose that the last statement of Theorem 2.1 is false. Then, there is a sequence $\{\overline{T}_k > 0\}_{k=1}^{\infty}$ with the following properties:

$$\overline{T}_k \uparrow T_0 \tag{2.20}$$

and, for each k, there is a weak solution $\overline{\mathbf{u}}^{(k)}$ to (1.19)–(1.28), satisfying Definition 2.1 with $T = \overline{T}_k$, such that the function $t \in [0, \overline{T}_k] \mapsto D_k(t) \in \mathcal{R}$, corresponding to this solution, satisfies the inequality

$$\limsup_{k\to+\infty} \operatorname{dist}\left\{\partial D_k(\overline{T}_k), S_0\right\} \geq 2d_0 > 0. \tag{2.21}$$

For $\epsilon > 0$, by (2.20) and (2.21), one can find a positive number $k = k(\epsilon, d_0)$ such that

$$T_0 - \epsilon < \overline{T}_k < T_0 \tag{2.22}$$

and

$$\operatorname{dist}\left\{\partial D_k(\overline{T}_k), S_0\right\} \geq \frac{3}{2} d_0. \tag{2.23}$$

Our goal is to choose a number ϵ in order to arrive at a contradiction with the definition of the number T_0. From Definition 2.1 and from (2.23) it follows that a number T_k exists such that

$$\overline{T}_k - \frac{\epsilon}{2} < T_k \leq \overline{T}_k, \quad \overline{\mathbf{u}}^{(k)}(\cdot, T_k) \in H^1(D_k(T_k)),$$

$$\operatorname{dist}\{\partial D_k(T_k), S_0\} \geq \frac{5}{4} d_0,$$

and

$$E_k + \nu \int_0^{T_k} \int_{D_+} |\varepsilon(\overline{\mathbf{u}}^{(k)})|^2 \, d\mathbf{y} dt \leq E_0 + E_{\mathbf{f}} T_k \leq E_0 + E_{\mathbf{f}} T_0, \tag{2.24}$$

where $E_k = E(\overline{\mathbf{u}}^{(k)}(\cdot, T_k))$. To obtain (2.24) from (2.15), we have used the definition of the number λ_+ and the Cauchy inequality.

By Theorem 2.2, there exists a weak solution $\widetilde{\mathbf{u}}^{(1)}$, defined on the interval $]T_k, \overline{T}^{(1)}[$, such that $\widetilde{\mathbf{u}}^{(1)}(\cdot, T_k) = \overline{\mathbf{u}}^{(k)}(\cdot, T_k)$, $\widetilde{D}^{(1)}(T_k) = D_k(T_k)$. Here $\overline{T}^{(1)} = T_k + T_*\Big(E_k, A_{\mathbf{f}}, \frac{5}{4} d_0\Big)$,

$$\begin{aligned} T_*\Big(E_k, A_{\mathbf{f}}, \frac{5}{4} d_0\Big) &= \min\left\{1, \frac{5 c_1 \, d_0}{8\sqrt{E_k + E_{\mathbf{f}}}}\right\} \\ &\geq (\text{ see (2.24) }) \\ &\geq T_*^{(1)} = \min\left\{1, \frac{c_1 \, d_0}{2\sqrt{E_0 + (T_0 + 1) E_{\mathbf{f}}}}\right\} \end{aligned}$$

where

$$c_1 = \frac{1}{\sqrt{2}} \left(\frac{\operatorname{diam} S_0}{\sqrt{\gamma_0}} + \frac{1}{\sqrt{m_s}} \right)^{-1}.$$

We also know that

$$\operatorname{dist}\{\partial \widetilde{D}^{(1)}(t), S_0\} \geq \frac{5}{8} d_0, \quad t \in [T_k, \overline{T}^{(1)}].$$

It should be noted that in order to apply Theorem 2.2, one needs to involve some fixed rotation and translation.

Next, there is a number $T^{(1)}$ such that

$$\overline{T}^{(1)} - \frac{\epsilon}{2^2} < T^{(1)} \leq \overline{T}^{(1)}, \quad \widetilde{\mathbf{u}}^{(1)}(\cdot, T^{(1)}) \in H^1(\widetilde{D}^{(1)}(T^{(1)})),$$

$$\operatorname{dist}\{\partial \widetilde{D}^{(1)}(T^{(1)}), S_0\} \geq \frac{9}{16} d_0,$$

$$E(\widetilde{\mathbf{u}}^{(1)}(\cdot, T^{(1)})) \leq E_k + (T^{(1)} - T_k) E_{\mathbf{f}} \leq E_0 + (T_0 + 1) E_{\mathbf{f}}.$$

Clearly, the functions

$$\mathbf{u}^{(1)}(\cdot, t) = \begin{cases} \overline{\mathbf{u}}^{(k)}(\cdot, t), & 0 \leq t \leq T_k \\ \widetilde{\mathbf{u}}^{(1)}(\cdot, t), & T_k \leq t \leq T^{(1)} \end{cases} \quad \text{and} \quad D^{(1)}(t) = \begin{cases} D_k(t), & 0 \leq t \leq T_k \\ \widetilde{D}^{(1)}(t), & T_k \leq t \leq T^{(1)} \end{cases}$$

provide a weak solution to (1.19)–(1.28), satisfying Definition 2.1 on the interval $]0, T^{(1)}[$.

Proceeding in such a way, we obtain that, for any m, there are two functions $\mathbf{u}^{(m)}(\cdot, t)$ and $D^{(m)}(t)$, satisfying Definition 2.1 on the interval $]0, T^{(m)}[$. Here, the number $T^{(m)}$ can be estimated from below in the following way:

$$T^{(m)} \geq \overline{T}_k - \frac{\epsilon}{2}\sum_{i=0}^{m}\frac{1}{2^i} + \sum_{i=1}^{m} T_*^{(i)} \geq \overline{T}_k - \epsilon + \sum_{i=1}^{m} T_*^{(i)},$$

where

$$T_*^{(i)} = \min\left\{1, \frac{c_1\, d_0}{2^i(E_0 + (T_0 + i)E_{\mathbf{f}})}\right\}.$$

Moreover,

$$\operatorname{dist}\left\{\partial D^{(m)}(T^{(m)}), S_0\right\} \geq \frac{d_0}{2^m}.$$

Thus, setting

$$T_0' \stackrel{\text{def}}{=} \overline{T}_k - \epsilon + \sum_{i=1}^{\infty} T_*^{(i)},$$

we see that, for any $T < T_0'$, there exists a weak solution to (1.19)–(1.28), satisfying Definition 2.1 on the interval $]0, T[$. But, according to (2.22), we have

$$T_0' > \sum_{i=1}^{\infty} T_*^{(i)} + T_0 - 2\epsilon,$$

To obtain the contradiction, we take $\epsilon = \frac{1}{4}\sum_{i=1}^{\infty} T_*^{(i)}$. Theorem 2.1 is proved. □

Theorem 2.2 will be proved with the help of a special regularization. To describe it, we take an even non-negative function $k \in C_0^\infty(\mathbb{R}^1)$ such that $k(s) = 0$ if $|s| \geq 1$. Then, we let

$$K_{1\mu}(t) = c_{*1}\frac{1}{\mu}k\left(\left|\frac{t}{\mu}\right|^2\right), \quad t \in \mathbb{R}^1,$$

$$K_{2\mu}(\mathbf{y}) = c_{*2}\frac{1}{\mu^3}k\left(\left|\frac{\mathbf{y}}{\mu}\right|^2\right), \quad \mathbf{y} \in \mathbb{R}^3,$$

$$K_\mu(\mathbf{y}, t) = K_{1\mu}(t)\, K_{2\mu}(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^3,\ t \in \mathbb{R}^1,$$

where μ is a positive parameter. The constants c_{*1} and c_{*2} are chosen so that

$$\int_{-\infty}^{+\infty} K_{1\mu}(t)\, dt = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^3} K_{2\mu}(\mathbf{y})\, d\mathbf{y} = 1.$$

Theorem 2.3. *Suppose that all conditions of Theorem 2.1 hold and let, in addition,*

$$\mathbf{u}_0 \in W_2^2\left(D_0 \backslash S_0; \mathbb{R}^3\right) \cap H^1(D_0). \tag{2.25}$$

Then, there is a function $\mathbf{u}$ *having the following properties:*

$$\begin{cases} \mathbf{u} \in L_\infty\left(0, T_*; W_2^1(D_+; \mathbb{R}^3)\right) \cap C\left([0, T_*]; L_2(D_+; \mathbb{R}^3)\right) \\ \partial_t \mathbf{u} \in L_2\left(0, T_*; L_2(D_+; \mathbb{R}^3)\right), \end{cases} \tag{2.26}$$

where $T_* = T_*(E_0, A_{\mathbf{f}}, d)$. *The function* $\mathbf{u}$ *satisfies the relations* (2.8), (2.9) *and* (2.11)–(2.13) *with*

$$Q \in C^1\left([0, T_*]; M^{3\times 3}\right) \qquad \textit{and} \qquad \overline{\mathbf{x}} \in C^1\left([0, T_*]; \mathbb{R}^3\right). \tag{2.27}$$

For almost all $t \in [0, T_*]$, *the function* $\mathbf{u}$ *satisfies the variational identity*

$$\begin{aligned} &\int_{D(t)} \rho(\mathbf{y}) \Big[\Big\{ (\partial_t \mathbf{u}(\mathbf{y}, t) + \boldsymbol{\omega}(t) \wedge \mathbf{u}(\mathbf{y}, t)) \cdot \mathbf{U}(\mathbf{y}) \\ &\qquad + \nabla \mathbf{u}(\mathbf{y}, t) : \big(\mathbf{U}(\mathbf{y}) \otimes ((\mathbf{u})_\mu(\mathbf{y}, t) - \boldsymbol{\omega}(t) \wedge \mathbf{y} - \mathbf{w}(t)) \big) \Big\} \Big] \, d\mathbf{y} \\ &+ \frac{1}{2}(\rho^f - \rho^s) \int_{\partial S_0} \big[\mathbf{u}(\mathbf{y}, t) \cdot \mathbf{U}(\mathbf{y}) \big((\mathbf{u})_\mu(\mathbf{y}, t) - \boldsymbol{\omega}(t) \wedge \mathbf{y} - \mathbf{w}(t) \big) \cdot \mathbf{n}(\mathbf{y}) \big] \, ds \\ &+ \int_{D(t)} \big[2\nu\, \varepsilon(\mathbf{u}(\mathbf{y}, t)) : \varepsilon(\mathbf{U}(\mathbf{y})) - \rho(\mathbf{y})\, Q^*(t) \mathbf{f}(Q(t)\mathbf{y} + \overline{\mathbf{x}}(t), t) \cdot \mathbf{U}(\mathbf{y}) \big] \, d\mathbf{y} \\ &= 0 \end{aligned} \tag{2.28}$$

for any $\mathbf{U} \in H^1(D(t))$. *Here* $\mathbf{n}$ *is the unit outward normal vector to the surface* S_0, *and*

$$(\mathbf{u})_\mu(\mathbf{y}, t) \stackrel{\text{def}}{=} \int\limits_{Q_+} K_\mu(\mathbf{y} - \mathbf{y}', t - t') \mathbf{u}(\mathbf{y}', t') d\mathbf{y}' dt', \quad Q_+ \stackrel{\text{def}}{=} D_+ \times]0, T_*[.$$

For all $t \in [0, T]$, *the energy inequality* (2.15) *holds. The initial conditions are satisfied in the following sense:*

$$\| \mathbf{u}(\cdot, t) - \mathbf{u}_0(\cdot) \|_{L_2(D_+, \mathbb{R}^3)} \to 0 \qquad \textit{as } t \downarrow 0. \tag{2.29}$$

3. Proof of Theorem 2.3

In this section, we prove Theorem 2.3. To this end, we introduce a partition of the segment $[0, T_*]$ determined by the points $t_k = k\Delta$, $\Delta = \dfrac{T_*}{N}$, $k = 0, 1, \ldots, N$, into N segments $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$. We look for a function $\mathbf{u}^N$, having the form

$$\mathbf{u}^N(\mathbf{y}, t) = \mathbf{u}^{(k-1)}(\mathbf{y}) + \Big(\mathbf{u}^{(k)}(\mathbf{y}) - \mathbf{u}^{(k-1)}(\mathbf{y}) \Big) \frac{t - t_{k-1}}{\Delta} \tag{3.1}$$

for $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$ and $\mathbf{y} \in D_+$. The functions $\mathbf{u}^{(k)}$, $k = 0, 1, \dots, N$, are required to satisfy the following system of equations:

$$\mathbf{u}^{(0)} = \mathbf{u}_0, \tag{3.2}$$

$$\mathbf{u}^{(k)}(\mathbf{y}) = \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} + \mathbf{w}^{(k)} \quad \text{for all } \mathbf{y} \in S_0 \quad \text{and for} \quad k = 0, 1, \dots, N, \tag{3.3}$$

$$\mathbf{u}^{(k)} \in H^1(D_k), \quad k = 1, 2, \dots, N, \tag{3.4}$$

$$\begin{cases} D_k = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = (Q^N)^*(t_k)(\mathbf{x} - \overline{\mathbf{x}}^N(t_k)),\ \mathbf{x} \in D_0 \right\} \\ \operatorname{dist}\{\partial D_k, S_0\} > 0, \quad k = 1, 2, \dots, N, \end{cases} \tag{3.5}$$

$$\begin{cases} (Q^N)^*(t)\,\dot{Q}^N(t)\mathbf{z} = \boldsymbol{\omega}^{(k-1)} \wedge \mathbf{z} \quad \text{for all } \mathbf{z} \in \mathbb{R}^3,\ t \in [t_{k-1}, t_k] \\ \overline{\mathbf{x}}^N(t) = \overline{\mathbf{x}}^N(t_{k-1}) + \displaystyle\int_{t_{k-1}}^{t} Q^N(\tau)\mathbf{w}^{(k-1)}\, d\tau, \quad t \in [t_{k-1}, t_k] \\ \qquad\qquad \text{for } k = 1, 2, \dots, N, \end{cases} \tag{3.6}$$

$$\begin{cases} Q^N(t_k) = Q^N(t_k - 0) = Q^N(t_k + 0) \in SO(3) \\ \overline{\mathbf{x}}^N(t_k) = \overline{\mathbf{x}}^N(t_k - 0) = \overline{\mathbf{x}}^N(t_k + 0) \quad \text{for } k = 1, 2, \dots, N-1, \end{cases} \tag{3.7}$$

$$Q^N(0) = I, \qquad \overline{\mathbf{x}}^N(0) = \mathbf{0}, \tag{3.8}$$

and

$$\begin{aligned} &\int_{D_+} \rho \left[\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \boldsymbol{\omega}^{(k)} \wedge \mathbf{u}^{(k)} \right) \cdot \mathbf{U} \right. \\ &\qquad\qquad \left. + \nabla \mathbf{u}^{(k)} : \Big(\mathbf{U} \otimes \Big((\mathbf{u}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \Big) \Big) \right] d\mathbf{y} \\ &+ \frac{1}{2}(\rho^f - \rho^s) \int_{\partial S_0} \mathbf{u}^{(k)} \cdot \mathbf{U} \Big((\mathbf{u}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \Big) \cdot \mathbf{n}\, ds \\ &+ \int_{D_+} \Big[2\nu\, \varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{U}) - \rho\, (Q^N)^*(t_k)\mathbf{f}(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k), t_k) \cdot \mathbf{U} \Big] d\mathbf{y} \\ &= 0 \qquad\qquad \text{for all } \mathbf{U} \in H^1(D_k), \quad k = 1, 2, \dots, N. \end{aligned} \tag{3.9}$$

Let us briefly explain how to prove the existence of a solution $\mathbf{u}^N$ to the problem (3.1)–(3.9). First of all, we derive from (3.9) an *a priori* estimate by substituting $\mathbf{U} = \mathbf{u}^{(k)}$ and then integrating by parts. As a result, we have

$$\begin{aligned} &\int_{D_+} \left\{ \rho\, \frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} \cdot \mathbf{u}^{(k)} + 2\nu |\varepsilon(\mathbf{u}^{(k)})|^2 \right\} d\mathbf{y} \\ &\qquad\qquad = \int_{D_+} \rho\, (Q^N)^*(t_k)\mathbf{f}(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k), t_k) \cdot \mathbf{u}^{(k)}\, d\mathbf{y}. \end{aligned}$$

We again use the definition of the number λ_+ and the Cauchy inequality to obtain from the last relation the following estimate:

$$\int_{D_+} \left\{ \rho \, \frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} \cdot \mathbf{u}^{(k)} + \nu |\varepsilon(\mathbf{u}^{(k)})|^2 \right\} d\mathbf{y} \le E_{\mathbf{f}}.$$

Summation by parts gives us

$$\begin{aligned} &E(\mathbf{u}^{(m)}) + \nu \sum_{k=1}^{m} \Delta \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y} + \frac{1}{2} \sum_{k=1}^{m} \int_{D_+} \rho \, |\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}|^2 \, d\mathbf{y} \\ &\le E_0 + E_{\mathbf{f}} t_m \le E_0 + E_{\mathbf{f}} T_* \le E_0 + E_{\mathbf{f}} \end{aligned} \tag{3.10}$$

for any $m = 1, 2, \ldots, N$.

The estimate (3.10) provides the inequality (see (2.3) and the definition of E)

$$\frac{1}{2} \left(m_s |\mathbf{w}^{(m)}|^2 + \gamma_0 |\boldsymbol{\omega}^{(m)}|^2 \right) \le E_0 + E_{\mathbf{f}} \, T_* \le E_0 + E_{\mathbf{f}}. \tag{3.11}$$

It follows from (3.6) and (3.11) that

$$\begin{aligned} &|\overline{\mathbf{x}}^N(t)| \le \sqrt{\frac{2(E_0 + E_{\mathbf{f}})}{m_s}} \, T_*, \\ &|\dot{Q}^N(t)| \le \sqrt{\frac{2(E_0 + E_{\mathbf{f}})}{\gamma_0}} \quad \text{implies} \quad |Q^N(t) - I| \le \sqrt{\frac{2(E_0 + E_{\mathbf{f}})}{\gamma_0}} \, T_*. \end{aligned} \tag{3.12}$$

Thus, if we introduce

$$D^N(t) = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = (Q^N)^*(t)(\mathbf{x} - \overline{\mathbf{x}}^N(t)), \mathbf{x} \in D_0 \right\}, \tag{3.13}$$

then

$$\begin{aligned} &\operatorname{dist}\{\partial D^N(t), S_0\} = \operatorname{dist}\{\partial D_0, S^N(t)\} \\ &= \inf \left\{ |\mathbf{x} - \widetilde{\mathbf{x}}| \mid \mathbf{x} \in \partial D_0, \ \widetilde{\mathbf{x}} = Q^N(t)\mathbf{y} + \overline{\mathbf{x}}^N(t), \ \mathbf{y} \in S_0 \right\} \\ &\ge \inf \left\{ |\mathbf{x} - \mathbf{y}| \mid \mathbf{x} \in \partial D_0, \ \mathbf{y} \in S_0 \right\} - \left(|Q^N(t) - I| \operatorname{diam} S_0 + |\overline{\mathbf{x}}^N(t)| \right) \\ &\ge d - T_* \sqrt{2(E_0 + E_{\mathbf{f}})} \left(\frac{\operatorname{diam} S_0}{\sqrt{\gamma_0}} + \frac{1}{\sqrt{m_s}} \right) \ge (\text{see}(2.18)) \ge \frac{d}{2} \end{aligned} \tag{3.14}$$

for all $t \in [0, T]$ and for all N.

Having in hand the estimates (3.10) and (3.14), one can prove the existence of a solution to problem (3.1)–(3.9). To this end, let us consider the following problem. For a given function

$$\mathbf{v}^N(\mathbf{y}, t) = \mathbf{v}^{(k-1)}(\mathbf{y}) + \left(\mathbf{v}^{(k)}(\mathbf{y}) - \mathbf{v}^{(k-1)}(\mathbf{y}) \right) \frac{t - t_{k-1}}{\Delta}, \qquad t \in [t_{k-1}, t_k], \tag{3.15}$$

where $k = 1, 2, \ldots, N$, and $\mathbf{v}^{(i)} \in L_2(D_+; \mathbb{R}^3)$, $\operatorname{div} \mathbf{v}^{(i)} = 0$ in D_+, $\mathbf{v}^{(i)} = \mathbf{0}$ in some neighborhood of ∂D_+, $i = 0, 1, \ldots, N$, we are looking for a function $\mathbf{u}^N$, satisfying (3.1)–(3.8) and the variational identity

$$\begin{aligned}
&\int_{D_+} \rho \left[\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \boldsymbol{\omega}^{(k)} \wedge \mathbf{u}^{(k)}\right) \cdot \mathbf{U}\right. \\
&\qquad\qquad \left. + \nabla \mathbf{u}^{(k)} : \left(\mathbf{U} \otimes \left((\mathbf{v}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\right)\right)\right] d\mathbf{y} \\
&+ \frac{1}{2}(\rho^f - \rho^s) \int_{\partial S_0} \mathbf{u}^{(k)} \cdot \mathbf{U}\Big((\mathbf{v}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\Big) \cdot \mathbf{n}\, ds \\
&+ \int_{D_+} \left[2\nu\, \varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{U}) - \rho\, (Q^N)^*(t_k) \mathbf{f}(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k), t_k) \cdot \mathbf{U}\right] d\mathbf{y} \\
&= 0
\end{aligned} \tag{3.16}$$

for any $\mathbf{U} \in H^1(D_k)$, $k = 1, 2, \ldots, N$. Since $\operatorname{div}(\mathbf{v}^N)_\mu = 0$, the estimates (3.10) and (3.14) hold for any solution to the problem (3.1)–(3.8) and (3.16). Solvability of this problem can be proved easily if we take into account (3.6) and (3.16). They show that to find $\mathbf{u}^{(k)}$ and D_k we need only know $\mathbf{u}^{(k-1)}$ and D_{k-1}. Then, marching from $k = 0$ to $k = N$, we find $\mathbf{u}^N$.

Now, we see that problem (3.1)–(3.8) and (3.16) defines multivalued mapping $\mathbf{v}^N \mapsto \mathbf{u}^N$. Since $\mathbf{u}^N$ satisfies (3.10) and (3.16), we easily can show that this mapping has a fixed point. It is a solution to problem (3.1)–(3.9).

In this remainder of this section, we shall denote by c, c_1, etc. all constants which depends only on $\mathbf{u}_0$, $A_\mathbf{f}$, d, D_0, S_0, and μ. At times, different constants will be denoted by the same symbol.

To prove Theorem 2.3, we will need the following result.

Lemma 3.1. *For any solution to problem* (3.1)–(3.9), *the following estimate holds*

$$\begin{aligned}
\int_{D_+} |\,\varepsilon(\mathbf{u}^{(m)})|^2\, d\mathbf{y} + \sum_{k=1}^{m} \Delta \int_{D_+} \left|\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right|^2 d\mathbf{y} \\
+ \frac{1}{\Delta} \int_{D_+} \Delta \left|\,\varepsilon\left(\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}\right)\right|^2 d\mathbf{y} \le c
\end{aligned} \tag{3.17}$$

for $m = 1, 2, \ldots, N$.

Proof. Unfortunately, we can not substitute $\mathbf{U} = \dfrac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}$ into (3.9) because $\mathbf{u}^{(k-1)}$ belongs to $H^1(D_{k-1})$ but not to $H^1(D_k)$. To overcome this, we introduce the following mapping $\mathbf{z}$ from D_k into D_{k-1}:

$$D_{k-1} \ni \mathbf{z}(\mathbf{y}) = (Q^N)^*(t_{k-1}) \left(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k) - \overline{\mathbf{x}}^N(t_{k-1})\right), \quad \mathbf{y} \in D_k.$$

Clearly, by (3.6) and (3.11),

$$\sup_{\mathbf{y} \in D_k} |\mathbf{z}(\mathbf{y}) - \mathbf{y}| \le c_1\, \Delta. \tag{3.18}$$

Now, let us introduce the function

$$\mathbf{u}_+^{(k-1)}(\mathbf{y}) = \begin{cases} (Q^N)^*(t_k)Q^N(t_{k-1})\mathbf{u}^{(k-1)}(\mathbf{z}(\mathbf{y})), & \mathbf{y} \in D_k \\ \mathbf{0}, & \mathbf{y} \in D_+ \backslash D_k. \end{cases}$$

We see that

$$\mathbf{u}_+^{(k-1)} \in W_2^1(D_+;\mathbb{R}^3) \qquad \text{and} \qquad \operatorname{div} \mathbf{u}_+^{(k-1)} = 0 \quad \text{in } D_+\,. \tag{3.19}$$

However, the function $\mathbf{u}_+^{(k-1)}$ still does not belong to $H^1(D_k)$.

We introduce the set

$$S_0^{d/4} = \left\{ \mathbf{y} \in \mathbb{R}^3 \mid \operatorname{dist}\{S_0, \mathbf{y}\} < \frac{d}{4} \right\};$$

by (3.14),

$$\overline{S_0^{d/4}} \cap \partial D_k = \emptyset, \qquad k = 0, 1, \dots, N. \tag{3.20}$$

Next, we choose a cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ with the following properties:

$$0 \le \varphi \le 1 \text{ in } \mathbb{R}^3 \qquad \text{and} \qquad \varphi = \begin{cases} 1 & \text{in } S_0 \\ 0 & \text{in } \mathbb{R}^3 \backslash S_0^{d/4}. \end{cases}$$

We then have

$$\begin{aligned} \int_{S_0^{d/4}\backslash \overline{S_0}} \nabla\varphi \cdot \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) d\mathbf{y} &= \int_{S_0^{d/4}\backslash \overline{S_0}} \operatorname{div} \left(\varphi \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) \right) d\mathbf{y} \\ = -\int_{\partial S_0} \mathbf{n} \cdot \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) d\mathbf{y} &= -\int_{S_0} \operatorname{div} \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) d\mathbf{y} = 0. \end{aligned}$$

Here, $\mathbf{n}$ is the outward unit normal vector to the surface ∂S_0. Then (see [7]), there exists a function $\mathbf{v}^{(k-1)} \in \overset{o}{W}{}_2^1(S_0^{d/4}\backslash \overline{S_0})$ such that

$$\operatorname{div} \mathbf{v}^{(k-1)} = \nabla\varphi \cdot \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) \qquad \text{in } S_0^{d/4}\backslash \overline{S_0}$$

and

$$\begin{aligned} \| \mathbf{v}^{(k-1)} \|_{L_2(S_0^{d/4}\backslash \overline{S_0})} &+ \| \nabla \mathbf{v}^{(k-1)} \|_{L_2(S_0^{d/4}\backslash \overline{S_0})} \\ &\le c_1(d, S_0) \| \mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \|_{L_2(S_0^{d/4}\backslash \overline{S_0})}. \end{aligned}$$

We extend $\mathbf{v}^{(k-1)}$ by zero to D_+. Then, we obtain

$$\begin{gathered} \mathbf{v}^{(k-1)} \in \overset{o}{W}{}_2^1(D_+;\mathbb{R}^3), \quad \operatorname{div} \mathbf{v}^{(k-1)} = \nabla\varphi \cdot \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) \quad \text{in } D_+, \\ \| \mathbf{v}^{(k-1)} \|_{L_2(D_+)} + \| \nabla \mathbf{v}^{(k-1)} \|_{L_2(D_+)} \le c_1(d, S_0) \, \| \mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \|_{L_2(D_+)}. \end{gathered} \tag{3.21}$$

Now, we let

$$\mathbf{u}_*^{(k-1)} = \mathbf{u}^{(k-1)} + (1 - \varphi) \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)} \right) + \mathbf{v}^{(k-1)}\,;$$

by (3.19) and (3.21),

$$\mathbf{u}_*^{(k-1)} \in H^1(D_k), \qquad k = 1, 2, \ldots, N. \tag{3.22}$$

We introduce the additional notation

$$\frac{\mathbf{u}^{(k)} - \mathbf{u}_*^{(k-1)}}{\Delta} = \frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \mathbf{v}_*^{(k-1)}, \tag{3.23}$$

where

$$\mathbf{v}_*^{(k-1)} \stackrel{\text{def}}{=} -\frac{1}{\Delta}(1-\varphi)\left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}\right) - \frac{1}{\Delta}\mathbf{v}^{(k-1)}.$$

Now, by (3.22), one may substitute $\dfrac{\mathbf{u}^{(k)} - \mathbf{u}_*^{(k-1)}}{\Delta}$ for $\mathbf{U}$ in (3.9). As a result, we have

$$\begin{aligned}
&\int_{D_+} \rho \left|\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right|^2 d\mathbf{y} + 2\nu \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right) d\mathbf{y} \\
&\qquad \stackrel{\text{def}}{=} I_0^{(k)} + 2\nu \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right) d\mathbf{y} \\
&\qquad = \textstyle\sum_{i=1}^6 I_i.
\end{aligned} \tag{3.24}$$

Here,

$$I_1 \stackrel{\text{def}}{=} -\int_{D_+} \rho \, \frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} \cdot \mathbf{v}_*^{(k-1)} \, d\mathbf{y} \le c_1 \, (I_0^{(k)})^{1/2} \left(\int_{D_+} \left|\mathbf{v}_*^{(k-1)}\right|^2 d\mathbf{y}\right)^{1/2},$$

$$\begin{aligned}
I_2 &\stackrel{\text{def}}{=} -\int_{D_+} \rho \left(\boldsymbol{\omega}^{(k)} \wedge \mathbf{u}^{(k)}\right) : \left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \mathbf{v}_*^{(k-1)}\right) d\mathbf{y} \\
&\le (\text{see (3.10) and (3.11) }) \le c_1 \left(I_0^{(k)} + \int_{D_+} \left|\mathbf{v}_*^{(k-1)}\right|^2 d\mathbf{y}\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
I_3 &\stackrel{\text{def}}{=} -\int_{D_+} \rho \nabla \mathbf{u}^{(k)} : \Bigg(\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \mathbf{v}_*^{(k-1)}\right) \\
&\qquad\qquad \otimes \left((\mathbf{u}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\right) \Bigg) d\mathbf{y} \\
&\le c_1 \left(\int_{D_+} |\nabla \mathbf{u}^{(k)}|^2 \, d\mathbf{y}\right)^{1/2} \left(I_0^{(k)} + \int_{D_+} |\mathbf{v}_*^{(k-1)}|^2 \, d\mathbf{y}\right)^{1/2} \\
&\qquad \times \left(\sup\nolimits_{\mathbf{y} \in D_+} \left|(\mathbf{u}^N)_\mu(\mathbf{y}, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\right|\right) \\
&\le c_2 \left(\int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y}\right)^{1/2} \left(I_0^{(k)} + \int_{D_+} \left|\mathbf{v}_*^{(k-1)}\right|^2 d\mathbf{y}\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
I_4 &\stackrel{\text{def}}{=} -\frac{1}{2}(\rho^f - \rho^s) \int_{\partial S_0} \mathbf{u}^{(k)} \cdot \left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \mathbf{v}_*^{(k-1)} \right) \\
&\qquad\qquad \times \left((\mathbf{u}^N)_\mu(\cdot, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \right) \cdot \mathbf{n}\, ds \\
&\le c_1 \sup\nolimits_{\mathbf{y}\in D_+} \left| (\mathbf{u}^N)_\mu(\mathbf{y}, t_k) - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \right| \\
&\qquad \times \frac{1}{\Delta} \int_{\partial S_0} \left| \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} + \mathbf{w}^{(k)} \right| \left| (\boldsymbol{\omega}^{(k)} - \boldsymbol{\omega}^{(k-1)}) \wedge \mathbf{y} + (\mathbf{w}^{(k)} - \mathbf{w}^{(k-1)}) \right| ds \\
&\le c_2 \left(I_0^{(k)} \right)^{1/2} \qquad \text{(where we used the fact that } \mathbf{v}_*^{(k-1)} = 0 \text{ on } \partial S_0\text{)}, \\
I_5 &\stackrel{\text{def}}{=} -2\nu \int_{D_+} \varepsilon\left(\mathbf{u}^{(k)}\right) : \varepsilon\left(\mathbf{v}_*^{(k-1)}\right) d\mathbf{y}, \\
I_6 &\stackrel{\text{def}}{=} \int_{D_+} \rho\, (Q^N)^*(t_k)\, \mathbf{f}\left(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k), t_k \right) \cdot \left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} + \mathbf{v}_*^{(k-1)} \right) d\mathbf{y} \\
&\le c_1 \left(I_0^{(k)} + \int_{D_+} \left| \mathbf{v}_*^{(k-1)} \right|^2 d\mathbf{y} \right)^{1/2}.
\end{aligned}$$

Using the Cauchy inequality, we derive from (3.24) and from the estimates given above, the following estimate:

$$\begin{aligned}
c_1\, I_0^{(k)} + 2\,\nu \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta} \right) d\mathbf{y} \\
\le I_5 + c_2 \left(1 + \int_{D_+} |\mathbf{v}_*^{(k-1)}|^2\, d\mathbf{y} + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y} \right).
\end{aligned} \tag{3.25}$$

Next, by (3.21) and (3.23),

$$\int_{D_+} |\mathbf{v}_*^{(k-1)}|^2\, d\mathbf{y} \le \frac{c_1}{\Delta} \int_{D_+} |\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}|^2\, d\mathbf{y}.$$

From the definition of $\mathbf{u}_+^{(k-1)}$ (see (3.19)), it follows that

$$\begin{aligned}
&|\mathbf{u}_+^{(k-1)}(\mathbf{y}) - \mathbf{u}^{(k-1)}(\mathbf{y})| \\
&= \left| \left((Q^N)^*(t_k) - (Q^N)^*(t_{k-1}) \right) Q^N(t_{k-1}) \mathbf{u}^{(k-1)}(\mathbf{z}(\mathbf{y})) + \mathbf{u}^{(k-1)}(\mathbf{z}(\mathbf{y})) - \mathbf{u}^{(k-1)}(\mathbf{y}) \right| \\
&\le c_1 \Delta |\mathbf{u}^{(k-1)}(\mathbf{z}(\mathbf{y}))| + \left| \int_0^1 \nabla \mathbf{u}^{(k-1)}(\mathbf{y} + \theta(\mathbf{z}(\mathbf{y}) - \mathbf{y}))(\mathbf{z}(\mathbf{y}) - \mathbf{y})\, d\theta \right|.
\end{aligned}$$

Taking into account the estimate (3.18), one can derive from the last two inequalities the following relations:

$$\begin{aligned}
\frac{1}{\Delta} \int_{D_+} |\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}|^2\, d\mathbf{y} &\le c_1 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2\, d\mathbf{y} \right) \\
\int_{D_+} |\mathbf{v}_*^{(k-1)}|^2\, d\mathbf{y} &\le c_2 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2\, d\mathbf{y} \right).
\end{aligned} \tag{3.26}$$

Now, let us estimate I_5 in (3.25). We have

$$I_5 = J_1 + J_2, \tag{3.27}$$

where

$$J_1 = 2\nu \frac{1}{\Delta} \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{v}^{(k-1)})\, d\mathbf{y}$$
$$J_2 = 2\nu \frac{1}{\Delta} \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left((1-\varphi)(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)})\right)\, d\mathbf{y}.$$

The integral J_1 can be estimated easily (see (3.21) and (3.26)):

$$\begin{aligned} J_1 &\le \frac{2\nu}{\Delta} \left(\int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y}\right)^{1/2} \left(\int_{D_+} |\varepsilon(\mathbf{v}^{(k-1)})|^2\, d\mathbf{y}\right)^{1/2} \\ &\le c_1 \frac{1}{\Delta} \left(\int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y}\right)^{1/2} \left(\int_{D_+} |\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}|^2\, d\mathbf{y}\right)^{1/2} \\ &\le c_2 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2 d\mathbf{y}\right). \end{aligned} \tag{3.28}$$

To estimate J_2, we first integrate by parts and obtain

$$J_2 = J_2{}' + J_2{}'', \tag{3.29}$$

where

$$J_2{}' = 2\nu \frac{1}{\Delta} \int_{\partial D_k} \left(\varepsilon(\mathbf{u}^{(k)})\mathbf{n}\right) \cdot \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}\right)\, ds$$

$$J_2{}'' = -2\nu \frac{1}{\Delta} \int_{D_k \setminus S_0} \operatorname{div} \varepsilon(\mathbf{u}^{(k)}) \cdot (1-\varphi) \left(\mathbf{u}_+^{(k-1)} - \mathbf{u}^{(k-1)}\right)\, d\mathbf{y}.$$

For $J_2{}'$, we have ($\mathbf{u}_+^{(k-1)} = \mathbf{0}$ on ∂D_k)

$$\begin{aligned} |J_2{}'| &= \frac{2\nu}{\Delta} \left| \int_{\partial D_k} \left(\varepsilon(\mathbf{u}^{(k)})\mathbf{n}\right) \cdot \mathbf{u}^{(k-1)}\, ds \right| \\ &\le \frac{2\nu}{\Delta} \left(\int_{\partial D_k} |\varepsilon(\mathbf{u}^{(k)})|^2\, ds\right)^{1/2} \left(\int_{\partial D_k} |\mathbf{u}^{(k-1)}|^2\, ds\right)^{1/2}. \end{aligned} \tag{3.30}$$

Now, we are going to exploit (3.20). By (3.20) and by the fact that the domain D_k is the result of translations and rotations of the domain D_0, we claim that for any $\gamma \in]0,1[$ the estimate

$$\int_{\partial D_k} |\varepsilon(\mathbf{u}^{(k)})|^2\, ds \le c \left(\gamma \int_{D_k \setminus S_0} |\nabla^2 \mathbf{u}^{(k)}|^2\, d\mathbf{y} + \frac{1}{\gamma} \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y}\right) \tag{3.31}$$

holds. The constant c in (3.31) depends only on D_0 and d.

To estimate the second factor on the right hand side of (3.30), we use Lemma A.1 of the Appendix and obtain the inequality

$$\begin{aligned}\frac{1}{\Delta^2}\int_{\partial D_k}|\mathbf{u}^{(k-1)}|^2\,ds &\le c\left(\int_{\partial D_{k-1}}|\nabla\mathbf{u}^{(k-1)}|^2\,ds+\int_{D_{k-1}\setminus S_0}|\nabla^2\mathbf{u}^{(k-1)}|^2\,d\mathbf{y}\right)\\ &\le c_1\left(\int_{D_{k-1}}|\varepsilon(\mathbf{u}^{(k-1)})|^2\,d\mathbf{y}+\int_{D_{k-1}\setminus S_0}|\nabla^2\mathbf{u}^{(k-1)}|^2\,d\mathbf{y}\right)\end{aligned} \tag{3.32}$$

with a constant c_1 depending only on D_0.

Next, we evaluate $J_2{}''$ with the help of (3.26)

$$|J_2{}''|\le c\left(\int_{D_k\setminus S_0}|\nabla^2\mathbf{u}^{(k)}|^2\,d\mathbf{y}\right)^{1/2}\left(1+\int_{D_+}|\varepsilon(\mathbf{u}^{(k-1)})|^2\,d\mathbf{y}\right)^{1/2}. \tag{3.33}$$

For $k=1,2,\dots,N$, the function $\mathbf{u}^{(k)}$ satisfies the Stokes equations

$$\begin{gathered}-2\nu\,\mathrm{div}\,\varepsilon(\mathbf{u}^{(k)})+\nabla p_k=\mathbf{g}^k\quad\text{in } D_k\setminus\overline{S_0},\\ \mathrm{div}\,\mathbf{u}^{(k)}=0\quad\text{in } D_k\setminus\overline{S_0},\\ \mathbf{u}^{(k)}=0\quad\text{on }\partial D_k,\\ \mathbf{u}^{(k)}(\mathbf{y})=\boldsymbol{\omega}^{(k)}\wedge\mathbf{y}+\mathbf{w}^{(k)},\quad \mathbf{y}\in\partial S_0,\end{gathered}$$

where

$$\begin{aligned}\mathbf{g}^{(k)}(\mathbf{y})=\rho^f\Big\{&(Q^N)^*(t_k)\mathbf{f}\left(Q^N(t_k)\mathbf{y}+\overline{\mathbf{x}}^N(t_k),t_k\right)-\frac{\mathbf{u}^{(k)}(\mathbf{y})-\mathbf{u}^{(k-1)}(\mathbf{y})}{\Delta}\\ &-\boldsymbol{\omega}^{(k)}\wedge\mathbf{u}^{(k)}(\mathbf{y})-\nabla\mathbf{u}^{(k)}(\mathbf{y})\left((\mathbf{u}^N)_\mu(\mathbf{y},t_k)-\boldsymbol{\omega}^{(k)}\wedge\mathbf{y}-\mathbf{w}^{(k)}\right)\Big\}.\end{aligned}$$

By Lemma A.2, there exists a constant c, depending on S_0, D_0 and d, such that

$$\int_{D_k\setminus S_0}|\nabla^2\mathbf{u}^{(k)}|^2\,d\mathbf{y}\le c\left(\int_{D_k\setminus S_0}|\mathbf{g}^{(k)}|^2\,d\mathbf{y}+|\boldsymbol{\omega}^{(k)}|^2+|\mathbf{w}^{(k)}|^2\right).$$

Using (3.10) and (3.11), we obtain

$$\int_{D_k\setminus S_0}|\nabla^2\mathbf{u}^{(k)}|^2\,d\mathbf{y}\le c_1\left(1+\int_{D_+}|\varepsilon(\mathbf{u}^{(k)})|^2\,d\mathbf{y}+\int_{D_+}\left|\frac{\mathbf{u}^{(k)}-\mathbf{u}^{(k-1)}}{\Delta}\right|^2\,d\mathbf{y}\right). \tag{3.34}$$

Thus, combining (3.29)–(3.34), we obtain

$$
\begin{aligned}
|J_2| \leq c_1 & \left(\gamma \int_{D_+} \left|\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right|^2 d\mathbf{y} + 1 + \frac{1}{\gamma}\int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y}\right)^{1/2} \\
& \times \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2\, d\mathbf{y} + \int_{D_+} \left|\frac{\mathbf{u}^{(k-1)} - \mathbf{u}^{(k-2)}}{\Delta}\right|^2 d\mathbf{y}\right)^{1/2} \\
& + c_2 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2\, d\mathbf{y} + \int_{D_+} \left|\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right|^2 d\mathbf{y}\right)^{1/2} \\
& \times \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2\, d\mathbf{y}\right)^{1/2}
\end{aligned}
\tag{3.35}
$$

for $k = 2, 3, \dots, N$, and

$$
\begin{aligned}
|J_2| \leq c_1 & \left(\gamma \int_{D_+} \left|\frac{\mathbf{u}^{(1)} - \mathbf{u}^{(0)}}{\Delta}\right|^2 d\mathbf{y} + 1 + \frac{1}{\gamma}\int_{D_+} |\varepsilon(\mathbf{u}^{(1)})|^2\, d\mathbf{y}\right)^{1/2} \\
& \times \left(\int_{D_0} |\varepsilon(\mathbf{u}_0)|^2\, d\mathbf{y} + \int_{D_0 \setminus S_0} |\nabla^2 \mathbf{u}_0|^2\, d\mathbf{y}\right)^{1/2} \\
& + c_2 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(1)})|^2\, d\mathbf{y} + \int_{D_+} \left|\frac{\mathbf{u}^{(1)} - \mathbf{u}^{(0)}}{\Delta}\right|^2 d\mathbf{y}\right)^{1/2} \\
& \times \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}_0)|^2\, d\mathbf{y}\right)^{1/2}
\end{aligned}
\tag{3.36}
$$

for $k = 1$.

From (3.35), (3.36), and (3.26)–(3.28), it follows that

$$
\begin{aligned}
& c_1 I_0^{(k)} + 2\nu \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right) d\mathbf{y} \\
& \leq c_2 \left(1 + \int_{D_+} \left(|\varepsilon(\mathbf{u}^{(k-1)})|^2 + |\varepsilon(\mathbf{u}^{(k)})|^2\right) d\mathbf{y}\right) \\
& + c_3 \left(\gamma I_0^{(k)} + 1 + \frac{1}{\gamma}\int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 d\mathbf{y}\right)^{1/2} \left(1 + I_0^{(k-1)} + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2 d\mathbf{y}\right)^{1/2} \\
& + c_4 \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 d\mathbf{y} + I_0^{(k)}\right)^{1/2} \left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2 d\mathbf{y}\right)^{1/2}
\end{aligned}
\tag{3.37}
$$

for $k = 2, 3, \dots, N$, and

$$
c_1 I_0^{(1)} + 2\nu \int_{D_+} \varepsilon(\mathbf{u}^{(1)}) : \varepsilon\left(\frac{\mathbf{u}^{(1)} - \mathbf{u}^{(0)}}{\Delta}\right) d\mathbf{y} \leq \frac{c_2}{\gamma}\left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(1)})|^2\, d\mathbf{y}\right) \tag{3.38}
$$

for $k = 1$. To obtain (3.38), we have used the Cauchy inequality. Applying the Cauchy inequality once more, we derive from (3.37) the following estimate:

$$\begin{aligned} & c_1 I_0^{(k)} + 2\nu \int_{D_+} \varepsilon(\mathbf{u}^{(k)}) : \varepsilon\left(\frac{\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}}{\Delta}\right) d\mathbf{y} \\ & \leq \frac{c_2}{\gamma}\left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y} + \int_{D_+} |\varepsilon(\mathbf{u}^{(k-1)})|^2 \, d\mathbf{y}\right) \\ & \quad + \frac{c_3}{\sqrt{\gamma}}\left(1 + \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y}\right)^{1/2} \left(I_0^{(k-1)}\right)^{1/2} + c_4\sqrt{\gamma}\left(I_0^{(k)} I_0^{(k-1)}\right)^{1/2} \end{aligned} \tag{3.39}$$

for $k = 2, 3, \dots, N$.

Note that

$$\varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)}) = \frac{1}{2}\left(|\varepsilon(\mathbf{u}^{(k)})|^2 - |\varepsilon(\mathbf{u}^{(k-1)})|^2 + |\varepsilon(\mathbf{u}^{(k)} - \mathbf{u}^{(k-1)})|^2\right).$$

Taking into account this identity, multiplying by Δ, utilizing (3.38) and (3.39), and choosing $\gamma \in]0, 1[$ in an appropriate way, we obtain (3.17). Lemma 3.1 is proved. □

Now we are prepared to complete the proof of Theorem 2.3. We wish to take the limit in (3.1)–(3.9) as $N \to +\infty$. We introduce the function

$$\mathbf{v}^N(\mathbf{y}, t) \stackrel{\text{def}}{=} \mathbf{u}^{(k)}(\mathbf{y}), \quad t_{k-1} < t \leq t_k, \quad k = 1, 2, \dots, N. \tag{3.40}$$

Comparing (3.1) and (3.40), we see that, by (3.10) and (3.17),

$$\int_{Q_+} |\mathbf{u}^N - \mathbf{v}^N|^2 \, d\mathbf{y}dt + \int_{Q_+} |\varepsilon(\mathbf{u}^N - \mathbf{v}^N)|^2 \, d\mathbf{y}dt < c\Delta, \tag{3.41}$$

where $Q_+ \stackrel{\text{def}}{=} D_+ \times]0, T_*[$. We also let

$$\begin{aligned} \mathbf{F}^N(\mathbf{y}, t) = (Q^N)^*(t_k) \; & \mathbf{f}\left(Q^N(t_k)\mathbf{y} + \overline{\mathbf{x}}^N(t_k), t_k\right), \\ & \mathbf{y} \in D_+, \quad t \in]t_{k-1}, t_k], \quad k = 1, 2, \dots, N. \end{aligned} \tag{3.42}$$

The relations (3.5) and (3.6) and the estimate (3.10) allow us to state that:

$$\begin{aligned} &\{Q^N\} \;\; \text{is bounded in } W^1_\infty(0, T_*; M^{3\times 3}), \\ &\{\overline{\mathbf{x}}^N\} \;\; \text{is bounded in } W^1_\infty(0, T_*; \mathbb{R}^3) \end{aligned}$$

and, therefore,

$$Q^N \to Q \text{ in } C([0, T_*]; M^{3\times 3}), \quad \overline{\mathbf{x}}^N \to \overline{\mathbf{x}} \text{ in } C([0, T_*]; \mathbb{R}^3) \tag{3.43}$$

and

$$Q(t) \subset SO(3) \text{ for all } t \in [0, T_*], \qquad Q(0) = I. \tag{3.44}$$

We let

$$D(t) = \left\{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = Q^*(t)(\mathbf{x} - \overline{\mathbf{x}}(t)), \; \mathbf{x} \in D_0\right\}. \tag{3.45}$$

By (3.43) and (3.14), one can obtain

$$\operatorname{dist}\{\partial D(t), S_0\} \geq \frac{d}{2} \text{ for all } t \in [0, T_*]. \tag{3.46}$$

Fix $t_0 \in]0, T_*[$ and a function $\mathbf{U} \in C_\infty(D(t_0))$ with $\operatorname{dist}\{\partial D(t_0), \operatorname{spt}|\mathbf{U}|\} = l_0 > 0$. Let the function $\psi \in C_0^1(0, T_*)$ satisfy the conditions

$$0 \leq \psi \leq 1 \quad \text{in }]0, T_*[,$$
$$\psi \equiv 1 \quad \text{in }]t_0 - \delta, t_0 + \delta[,$$
$$\psi \equiv 0 \quad \text{outside of } \ ]t_0 - 2\delta, t_0 + 2\delta[$$

for some $\delta > 0$. We have that

$$\operatorname{dist}\{\partial D(t), \operatorname{spt}|\widetilde{\mathbf{U}}(\cdot, t)|\} \geq \frac{l_0}{2} \tag{3.47}$$

for all $t \in [0, T_*]$ and all $0 < \delta \leq \delta_0$, where $\widetilde{\mathbf{U}}(\cdot, t) = \psi(t)\mathbf{U}(\cdot)$. Next, we set

$$\widetilde{\mathbf{U}}^N(\cdot, t) = \widetilde{\mathbf{U}}(\cdot, t_k), \quad t_{k-1} < t \leq t_k, \quad k = 1, 2, \ldots, N. \tag{3.48}$$

It is clear that

$$\widetilde{\mathbf{U}}^N \to \widetilde{\mathbf{U}} \quad \text{in } L_\infty(0, T_*; W_2^1(D_+; \mathbb{R}^3)) \tag{3.49}$$

and

$$\mathbf{F}^N \to \mathbf{F} \quad \text{in } L_\infty(0, T_*; L_2(D_+; \mathbb{R}^3)), \tag{3.50}$$

where

$$\mathbf{F}(\mathbf{y}, t) = Q^*(t)\mathbf{f}(Q(t)\mathbf{y} + \overline{\mathbf{x}}(t), t).$$

Moreover, we have

$$\widetilde{\mathbf{U}}^N(\cdot, t_k) \in H^1(D_k), \quad k = 1, 2, \ldots, N. \tag{3.51}$$

Now, from (3.9), (3.10), (3.17), and (3.51), we obtain

$$\begin{aligned}
&\int_{Q_+} \rho \Big[\big(\partial_t \mathbf{u}^N + \boldsymbol{\omega}_{\mathbf{v}}^N \wedge \mathbf{v}^N\big) \cdot \widetilde{\mathbf{U}}^N \\
&\qquad\qquad + \nabla \mathbf{v}^N : \Big(\widetilde{\mathbf{U}}^N \otimes \big(\mathbf{v}^{N,\mu} - \boldsymbol{\omega}_{\mathbf{v}}^N \wedge \mathbf{y} - \mathbf{w}_{\mathbf{v}}^N\big)\Big) \Big]\, d\mathbf{y}dt \\
&+ \frac{1}{2}(\rho^f - \rho^s) \int_0^{T_*} \int_{\partial S_0} \mathbf{v}^N \cdot \widetilde{\mathbf{U}}^N \big(\mathbf{v}^{N,\mu} - \boldsymbol{\omega}_{\mathbf{v}}^N \wedge \mathbf{y} - \mathbf{w}_{\mathbf{v}}^N\big) \cdot \mathbf{n}\, ds \\
&+ \int_{Q_+} \big[2\nu\, \varepsilon(\mathbf{v}^N) : \varepsilon(\widetilde{\mathbf{U}}^N) - \rho\, \mathbf{F}^N \cdot \widetilde{\mathbf{U}}^N\big]\, d\mathbf{y}dt = 0,
\end{aligned} \tag{3.52}$$

where $\mathbf{v}^N = \boldsymbol{\omega}_{\mathbf{v}}^N \wedge \mathbf{y} + \mathbf{w}_{\mathbf{v}}^N$ on S_0, and $\mathbf{v}^{N,\mu}(\cdot, t) \overset{\text{def}}{=} (\mathbf{u}^N)_\mu(\cdot, t_i)$, $t \in]t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$,

$$E(\mathbf{v}^N(\cdot, t)) + \nu \int_0^t \int_{D_+} |\varepsilon(\mathbf{v}^N)|^2\, d\mathbf{y}d\tau \leq E_0 + E_{\mathbf{f}} \tag{3.53}$$

for all $t \in]0,T]$, and

$$\int_{D_+} |\varepsilon(\mathbf{v}^N(\mathbf{y},t))|^2\,d\mathbf{y} + \int_0^t \int_{D_+} |\partial_t \mathbf{u}^N|^2 d\mathbf{y} d\tau \le c \tag{3.54}$$

for all $t \in]0,T]$.

Selecting if it is necessary subsequences (still denoted by the same symbol as the whole sequence), we obtain from (3.53), (3.54), and (3.41) that

$$\begin{aligned} &\mathbf{u}^N \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L_\infty(0,T_*;W_2^1(D_+;\mathbb{R}^3)) \\ &\mathbf{v}^N \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L_\infty(0,T_*;W_2^1(D_+;\mathbb{R}^3)) \\ &\partial_t\mathbf{u}^N \rightharpoonup \partial_t\mathbf{u} \quad \text{in } L_2(Q_+;\mathbb{R}^3) \\ &\mathbf{u}^N \to \mathbf{u} \quad \text{in } L_2(Q_+;\mathbb{R}^3) \\ &\mathbf{v}^N \to \mathbf{u} \quad \text{in } L_2(Q_+;\mathbb{R}^3). \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbf{u}(\cdot,t) \in H^1(D(t)) \quad \text{for a.a. } t \in [0,T_*] \\ &\mathbf{u} \in C([0,T_*];L_2(D_+;\mathbb{R}^3)) \\ &\mathbf{u}(\cdot,0) = \mathbf{u}_0 \\ &\boldsymbol{\omega}^N \to \boldsymbol{\omega} \quad \text{in } C([0,T_*];\mathbb{R}^3) \\ &\mathbf{w}^N \to \mathbf{w} \quad \text{in } C([0,T_*];\mathbb{R}^3), \end{aligned}$$

where $\mathbf{u}^N = \boldsymbol{\omega}^N \wedge \mathbf{y} + \mathbf{w}^N$ and $\mathbf{u} = \boldsymbol{\omega} \wedge \mathbf{y} + \mathbf{w}$ on S_0, and

$$\begin{aligned} &\boldsymbol{\omega}_{\mathbf{v}}^N \to \boldsymbol{\omega} \quad \text{in } L_2(0,T_*;\mathbb{R}^3) \\ &\mathbf{w}_{\mathbf{v}}^N \to \mathbf{w} \quad \text{in } L_2(0,T_*;\mathbb{R}^3). \end{aligned}$$

All this information together with (3.49), (3.50), and (3.43)–(3.47) allows us to take limit in (3.52) for fixed μ and obtain

$$\begin{aligned} \int_0^{T_*} \psi(t)\,dt \Big\{ &\int_{D_+} \rho\,[(\partial_t\mathbf{u} + \boldsymbol{\omega}\wedge\mathbf{v})\cdot\mathbf{U} + \nabla\mathbf{u} : (\mathbf{U}\otimes((\mathbf{u})_\mu - \boldsymbol{\omega}\wedge\mathbf{y} - \mathbf{w}))]\,d\mathbf{y} \\ &+\frac{1}{2}(\rho^f - \rho^s)\int_{\partial S_0} \mathbf{u}\cdot\mathbf{U}\,((\mathbf{u})_\mu - \boldsymbol{\omega}\wedge\mathbf{y} - \mathbf{w})\cdot\mathbf{n}\,ds \\ &+\int_{D_+} \big[2\nu\,\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{U}) - \rho\,Q^*(t)\mathbf{f}(Q(t)\mathbf{y} + \overline{\mathbf{x}}(t),t)\cdot\mathbf{U}\big]\,d\mathbf{y}\Big\} = 0. \end{aligned}$$

By the arbitrariness of ψ, we obtain the identity (2.28) for all Lebesque points of the functions $t \mapsto \partial_t\mathbf{u}(\cdot,t)$, $t \mapsto \nabla\mathbf{u}(\cdot,t)$, $t \mapsto \varepsilon(\mathbf{u})(\cdot,t)$ and for all functions $\mathbf{U}$ from $C_\infty(D(t))$. Thus, (2.28) now follows from the definition of the space $H^1(D(t))$. The relations (2.9), (2.11)–(2.13), and (2.27) are obtained from (3.6), (3.45), (3.46), and (2.26) which has been already proved. Theorem 2.3 is proved.

4. Proof of Theorem 2.2

The first claim of Theorem 2.2 follows from inequality (2.15) and the definition of the numbers λ_+ and T_*; see the proof of Theorem 2.3 for a similar situation. Thus, it remains to prove the second claim only.

Let D be a smooth domain such that $S_0 \subset\subset D \subset\subset D_+$ and let $\{\mathbf{e}_j \in H^1(D)\}_{j=1}^{\infty}$ and $\{\lambda_j\}_{j=1}^{\infty}$ be eigenvectors and eigenvalues of the boundary value problem

$$\begin{aligned} &2\int_{D_+} \varepsilon(\mathbf{e}_j) : \varepsilon(\mathbf{v})\, d\mathbf{y} = \lambda_j \int_{D_+} \rho\, \mathbf{e}_j \cdot \mathbf{v}\, d\mathbf{y}, \quad \mathbf{v} \in H^1(D), \\ &\int_{D_+} \rho\, \mathbf{e}_i \cdot \mathbf{e}_j\, d\mathbf{y} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned} \tag{4.1}$$

The spaces $H(D)$ and $H^1(D)$ are defined in the same way as in the case $D \subset \mathcal{R}$. For them, all the properties of (2.2) are valid as well. By (2.2), the system $\{\mathbf{e}_j\}_{j=1}^{\infty}$ forms an orthogonal basis in $H(D)$ with respect to the scalar product

$$(\mathbf{u}, \mathbf{v})_{H(D)} \stackrel{\text{def}}{=} \int_{D_+} \rho\, \mathbf{u} \cdot \mathbf{v}\, d\mathbf{y}, \qquad \mathbf{u} \in H(D), \quad \mathbf{v} \in H(D).$$

Clearly, solutions to (4.1) belong to $W_2^2(D \setminus \overline{S_0})$.

For $D = D_0$, we set $\mathbf{e}_j^0 \stackrel{\text{def}}{=} \mathbf{e}_j$, $j = 1, 2, \ldots$. Now, let $\mu_k \to 0$ as $k \to +\infty$ and

$$\mathbf{u}_0^{(k)} = \sum_{i=1}^{k} a_i^0 \mathbf{e}_j^0, \qquad \text{where} \qquad a_j^0 = (\mathbf{e}_j^0, \mathbf{u}_0)_{H(D_0)}.$$

Obviously,

$$E_0^{(k)} \stackrel{\text{def}}{=} E(\mathbf{u}_0^{(k)}) = \frac{1}{2}\int_{D_+} \rho\, \mathbf{u}_0^{(k)} \cdot \mathbf{u}_0^{(k)}\, d\mathbf{y} \leq \frac{1}{2}\int_{D_+} \rho\, \mathbf{u}_0 \cdot \mathbf{u}_0\, d\mathbf{y} = E_0. \tag{4.2}$$

We introduce

$$T_*^{(k)} \stackrel{\text{def}}{=} T_*(E_0^{(k)}, A_{\mathbf{f}}, d) = \min\left\{1, \frac{d}{2\sqrt{2}}\left(\left(\frac{\operatorname{diam} S_0}{\sqrt{\gamma_0}} + \frac{1}{\sqrt{m_s}}\right)\sqrt{E_0^{(k)} + E_{\mathbf{f}}}\right)^{-1}\right\}.$$

By (4.2), we have

$$T_*^{(k)} \geq T_*(E_0, A_{\mathbf{f}}, d) = T_* \quad \text{for all } k. \tag{4.3}$$

According to Theorem 2.3, there exists at least one function $\mathbf{u}^{(k)}$ with the following properties:

$$\begin{cases} \mathbf{u}^{(k)} \in L_\infty(0, T_*^{(k)}; W_2^1(D_+; \mathbb{R}^3)) \cap C([0, T_*^{(k)}]; L_2(D_+; \mathbb{R}^3)) \\ \partial_t \mathbf{u}^{(k)} \in L_2(0, T_*^{(k)}; L_2(D_+; \mathbb{R}^3)), \end{cases} \tag{4.4}$$

$$\mathbf{u}^{(k)}(\cdot, t) \in H^1(D^{(k)}(t)) \quad \text{for a.a. } t \in [0, T_*^{(k)}], \tag{4.5}$$

where

$$D^{(k)}(t) \stackrel{\text{def}}{=} \Big\{ \mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = (Q^{(k)})^*(t)(\mathbf{x} - \overline{\mathbf{x}}^{(k)}(t)), \quad \mathbf{x} \in D_0 \Big\} \in \mathcal{R}$$

for all $t \in [0, T_*^{(k)}]$ and, moreover,

$$\operatorname{dist}\Big\{\partial D^{(k)}(t), S_0\Big\} \geq \frac{d}{2} \quad \text{for all } t \in [0, T_*^{(k)}]. \tag{4.6}$$

The functions

$$Q^{(k)} \in C^1\Big([0, T_*^{(k)}]; M^{3\times 3}\Big) \qquad \text{and} \qquad \overline{\mathbf{x}}^{(k)} \in C^1\Big([0, T_*^{(k)}]; \mathbb{R}^3\Big) \tag{4.7}$$

satisfy the relations

$$\overline{\mathbf{x}}^{(k)}(t) = \int_0^t Q^{(k)}(\tau)\mathbf{w}^{(k)}(\tau)\, d\tau \quad \text{for all } t \in [0, T_*^{(k)}], \tag{4.8}$$

$$\begin{cases} \dot{Q}^{(k)}(t)\mathbf{z} = Q^{(k)}(t)(\boldsymbol{\omega}^{(k)}(t) \wedge \mathbf{z}) \quad \text{for all } t \in [0, T_*^{(k)}] \\ Q^{(k)}(t) \in SO(3) \quad \text{for all } t \in [0, T_*^{(k)}], \quad Q^{(k)}(0) = I, \end{cases} \tag{4.9}$$

and

$$\mathbf{u}^{(k)}(\mathbf{y}, t) = \boldsymbol{\omega}^{(k)}(t) \wedge \mathbf{y} + \mathbf{w}^{(k)}(t) \text{ for all } \mathbf{y} \in S_0 \text{ and for all } t \in [0, T_*^{(k)}]. \tag{4.10}$$

For almost all $t \in [0, T_*^{(k)}]$, the function $\mathbf{u}^{(k)}$ satisfies the variational identity

$$\begin{aligned} &\int_{D_+} \rho(\mathbf{y}) \Big[\Big(\partial_t \mathbf{u}^{(k)}(\mathbf{y}, t) + \boldsymbol{\omega}^{(k)}(t) \wedge \mathbf{u}^{(k)}(\mathbf{y}, t) \Big) \cdot \mathbf{U}(\mathbf{y}) \\ &\qquad + \nabla \mathbf{u}^{(k)}(\mathbf{y}, t) : \Big(\mathbf{U}(\mathbf{y}) \otimes \Big((\mathbf{u}^{(k)})_{\mu_k}(\mathbf{y}, t) - \boldsymbol{\omega}^{(k)}(t) \wedge \mathbf{y} - \mathbf{w}^{(k)}(t) \Big) \Big) \Big] \, d\mathbf{y} \\ &+ \frac{1}{2}(\rho^f - \rho^s) \int_{\partial S_0} \mathbf{u}^{(k)}(\mathbf{y}, t) \cdot \mathbf{U}(\mathbf{y}) \Big((\mathbf{u}^{(k)})_{\mu_k}(\mathbf{y}, t) - \boldsymbol{\omega}^{(k)}(t) \wedge \mathbf{y} - \mathbf{w}^{(k)}(t) \Big) \cdot \mathbf{n} \, ds \\ &+ \int_{D_+} \big[2\nu\, \varepsilon(\mathbf{u}^{(k)}(\mathbf{y}, t)) : \varepsilon(\mathbf{U}(\mathbf{y})) - \rho(\mathbf{y})\, \mathbf{F}^{(k)}(\mathbf{y}, t) \cdot \mathbf{U}(\mathbf{y}) \big] \, d\mathbf{y} = 0 \end{aligned} \tag{4.11}$$

for any $\mathbf{U} \in H^1(D^{(k)}(t))$, where $\mathbf{F}^{(k)}(\mathbf{y}, t) \stackrel{\text{def}}{=} (Q^{(k)})^*(t)\mathbf{f}(Q^{(k)}(t)\mathbf{y} + \overline{\mathbf{x}}^{(k)}(t), t)$. The function $\mathbf{u}^{(k)}$ satisfies the initial condition

$$\mathbf{u}^{(k)}(\cdot, 0) = \mathbf{u}_0^{(k)}(\cdot) \tag{4.12}$$

and the energy inequality

$$\begin{aligned} E(\mathbf{u}^{(k)}(\cdot, t)) + 2\nu \int_0^t \int_{D_+} & |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y} d\tau \\ & \leq E(\mathbf{u}_0^{(k)}) + \int_0^t \int_{D_+} \rho\, \mathbf{F}^{(k)} \cdot \mathbf{u}^{(k)} \, d\mathbf{y} d\tau \end{aligned} \tag{4.13}$$

for all $t \in [0, T_*^{(k)}]$.

Using (2.17) once again, we obtain from (4.2) and (4.13) the estimate

$$E(\mathbf{u}^{(k)}(\cdot,t)) + \nu \int_0^t \int_{D_+} |\varepsilon(\mathbf{u}^{(k)})|^2 \, d\mathbf{y} d\tau \le E_0^{(k)} + E_{\mathbf{f}} \le E_0 + E_{\mathbf{f}} \tag{4.14}$$

for all $t \in [0, T_*]$. By (4.14), one may select subsequences (still denoted in the same way) such that

$$\begin{aligned} &\mathbf{u}^{(k)} \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L_\infty(0,T_*; L_2(D_+;\mathbb{R}^3)) \\ &\mathbf{u}^{(k)} \rightharpoonup \mathbf{u} \quad \text{in } L_2(0,T_*; W_2^1(D_+;\mathbb{R}^3)) \\ &\boldsymbol{\omega}^{(k)} \overset{*}{\rightharpoonup} \boldsymbol{\omega} \quad \text{in } L_\infty(0,T_*;\mathbb{R}^3) \\ &\mathbf{w}^{(k)} \overset{*}{\rightharpoonup} \mathbf{w} \quad \text{in } L_\infty(0,T_*;\mathbb{R}^3) \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} &Q^{(k)} \overset{*}{\rightharpoonup} Q \text{ in } W_\infty^1(0,T_*;M^{3\times 3}) \quad \text{implies} \quad Q^{(k)} \to Q \text{ in } C([0,T_*];M^{3\times 3}) \\ &\overline{\mathbf{x}}^{(k)} \overset{*}{\rightharpoonup} \overline{\mathbf{x}} \text{ in } W_\infty^1(0,T_*;\mathbb{R}^3) \quad \text{implies} \quad \overline{\mathbf{x}}^{(k)} \to \overline{\mathbf{x}} \text{ in } C([0,T_*];\mathbb{R}^3). \end{aligned} \tag{4.16}$$

Moreover, it is not difficult to verify that the limit functions $\mathbf{u}$, $\boldsymbol{\omega}$, $\mathbf{w}$, Q, and $\overline{\mathbf{x}}$ satisfy the relations (2.7)–(2.13) and the energy inequality (2.15) and, therefore, the relation (2.19). It remains to prove that (2.14) and (2.16) hold.

First we note that, by (2.4), (4.15), and (4.16),

$$\mathbf{F}^{(k)} \to \mathbf{F} \text{ in } L_2(D_+ \times]0,T_*[;\mathbb{R}^3), \tag{4.17}$$

where $\mathbf{F}(\mathbf{y},t) = Q^*(t)\mathbf{f}(Q(t)\mathbf{y} + \overline{\mathbf{x}}(t), t)$.

In the remainder of this section, we shall denote by c, c_1, etc. all constants that depend only on $\mathbf{u}_0$, $A_{\mathbf{f}}$, d, D_0, and S_0.

Now, let us fix a positive number $\delta_1 < d/4$. Clearly, there exists is a number k_0, depending on δ_1 only, such that

$$D^{\delta_{1/2}}(t) \supset D^{(k)}(t) \supset D_{\delta_{1/2}}(t) \tag{4.18}$$

for all $t \in [0,T_*]$ and for all $k \ge k_0(\delta_1)$. Here, we use the notation

$$\begin{aligned} D^\delta(t) &\overset{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^3 \mid \operatorname{dist}\{\mathbf{y}, D(t)\} < \delta\right\} \\ D_\delta(t) &\overset{\text{def}}{=} \left\{\mathbf{y} \in D(t) \mid \operatorname{dist}\{\mathbf{y}, \partial D(t)\} > \delta\right\}. \end{aligned}$$

We again divide the segment $[0,T_*]$ into N segments $[t_{i-1},t_i]$, $t_i - t_{i-1} = \dfrac{T_*}{N} = \Delta$, $i = 1,2,\dots,N$. Let $\left\{\mathbf{e}_j^{i,\delta_1}\right\}_{j=1}^\infty$ be the orthogonal basis in $H(D_{\delta_1}(t_i))$, described at the beginning of the section. Consider piecewise linear functions in t

$$\mathbf{U}^{\delta_1}(\cdot,t) = \mathbf{e}_j^{i-1,\delta_1}(\cdot) + \frac{1}{\Delta}(t - t_{i-1})\left(\mathbf{e}_m^{i,\delta_1}(\cdot) - \mathbf{e}_j^{i-1,\delta_1}(\cdot)\right)$$

for $t \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots$, and $m = 1, 2, \ldots$. We denote the set of such functions by $\mathcal{A}_N$. Obviously, there is a number $N_0 = N_0(\delta_1)$ such that

$$\mathbf{U}^{\delta_1}(\cdot, t) = 0 \ \text{ outside of } D_{\delta_{1/2}}(t)$$

for all $t \in [0, T_*]$ and all $N \geq N_0$.

Let $\mathcal{A} = \bigcup\limits_{N_0}^{\infty} \mathcal{A}_N$. This is a countable set, i.e., $\mathcal{A} = \left\{\mathbf{U}^{\delta_1, r}\right\}_{r=1}^{\infty}$. It is important to note that functions belonging to this set possess the property (see (4.18))

$$\mathbf{U}^{\delta_1, r}(\cdot, t) \in H^1\left(D^{(k)}(t)\right)$$

for all $t \in [0, T_*]$, for all $r = 1, 2, \ldots$, and for all $k \geq k_0(\delta_1)$.

Now, let us substitute $\mathbf{U} = \mathbf{U}^{\delta_1, r}$ into (4.11) and integrate the result in time from t up to $t + \Delta$. As a result, we obtain

$$\begin{aligned}
&\int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y}, \tau) \cdot \mathbf{U}^{\delta_1, r}(\mathbf{y}, \tau)\, d\mathbf{y}\Big|_{\tau=t}^{\tau=t+\Delta} \\
&= \int_t^{t+\Delta} \int_{D_+} \rho\, \mathbf{u}^{(k)} \cdot \partial_t \mathbf{U}^{\delta_1, r}\, d\mathbf{y} d\tau - \int_t^{t+\Delta} \int_{D_+} \rho\, \Big[\Big(\boldsymbol{\omega}^{(k)} \wedge \mathbf{u}^{(k)}\Big) \cdot \mathbf{U}^{\delta_1, r} \\
&\qquad + \nabla \mathbf{u}^{(k)} : \big(\mathbf{U}^{\delta_1, r} \otimes \big((\mathbf{u}^{(k)})_{\mu_k} - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\big)\big)\Big]\, d\mathbf{y} d\tau \\
&- \frac{1}{2}(\rho^f - \rho^s) \int_t^{t+\Delta} \int_{\partial S_0} \mathbf{u}^{(k)} \cdot \mathbf{U}^{\delta_1, r} \Big((\mathbf{u}^{(k)})_{\mu_k} - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)}\Big) \cdot \mathbf{n}\, ds d\tau \\
&- \int_t^{t+\Delta} \int_{D_+} \big[2\nu\, \varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{U}^{\delta_1, r}) + \rho\, \mathbf{F}^{(k)} \cdot \mathbf{U}^{\delta_1, r}\big]\, d\mathbf{y} d\tau.
\end{aligned} \tag{4.19}$$

We wish to estimate the right-hand side of (4.19) with the help of (4.13). We set

$$I(t) \stackrel{\text{def}}{=} \left| \int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y}, \tau) \cdot \mathbf{U}^{\delta_1, r}(\mathbf{y}, \tau)\, d\mathbf{y}\Big|_{\tau=t}^{\tau=t+\Delta} \right| \leq \sum_{i=1}^{6} I_i(t), \tag{4.20}$$

where

$$\begin{aligned}
I_1(t) &\stackrel{\text{def}}{=} \left| \int_t^{t+\Delta} \int_{D_+} \rho\, \mathbf{u}^{(k)} \cdot \partial_t \mathbf{U}^{\delta_1, r}\, d\mathbf{y} d\tau \right| \\
&\leq \left(\int_t^{t+\Delta} \int_{D_+} \rho\, |\mathbf{u}^{(k)}|^2\, d\mathbf{y} d\tau \right)^{1/2} \left(\int_t^{t+\Delta} \int_{D_+} \rho\, |\partial_t \mathbf{U}^{\delta_1, r}|^2\, d\mathbf{y} d\tau \right)^{1/2} \\
&\leq C_1(r) \Delta^{1/2},
\end{aligned} \tag{4.21}$$

$$I_2(t) \stackrel{\text{def}}{=} \left| \int_t^{t+\Delta} \int_{D_+} \rho\, \left(\boldsymbol{\omega}^{(k)} \wedge \mathbf{u}^{(k)}\right) \cdot \mathbf{U}^{\delta_1, r}\, d\mathbf{y} d\tau \right| \leq C_2(r) \Delta, \tag{4.22}$$

$$
\begin{aligned}
& I_3(t) \\
& \stackrel{\text{def}}{=} \Big| \int_t^{t+\Delta} \int_{D_+} \rho \, \nabla \mathbf{u}^{(k)} : \Big(\mathbf{U}^{\delta_1, r} \otimes \Big((\mathbf{u}^{(k)})_{\mu_k} - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \Big) \Big) \, d\mathbf{y} d\tau \Big| \\
& \leq c_1 \left(\int_t^{t+\Delta} \int_{D_+} |\mathbf{U}^{\delta_1, r}|^2 \left(|(\mathbf{u}^k)_{\mu_k}|^2 + 1 \right) \, d\mathbf{y} d\tau \right)^{1/2} \\
& \leq c_2 \left[\left(\int_t^{t+\Delta} \int_{D_+} |\mathbf{U}^{\delta_1, r}|^2 \, d\mathbf{y} d\tau \right)^{1/2} \right. \\
& \quad \left. + \left(\int_t^{t+\Delta} \int_{D_+} |\mathbf{U}^{\delta_1, r}|^6 \, d\mathbf{y} d\tau \right)^{1/6} \left(\int_t^{t+\Delta} \int_{D_+} |(\mathbf{u}^{(k)})_{\mu_k}|^3 \, d\mathbf{y} d\tau \right)^{1/3} \right].
\end{aligned}
\tag{4.23}
$$

By the property of the smoothing kernel and the multiplicative inequality, we have

$$
\begin{aligned}
& \int_0^{T_*} \int_{D_+} |(\mathbf{u}^{(k)})_{\mu_k}|^3 \, d\mathbf{y} dt \leq \int_0^{T_*} \int_{D_+} |\mathbf{u}^{(k)}|^3 \, d\mathbf{y} dt \\
& \leq \int_0^{T_*} \left(\int_{D_+} |\mathbf{u}^{(k)}|^2 \, d\mathbf{y} \right)^{3/4} \left(\int_{D_+} |\nabla \mathbf{u}^{(k)}|^2 \, d\mathbf{y} \right)^{3/4} dt \\
& \leq c_3 \operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(\int_{D_+} |\mathbf{u}^{(k)}(\mathbf{y}, t)|^2 \, d\mathbf{y} \right)^{3/4} \left(\int_0^{T_*} \int_{D_+} |\nabla \mathbf{u}^{(k)}|^2 \, d\mathbf{y} d\tau \right)^{3/4} \\
& \leq c_4.
\end{aligned}
\tag{4.24}
$$

On the other hand, we know that

$$
\operatorname*{ess\,sup}_{0 \leq t \leq T_*} \left(\| \, \mathbf{U}^{\delta_1, r}(\cdot, t) \, \|_{L_2(D_+)} + \| \, \nabla \mathbf{U}^{\delta_1, r}(\cdot, t) \, \|_{L_2(D_+)} \right) \leq C_3(r) \tag{4.25}
$$

and

$$
\left(\int_{D_+} |\mathbf{U}^{\delta_1, r}(\mathbf{y}, t)|^6 d\mathbf{y} \right)^{1/6} \leq c_5 \left(\int_{D_+} |\nabla \mathbf{U}^{\delta_1, r}(\mathbf{y}, t)|^2 d\mathbf{y} \right)^{1/2}. \tag{4.26}
$$

Thus, we obtain from (4.23)–(4.26)

$$
I_3(t) \leq C_4(r) \left(\Delta^{1/2} + \Delta^{1/6} \right). \tag{4.27}
$$

Next, we have

$$
\begin{aligned}
I_4(t) & \stackrel{\text{def}}{=} \frac{1}{2} |\rho^f - \rho^s| \left| \int_t^{t+\Delta} \int_{\partial S_0} \mathbf{u}^{(k)} \cdot \mathbf{U}^{\delta_1, r} \Big((\mathbf{u}^{(k)})_{\mu_k} - \boldsymbol{\omega}^{(k)} \wedge \mathbf{y} - \mathbf{w}^{(k)} \Big) \cdot \mathbf{n} \, ds d\tau \right| \\
& \leq C_5(r) \Delta + c_6 \int_t^{t+\Delta} \int_{\partial S_0} |\mathbf{U}^{\delta_1, r}| \, |(\mathbf{u}^{(k)})_{\mu_k}| \, ds d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_5(r)\Delta + C_6(r)\Delta^{1/2}\left(\int_t^{t+\Delta}\int_{\partial S_0} |(\mathbf{u}^{(k)})_{\mu_k}|^2\, ds d\tau\right)^{1/2} \\
&\leq C_7(r)\Big[\Delta + \Delta^{1/2}\Big(\int_t^{t+\Delta}\int_{S_0}\big(|(\mathbf{u}^{(k)})_{\mu_k}|^2 + |\nabla(\mathbf{u}^{(k)})_{\mu_k}|^2\big)\, d\mathbf{y} d\tau\Big)^{1/2}\Big] \\
&\leq C_8(r)\left[\Delta + \Delta^{1/2}\left(\int_0^{T_*}\int_{D_+}\Big(|\mathbf{u}^{(k)}|^2 + |\nabla\mathbf{u}^{(k)}|^2\Big)\, d\mathbf{y} d\tau\right)^{1/2}\right] \\
&\leq C_9(r)\left(\Delta + \Delta^{1/2}\right),
\end{aligned} \tag{4.28}
$$

$$
I_5(t) \stackrel{\text{def}}{=} \left|\int_t^{t+\Delta}\int_{D_+} 2\nu\, \varepsilon(\mathbf{u}^{(k)}) : \varepsilon(\mathbf{U}^{\delta_1,r}) d\mathbf{y} d\tau\right| \leq C_{10}(r)\Delta^{1/2}, \tag{4.29}
$$

and

$$
I_6(t) \stackrel{\text{def}}{=} \left|\int_t^{t+\Delta}\int_{D_+} \rho\, \mathbf{F}^{(k)} \cdot \mathbf{U}^{\delta_1,r} d\mathbf{y} d\tau\right| \leq C_{11}(r)\Delta. \tag{4.30}
$$

Combining all the estimates (4.20)–(4.30), we obtain

$$
I(t) \leq C_{12}(r)\Delta^{\frac{1}{6}}. \tag{4.31}
$$

According to Arcela's criterion of compactness in $C([0,T_*])$, one can choose a subsequence (denoted in the same way; we will exploit this agreement in what follows) such that

$$
\int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}^{\delta_1,1}(\mathbf{y},t)\, d\mathbf{y} \to \int_{D_+} \rho(\mathbf{y})\, \mathbf{u}(\mathbf{y},t) \cdot \mathbf{U}^{\delta_1,1}(\mathbf{y},t)\, d\mathbf{y} \text{ in } C([0,T_*]).
$$

Using the diagonal Cantor procedure, one can find a subsequence such that

$$
\int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}^{\delta_1,r}(\mathbf{y},t)\, d\mathbf{y} \to \int_{D_+} \rho(\mathbf{y})\, \mathbf{u}(\mathbf{y},t) \cdot \mathbf{U}^{\delta_1,r}(\mathbf{y},t)\, d\mathbf{y} \text{ in } C([0,T_*]).
$$

for any $r = 1, 2, \dots$. Moreover, the energy estimate (4.13) yields

$$
\int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \to \int_{D_+} \rho(\mathbf{y})\, \mathbf{u}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \text{ in } C([0,T_*])
$$

for any $\mathbf{U} \in C([0,T_*]; L_2(D_+;\mathbb{R}^3))$ such that $\mathbf{U}(\cdot,t) \in H(D_{\delta_1}(t))$ for all $t \in [0,T_*]$. Next, replacing δ_1 by $\delta_{1/2}$, we may repeat the same arguments and obtain a subsequence with the property

$$
\begin{aligned}
&\int_{D_+} \rho(\mathbf{y})\, \mathbf{u}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \\
&\qquad \to \int_{D_+} \rho(\mathbf{y})\, \mathbf{u}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \text{ in } C([0,T_*])
\end{aligned} \tag{4.32}
$$

for any $\mathbf{U} \in C([0,T_*];L_2(D_+;\mathbb{R}^3))$ such that $\mathbf{U}(\cdot,t) \in H(D_\delta(t))$ for all $t \in [0,T_*]$ and for some $\delta > 0$.

In what follows, we shall use a fixed cut-off function $\varphi \in C_0^\infty(\mathbb{R}^3)$ with the following properties:

$$0 \le \varphi \le 1 \qquad \text{and} \qquad \varphi = \begin{cases} 1 & \text{in } S_0 \\ 0 & \text{in } \mathbb{R}^3 \backslash S_0^{d/4}. \end{cases} \tag{4.33}$$

Now, let us introduce additional notation. For a domain $D \subset\subset D_+$, we introduce the spaces

$$\begin{aligned} &\dot{C}^\infty(D) = \left\{ \mathbf{v} \in C_0^\infty(D_+,\mathbb{R}^3) \mid \operatorname{spt}\mathbf{v} \subset D,\ \operatorname{div}\mathbf{v} = 0 \text{ on } D \right\}, \\ &\mathcal{H}(D) \text{ is the closure of } \dot{C}^\infty(D) \text{ in } L_2(D_+;\mathbb{R}^3), \\ &\mathcal{H}^1(D) \text{ is the closure of } \dot{C}^\infty(D) \text{ in } W_2^1(D_+;\mathbb{R}^3). \end{aligned} \tag{4.34}$$

The scalar product in $\mathcal{H}(D)$ is defined as

$$\int_{D_+} \mathbf{u} \cdot \mathbf{v} \, d\mathbf{y}$$

for $\mathbf{u} \in \mathcal{H}(D)$ and for $\mathbf{v} \in \mathcal{H}(D)$, and let

$$P(D) \text{ be the orthogonal projector from } \mathcal{H}(D) \text{ onto } H(D). \tag{4.35}$$

Next, we assume that

$$S_0 \subset\subset D \subset\subset D_+ \quad \text{and} \quad \operatorname{dist}\{\partial D, S_0\} \ge \frac{d}{2}. \tag{4.36}$$

Let

$$\mathcal{L}_2(D_+) \stackrel{\text{def}}{=} \left\{ \mathbf{v} \in L_2(D_+;\mathbb{R}^3) \mid \operatorname{div}\mathbf{v} = 0 \text{ in } D_+ \right\}.$$

We take a cut-off function φ satisfying the conditions of (4.33). For an arbitrary function $\mathbf{v} \in \mathcal{L}_2(D_+)$, there exists a unique function $\mathbf{v}_* \in \overset{o}{W}{}_2^1(S_0^{d/4} \backslash \overline{S}_0;\mathbb{R}^3)$ such that

$$\operatorname{div}\mathbf{v}_* = -\nabla\varphi \cdot \mathbf{v} \quad \text{in } S_0^{d/4} \backslash \overline{S}_0$$

and

$$\int_{S_0^{d/4} \backslash \overline{S}_0} |\nabla \mathbf{v}_*|^2 \, d\mathbf{y} = \inf \left\{ \int_{S_0^{d/4} \backslash \overline{S}_0} |\nabla \widetilde{\mathbf{v}}|^2 \, d\mathbf{y} \;\middle|\; \widetilde{\mathbf{v}} \in \overset{o}{W}{}_2^1(S_0^{d/4} \backslash \overline{S}_0;\mathbb{R}^3), \quad \operatorname{div}\widetilde{\mathbf{v}} = -\nabla\varphi \cdot \mathbf{v} \text{ in } S_0^{d/4} \backslash \overline{S}_0 \right\}.$$

As it was shown in [7], the following estimate is valid:

$$\| \mathbf{v}_* \|^2_{L_2(S_0^{d/4} \backslash \overline{S_0})} + \| \nabla \mathbf{v}_* \|^2_{L_2(S_0^{d/4} \backslash \overline{S_0})} \le c_1 \| \nabla\varphi \cdot \mathbf{v} \|^2_{L_2(S_0^{d/4} \backslash \overline{S_0})},$$

where the positive constant c_1 depends only on S_0 and d.

We extend the function $\mathbf{v}_*$ by zero to the whole of D_+. Next, we introduce the operator $\Lambda : \mathcal{L}_2(D_+) \to \mathcal{L}_2(D_+)$ by setting

$$\mathbf{u} = \Lambda \mathbf{v} \stackrel{\text{def}}{=} \varphi \mathbf{v} + \mathbf{v}_*. \tag{4.37}$$

This operator has the properties:

$$\begin{aligned} &\mathbf{u} = \mathbf{v} \ \text{ in } S_0, \qquad \mathbf{u} = \mathbf{0} \ \text{ in } D_+ \backslash \overline{S_0^{d/4}}, \\ &\operatorname{div} \mathbf{u} = 0 \ \text{ in } D_+, \\ &\| \mathbf{u} \|_{L_2(D_+)} \leq c_2(\varphi, d, S_0) \| \mathbf{v} \|_{L_2(D_+)}. \end{aligned} \tag{4.38}$$

Moreover,

$$\text{if } \ \mathbf{v} \in C\left([0, T_*]; L_2(D_+; \mathbb{R}^3)\right), \quad \text{then } \ \Lambda \mathbf{v} \in C\left([0, T_*]; L_2(D_+; \mathbb{R}^3)\right) \tag{4.39}$$

and

$$\text{if } \ \mathbf{v} \in L_2(Q_+; \mathbb{R}^3), \quad \text{then } \ \Lambda \mathbf{v} \in L_2(Q_+; \mathbb{R}^3), \tag{4.40}$$

where $Q_+ \stackrel{\text{def}}{=} D_+ \times]0, T_*[$.

Lemma 4.1. *The relation* (4.32) *holds for any* $\mathbf{U} \in C\left([0, T_*]; L_2(D_+; \mathbb{R}^3)\right)$ *such that* $\mathbf{U}(\cdot, t) \in H(D(t))$ *for all* $t \in [0, T_*]$.

Proof. Let $\mathbf{U}$ be an arbitrary function satisfying the hypotheses of the lemma. We introduce the function

$$\widetilde{\mathbf{U}}(\mathbf{x}, t) = Q(t) \mathbf{U}(\mathbf{y}, t), \qquad \mathbf{x} = Q(t)\mathbf{y} + \overline{\mathbf{x}}(t).$$

Clearly, $\widetilde{\mathbf{U}} \in C([0, T_*]; \mathcal{H}(D_0))$.

For fixed $\epsilon > 0$, a number $N = N(\epsilon)$ exists such that

$$\| \widetilde{\mathbf{U}} - \widetilde{\mathbf{U}}^N \|_{C([0,T_*]; L_2(D_+; \mathbb{R}^3))} \leq \frac{\epsilon}{2},$$

where

$$\widetilde{\mathbf{U}}^N(\cdot, t) = \widetilde{\mathbf{U}}(\cdot, t_{k-1}) + \frac{t - t_{k-1}}{\Delta} \left(\widetilde{\mathbf{U}}(\cdot, t_k) - \widetilde{\mathbf{U}}(\cdot, t_{k-1}) \right),$$
$$t \in [t_{k-1}, t_k], \quad k = 1, 2, \ldots, N, \quad \Delta = \frac{T_*}{N}.$$

For each $k = 0, 1, \ldots, N$, there exists a function $\mathbf{v}_k \in \dot{C}^\infty(D_0)$ such that

$$\| \mathbf{v}_k - \widetilde{\mathbf{U}}(\cdot, t_k) \|_{L_2(D_0; \mathbb{R}^3)} < \frac{\epsilon}{2(N+1)}.$$

If we set

$$\widetilde{\mathbf{V}}^N(\cdot, t) = \mathbf{v}_{k-1}(\cdot) + \frac{t - t_{k-1}}{\Delta} (\mathbf{v}_k(\cdot) - \mathbf{v}_{k-1}(\cdot)), \quad t \in [t_{k-1}, t_k], \quad k = 1, 2, \ldots, N,$$

then

$$\| \widetilde{\mathbf{U}} - \widetilde{\mathbf{V}}^N \|_{C([0,T_*]; L_2(D_+; \mathbb{R}^3))} < \epsilon,$$

Moreover, $\widetilde{\mathbf{V}}^N(\cdot,t) = \mathbf{0}$ outside of $D_{0\delta}$ for all $[0,T_*]$ and for some $\delta < \delta(\epsilon)$, where

$$\delta(\epsilon) = \operatorname{dist}\left\{\partial D_0, \bigcup_{k=0}^{N} \operatorname{spt}\mathbf{v}_k\right\}.$$

Now, one can make inverse change of variables and obtain

$$\widetilde{\mathbf{V}}^N(\mathbf{x},t) = Q(t)\mathbf{V}^N(\mathbf{y},t).$$

Obviously, $\mathbf{V}^N \in C([0,T_*];L_2(D_+;\mathbb{R}^3))$, $\mathbf{V}^N(\cdot,t) \in \mathcal{H}(D_\delta(t))$ for all $t \in [0,T_*]$, and

$$\| \mathbf{U} - \mathbf{V}^N \|_{C([0,T_*];L_2(D_+;\mathbb{R}^3))} < \epsilon.$$

We now let

$$\mathbf{U}^N(\cdot,t) = \mathbf{V}^N(\cdot,t) + \Lambda(\mathbf{U}(\cdot,t) - \mathbf{V}^N(\cdot,t)).$$

By (4.39), $\mathbf{U}^N \in C([0,T_*];L_2(D_+;\mathbb{R}^3))$. By (4.38), $\mathbf{U}^N(\cdot,t) \in H(D_\delta(t))$ for all $t \in [0,T_*]$ and

$$\| \mathbf{U} - \mathbf{U}^N \|_{C([0,T_*];L_2(D_+;\mathbb{R}^3))} < c(d,S_0)\,\epsilon.$$

Now, the statement of the lemma easily follows from (4.32). Lemma 4.1 is proved. □

By Lemma 4.1, the function $\mathbf{u}$ satisfies (2.16).

Lemma 4.2.

$$\int_{D_+} \rho(\mathbf{y})\,\mathbf{u}^{(k)}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t)\,d\mathbf{y} \to \int_{D_+} \rho(\mathbf{y})\,\mathbf{u}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t)\,d\mathbf{y} \quad \textit{in } L_2(0,T_*)$$

for all $\mathbf{U} \in L_2(Q_+;\mathbb{R}^3)$ *such that* $\mathbf{U}(\cdot,t) \in H(D(t))$ *for almost all* $t \in [0,T_*]$.

Proof. For the function $\mathbf{U}$ satisfying the hypotheses of the lemma, we again use the change of variables

$$\widetilde{\mathbf{U}}(\mathbf{x},t) = Q(t)\mathbf{U}(\mathbf{y},t), \qquad \mathbf{x} = Q(t)\mathbf{y} + \overline{\mathbf{x}}(t).$$

Let us make use of the smoothing kernel, introduced in Section 2,

$$(\widetilde{\mathbf{U}})^\mu(\mathbf{x},t) = \int_0^{T_*} K_{1\mu}(t-t')\widetilde{\mathbf{U}}(\mathbf{x},t')\,dt'.$$

Now, we have

$$\widetilde{\mathbf{U}} \in L_2(Q_+;\mathbb{R}^3), \quad \widetilde{\mathbf{U}}(\cdot,t) \in \mathcal{H}(D_0) \quad \text{for a.a. } t \in [0,T_*]$$

$$(\widetilde{\mathbf{U}})^\mu \in C([0,T_*];L_2(D_+;\mathbb{R}^3)), \quad (\widetilde{\mathbf{U}})^\mu(\cdot,t) \in \mathcal{H}(D_0) \quad \text{for all } t \in [0,T_*]$$

and, moreover,

$$\| \widetilde{\mathbf{U}} - (\widetilde{\mathbf{U}})^\mu \|_{L_2(Q_+;\mathbb{R}^3)} \to 0 \quad \text{as } \mu \to 0.$$

Setting

$$\mathbf{V}^\mu(\mathbf{y},t) = Q^*(t)(\widetilde{\mathbf{U}})^\mu(\mathbf{x},t), \quad \mathbf{x} = Q(t)\mathbf{y} + \overline{\mathbf{x}}(t),$$

one can see that

$$\mathbf{V}^\mu \in C([0,T_*];L_2(D_+;\mathbb{R}^3)), \qquad \mathbf{V}^\mu(\cdot,t) \in \mathcal{H}(D(t)) \quad \text{for all } t \in [0,T_*],$$

and

$$\| \mathbf{U} - \mathbf{V}^\mu \|_{L_2(Q_+;\mathbb{R}^3)} \to 0 \quad \text{as } \mu \to 0.$$

Now, if we let

$$\mathbf{U}^\mu(\cdot,t) = \mathbf{V}^\mu(\cdot,t) + \Lambda((\mathbf{U})^\mu(\cdot,t) - \mathbf{V}^\mu(\cdot,t)),$$

where

$$(\mathbf{U})^\mu(\mathbf{y},t) = \int_0^{T_*} K_{1\mu}(t-t')\mathbf{U}(\mathbf{y},t')\,dt',$$

then, by (4.38) and (4.39),

$$\mathbf{U}^\mu \in C([0,T_*];L_2(D_+;\mathbb{R}^3)), \qquad \mathbf{U}^\mu(\cdot,t) \in H(D(t)) \quad \text{for all } t \in [0,T_*],$$

and

$$\| \mathbf{U} - \mathbf{U}^\mu \|_{L_2(Q_+;\mathbb{R}^3)} \to 0 \quad \text{as } \mu \to 0.$$

This property, Lemma 4.1, and the estimate (4.14) imply the required statement. Lemma 4.2 is proved. □

Our next goal is to prove that

$$\int_{Q_+} |\mathbf{u}^{(k)} - \mathbf{u}|^2\,d\mathbf{y}dt \to 0 \quad \text{as } k \to +\infty, \tag{4.41}$$

where $Q_+ = D_+ \times]0,T_*[$. To this end, we are going to exploit the constructions of Lemma 3.1. First, we introduce a new function $\mathbf{u}^{(k)}_{(1)}$ in the following way:

$$(Q^{(k)})^*(t)Q(t)\mathbf{u}^{(k)}_{(1)}(\mathbf{z},t) = \mathbf{u}^{(k)}(\mathbf{y},t)$$

$$\mathbf{y} = (Q^{(k)})^*(t)(Q(t)\mathbf{z} + \overline{\mathbf{x}}(t) - \overline{\mathbf{x}}^{(k)}(t)).$$

Clearly,

$$\begin{aligned}
&\operatorname{div}\mathbf{u}^{(k)}_{(1)}(\cdot,t) = 0 \text{ in } D_+\\
&\mathbf{u}^{(k)}_{(1)}(\cdot,t) = \mathbf{0} \text{ outside of } D(t)\\
&\mathbf{u}^{(k)}_{(1)} \in L_2(0,T_*;W^1_2(D_+;\mathbb{R}^3)) \cap L_\infty(0,T_*;L_2(D_+;\mathbb{R}^3)).
\end{aligned} \tag{4.42}$$

Using the same arguments as in Lemma 3.1, one can show that

$$\int_{Q_+} |\mathbf{u}^{(k)}_{(1)} - \mathbf{u}^{(k)}|^2\,d\mathbf{y}dt \to 0 \quad \text{as } k \to +\infty. \tag{4.43}$$

But the function $\mathbf{u}^{(k)}_{(1)}(\cdot,t)$ does not belong to $H^1(D(t))$. To provide this, we introduce

$$\mathbf{u}^{(k)}_{(2)}(\cdot,t) \stackrel{\text{def}}{=} \mathbf{u}^{(k)}_{(1)}(\cdot,t) + \Lambda\left(\mathbf{u}^{(k)}(\cdot,t) - \mathbf{u}^{(k)}_{(1)}(\cdot,t)\right). \tag{4.44}$$

Then, by (4.38), (4.40), (4.42), and (4.44),

$$\begin{cases} \left\{\mathbf{u}^{(k)}_{(2)}\right\}_{k=1}^{\infty} \text{ is bounded in } L_2(0,T_*;W^1_2(D_+;\mathbb{R}^3)) \cap L_\infty(0,T_*;L_2(D_+;\mathbb{R}^3)) \\ \mathbf{u}^{(k)}_{(2)}(\cdot,t) \in H^1(D(t)) \quad \text{for a.a. } t \in [0,T_*]. \end{cases} \tag{4.45}$$

By (4.41), (4.43),

$$\|\, \mathbf{u}^{(k)}_{(2)} - \mathbf{u}^{(k)} \,\|_{L_2(Q_+;\mathbb{R}^3)} \to 0 \text{ as } k \to +\infty \tag{4.46}$$

and, by (4.46) and Lemma 4.2, one can conclude that

$$\begin{aligned} &\int_{D_+} \rho(\mathbf{y}) \;\; \mathbf{u}^{(k)}_{(2)}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t)\,d\mathbf{y} \\ &\qquad \to \int_{D_+} \rho(\mathbf{y})\,\mathbf{u}(\mathbf{y},t)\cdot\mathbf{U}(\mathbf{y},t)\,d\mathbf{y} \quad \text{in } L_2(0,T_*) \end{aligned} \tag{4.47}$$

for any $\mathbf{U} \in L_\infty(0,T_*;L_2(D_+;\mathbb{R}^3))$ such that $\mathbf{U}(\cdot,t) \in H(D(t))$ for almost all $t \in [0,T_*]$.

We also introduce the operator

$$M(D) : \mathcal{L}_2(D_+) \mapsto W(D) = \left\{ q \in W^1_2(D\backslash\overline{S_0}) \mid \int_{D\backslash\overline{S_0}} q\,d\mathbf{y} = 0 \right\}$$

in the following way:

$$\begin{aligned} &W(D) \ni p = M(D)\mathbf{v},\ \mathbf{v} \in \mathcal{L}_2(D_+) \qquad \text{if and only if} \\ &\qquad \int_{D\backslash\overline{S_0}} \nabla p\cdot\nabla q\,d\mathbf{y} = \int_{D\backslash\overline{S_0}} \Lambda\mathbf{v}\cdot\nabla q\,d\mathbf{y} \quad \text{for any } q \in W(D). \end{aligned} \tag{4.48}$$

The variational identity in (4.48) is the weak form of the boundary value problem

$$\begin{aligned} &\Delta p = 0 \quad \text{in } D\backslash\overline{S_0}, \\ &\nabla p\cdot\mathbf{n} = 0 \quad \text{on } \partial D, \quad \nabla p\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} \quad \text{on } \partial S_0. \end{aligned} \tag{4.49}$$

Let us denote by H_r the finite dimensional space of rigid motions, i.e.,

$$H_r \stackrel{\text{def}}{=} \left\{\mathbf{v} \mid \mathbf{v}(\mathbf{y}) = \boldsymbol{\omega}\wedge\mathbf{y} + \mathbf{w}, \mathbf{y} \in \mathbb{R}^3,\ \text{for some } \boldsymbol{\omega} \in \mathbb{R}^3,\ \text{and for some } \mathbf{w} \in \mathbb{R}^3\right\}.$$

In this space, we shall use the scalar product

$$\langle \mathbf{u},\mathbf{v}\rangle_D \stackrel{\text{def}}{=} \rho^s \int_{S_0} \mathbf{u}\cdot\mathbf{v}\,d\mathbf{y} + \rho^f \int_{D\backslash\overline{S}_0} \nabla p\cdot\nabla q\,d\mathbf{y}, \qquad \mathbf{u} \in H_r, \mathbf{v} \in H_r,$$

where $p = M(D)\mathbf{u}$, $q = M(D)\mathbf{v}$, and the corresponding norm $\|\cdot\|_D = \sqrt{\langle\cdot,\cdot\rangle_D}$. If we assume that D satisfies (4.36), then

$$\int_{D\setminus\overline{S}_0} |\nabla q|^2\, d\mathbf{y} \le \int_{D\setminus\overline{S}_0} |\Lambda\mathbf{v}|^2\, d\mathbf{y}$$

and, therefore, by (4.38), we have

$$\rho^s \int_{S_0} |\mathbf{v}|^2\, d\mathbf{y} \le \|\,\mathbf{v}\,\|_D^2 \le c_3(\varphi, d, S_0) \int_{S_0} |\mathbf{v}|^2\, d\mathbf{y}, \qquad \mathbf{v} \in H_r. \tag{4.50}$$

Let

$$\begin{aligned}
&\overline{\mathbf{u}}^{(k)} \stackrel{\text{def}}{=} \mathbf{u}^{(k)}_{(2)} - \mathbf{u}\\
&\overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \stackrel{\text{def}}{=} \overline{\boldsymbol{\omega}}^{(k)}(t) \wedge \mathbf{y} + \overline{\mathbf{w}}^{(k)}(t),\ \mathbf{y} \in S_0, t \in [0,T_*],\\
&\overline{\mathbf{v}}^{(k)}(\mathbf{y},t) \stackrel{\text{def}}{=} \overline{\boldsymbol{\omega}}^{(k)}(t) \wedge \mathbf{y} + \overline{\mathbf{w}}^{(k)}(t),\ \mathbf{y} \in \mathbb{R}^3, t \in [0,T_*].
\end{aligned}$$

By (4.47),

$$\int_{D_+} \rho(\mathbf{y})\, \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \to 0 \quad \text{in } L_2(0,T_*) \tag{4.51}$$

for any $\mathbf{U} \in L_\infty(0,T_*; L_2(D_+;\mathbb{R}^3))$ such that $\mathbf{U}(\cdot,t) \in H(D(t))$ for almost all $t \in [0,T_*]$. Let us take $\mathbf{U} = \mathbf{U}^i$, where

$$\mathbf{U}^i(\mathbf{y},t) \stackrel{\text{def}}{=} \begin{cases} \mathbf{g}^i(\mathbf{y}) & \mathbf{y} \in S_0 \\ \nabla p^i(\mathbf{y},t) & \mathbf{y} \in D(t)\setminus\overline{S_0} \\ 0 & \mathbf{y} \in D_+\setminus\overline{D}(t), \end{cases}$$

$$p^i(\cdot,t) \stackrel{\text{def}}{=} M(D(t))\mathbf{g}^i(\cdot), \qquad \mathbf{g}^i \in H_r,\ i = 1,2,\ldots,6,$$

and

$$\rho^s \int_{S_0} \mathbf{g}^i \cdot \mathbf{g}^j\, d\mathbf{y} = \delta_{ij} \stackrel{\text{def}}{=} \begin{cases} 1 & i = j \\ 0 & i \ne j. \end{cases}$$

By the definition of the operators Λ and M, we have (see (4.51))

$$\begin{aligned}
&\int_{D_+} \rho(\mathbf{y})\, \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}^i(\mathbf{y},t)\, d\mathbf{y}\\
&= \rho^s \int_{S_0} \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{g}^i(\mathbf{y})\, d\mathbf{y} + \rho^f \int_{D(t)\setminus\overline{S}_0} \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \nabla p^i(\mathbf{y},t)\, d\mathbf{y}\\
&= \rho^s \int_{S_0} \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{g}^i(\mathbf{y})\, d\mathbf{y}\\
&+ \rho^f \left(\int_{\partial D(t)} \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{n}(\mathbf{y},t)\, p^i(\mathbf{y},t)\, ds + \int_{\partial S_0} \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{n}(\mathbf{y})\, p^i(\mathbf{y},t)\, ds \right)\\
&= \langle \overline{\mathbf{v}}^{(k)}(\cdot,t), \mathbf{g}^i(\cdot) \rangle_{D(t)} \stackrel{\text{def}}{=} \gamma_i^{(k)}(t) \to 0 \quad \text{in } L_2(0,T_*).
\end{aligned} \tag{4.52}$$

We are seeking $\beta_i^{(k)}(t)$ so that

$$\overline{\mathbf{v}}^{(k)}(\mathbf{y},t) = \sum_{i=1}^{6} \beta_i^{(k)}(t)\mathbf{g}^i(\mathbf{y}). \tag{4.53}$$

Let $\boldsymbol{\beta}^{(k)}(t) = (\beta_1^{(k)}(t),\dots,\beta_6^{(k)}(t))$ and $\boldsymbol{\gamma}^{(k)}(t) = (\gamma_1^{(k)}(t),\dots,\gamma_6^{(k)}(t))$. Then, we have

$$A^{(k)}(t)\boldsymbol{\beta}^{(k)}(t) = \boldsymbol{\gamma}^{(k)}(t), \tag{4.54}$$

where

$$A_{ij}^{(k)}(t) \stackrel{\text{def}}{=} \langle \mathbf{g}^i(\cdot), \mathbf{g}^j(\cdot)\rangle_{D(t)} = \delta_{ij} + \rho^f \int_{D(t)\setminus\overline{S}_0} \nabla p^i \cdot \nabla p^j \, d\mathbf{y}, \quad i,j = 1,2,\dots,6$$

and, therefore,

$$\det A^{(k)}(t) \geq 1, \quad t \in [0,T_*], \tag{4.55}$$

for all $k = 1,2,\dots$. Moreover, by (4.50) for $\mathbf{v} = \mathbf{g}^i$ and $D = D(t)$, and, by (4.54), (4.55) , we obtain the estimate

$$|\boldsymbol{\beta}^{(k)}(t)| \leq c_4(\varphi, d, S_0)|\boldsymbol{\gamma}^{(k)}(t)|. \tag{4.56}$$

Now, from (4.52), (4.53), and (4.56), it follows that

$$\begin{aligned}\rho^s \int_0^{T_*} &\int_{S_0} |\overline{\mathbf{v}}^{(k)}(\mathbf{y},t)|^2 \, d\mathbf{y}dt \\ &= \int_0^{T_*} \Big(m_s |\overline{\mathbf{w}}^{(k)}(t)|^2 + \overline{\boldsymbol{\omega}}^{(k)}(t) \cdot J\overline{\boldsymbol{\omega}}^{(k)}(t)\Big) \, d\mathbf{y}dt \to 0.\end{aligned} \tag{4.57}$$

Now, for any $\mathbf{U} \in L_\infty(0,T_*; L_2(D_+;\mathbb{R}^3))$ such that $\mathbf{U}(\cdot,t) \in H(t)$ for almost all $t \in [0,T_*]$, we have

$$\begin{aligned}\int_{D_+} &\rho(\mathbf{y})\, \overline{\mathbf{u}}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \\ &= \rho^f \int_{D_+\setminus\overline{S}_0} \Big(\overline{\mathbf{u}}^{(k)}(\mathbf{y},t) - \Lambda\overline{\mathbf{v}}^{(k)}(\mathbf{y},t)\Big) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \\ &\qquad + \int_{D_+} \rho(\mathbf{y})\, \Lambda\overline{\mathbf{v}}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y}.\end{aligned} \tag{4.58}$$

By (4.38), (4.40), and (4.57),

$$\int_{D_+} \rho(\mathbf{y})\, \Lambda\overline{\mathbf{v}}^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \to 0 \text{ in } L_2(0,T_*),$$

and, therefore, setting $\overline{\mathbf{u}}_*^{(k)} = \overline{\mathbf{u}}^{(k)} - \Lambda\overline{\mathbf{v}}^{(k)}$, we obtain (see (4.51))

$$\int_{D_+} \overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t) \cdot \mathbf{U}(\mathbf{y},t)\, d\mathbf{y} \to 0 \text{ in } L_2(0,T_*) \tag{4.59}$$

for the same test functions $\mathbf{U}$ as in (4.58). Taking into account (2.8), (4.45), and the construction of the operator Λ, one can assert

$$\begin{aligned}&\left\{\overline{\mathbf{u}}_*^{(k)}\right\}_{k=1}^{\infty} \text{ is bounded in } L_\infty(0,T_*;L_2(D_+;\mathbb{R}^3)) \cap L_2(0,T_*;W_2^1(D_+;\mathbb{R}^3)),\\ &\overline{\mathbf{u}}_*^{(k)}(\cdot,t) \in H^1(D(t)) \quad \text{for a.a. } t\in[0,T_*].\end{aligned} \tag{4.60}$$

Let $\{\mathbf{h}_j^0\}_{j=1}^{\infty}$ and $\{\overline{\lambda}_j\}_{j=1}^{\infty}$ be the eigenvectors and eigenvalues of the Stokes problem

$$\begin{aligned}-\Delta \mathbf{h}_j^0 + \nabla p_j^0 &= \overline{\lambda}_j \mathbf{h}_j^0 \quad \text{in } D_0,\\ \operatorname{div} \mathbf{h}_j^0 &= 0 \quad \text{in } D_0,\\ \mathbf{h}_j^0 &= \mathbf{0} \quad \text{on } \partial D_0,\end{aligned}$$

and

$$\int_{D_0} \mathbf{h}_i^0 \cdot \mathbf{h}_j^0 \, d\mathbf{x} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j\\ 0 & \text{if } i\neq j.\end{cases}$$

We extend the functions $\mathbf{h}_j^0$ by zero to the whole of $\mathbb{R}^3$. It is easy to verify that

$$\begin{aligned}-\Delta \mathbf{h}_j(\mathbf{y},t) + \nabla p_j(\mathbf{y},t) &= \overline{\lambda}_j \mathbf{h}_j(\mathbf{y},t) \quad \mathbf{y}\in D(t),\\ \operatorname{div} \mathbf{h}_j(\mathbf{y},t) &= 0 \quad \mathbf{y}\in D(t),\\ \mathbf{h}_j(\mathbf{y},t) &= \mathbf{0} \quad \mathbf{y}\in \partial D(t),\end{aligned}$$

and

$$\int_{D(t)} \mathbf{h}_i(\mathbf{y},t)\cdot \mathbf{h}_j(\mathbf{y},t)\, d\mathbf{y} = \delta_{ij},$$

if

$$\mathbf{h}_j^0(\mathbf{x}) = Q(t)\mathbf{h}_j(\mathbf{y},t), \qquad p_j^0(\mathbf{x}) = p_j(\mathbf{y},t), \qquad \mathbf{x} = Q(t)\mathbf{y} + \overline{\mathbf{x}}. \tag{4.61}$$

Obviously, by (4.61),

$$\mathbf{h}_j \in C([0,T_*];L_2(D_+;\mathbb{R}^3)), \quad \mathbf{h}_j(\cdot,t)\in \mathcal{H}(D(t)) \text{ for all } t\in[0,T_*]$$

and, according to Lemma A.3,

$$\widehat{\mathbf{h}}_j \in L_\infty(0,T_*;L_2(D_+;\mathbb{R}^3)) \tag{4.62}$$

for each $j=1,2,\dots$, where

$$\widehat{\mathbf{h}}_j(\cdot,t) \stackrel{\text{def}}{=} P(D(t))\mathbf{h}_j(\cdot,t).$$

By (4.59) and (4.62), one can state

$$\int_{D_+} \overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)\cdot \widehat{\mathbf{h}}_j(\mathbf{y},t)\, d\mathbf{y} \to 0 \text{ in } L_2(0,T_*). \tag{4.63}$$

But, by (4.60), we see that

$$c_i^{(k)}(t) \overset{\text{def}}{=} \int_{D_+} \overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)\cdot \mathbf{h}_j(\mathbf{y},t)\,d\mathbf{y} = \int_{D_+} \overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)\cdot \widehat{\mathbf{h}}_j(\mathbf{y},t)\,d\mathbf{y} \tag{4.64}$$

for each $i = 1, 2, \ldots$.

Let us take an arbitrary number $\epsilon > 0$ and fix it. Since $\overline{\lambda}_j \to +\infty$ as $j \to +\infty$, one can find a natural number $N = N(\epsilon)$ such that

$$\epsilon\,\overline{\lambda}_N \le 1 \qquad \text{and} \qquad \epsilon\overline{\lambda}_{N+1} > 1.$$

Then, again by (4.60), we obtain

$$\int_{D_+} |\overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)|^2\,d\mathbf{y} = \sum_{j=1}^{\infty} |c_j^{(k)}(t)|^2$$

and

$$\int_{D_+} |\nabla\overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)|^2\,d\mathbf{y} = \sum_{j=1}^{\infty} \overline{\lambda}_j |c_j^{(k)}(t)|^2$$

and thus

$$\int_{D_+} |\overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)|^2\,d\mathbf{y} \le \epsilon \int_{D_+} |\nabla\overline{\mathbf{u}}_*^{(k)}(\mathbf{y},t)|^2\,d\mathbf{y} + \sum_{j=1}^{N} |c_j^{(k)}(t)|^2.$$

If we integrate the last inequality in t over the set $[0, T_*]$, use (4.60), (4.63), (4.64) and the arbitrariness of ϵ, we obtain

$$\int_{Q_+} |\overline{\mathbf{u}}_*^{(k)}|^2\,d\mathbf{y}dt \to 0 \quad \text{as } k \to +\infty.$$

The last relation together with (4.46) and (4.57) implies (4.41).

To obtain the variational identity (2.14), we substitute the test function with the properties described in (2.14) into (4.11), integrate in t by parts and, using (4.15) and (4.41), take the limit as $k \to +\infty$. As a result, we easily obtain (2.14). Theorem 2.2 is proved.

A. Appendix

In this section, we gather some technical results that were used in the proofs of the main results of the paper.

Suppose we are given a bounded domain D of class C^2. This means that there are positive numbers $R < 1$ and M having the following property. For any point $\mathbf{y}_0 \in \partial D$, there is a matrix $Q_{\mathbf{y}_0} \in SO(3)$, a function $\Psi_{\mathbf{y}_0} \in C^2(\overline{B}(R))$, and a Cartesian system of coordinates $\mathbf{z}$ such that

1. $\Psi_{\mathbf{y}_0}(0) = 0, \quad \nabla\Psi_{\mathbf{y}_0}(0) = 0,$
2. $|\nabla^2\Psi_{\mathbf{y}_0}(\mathbf{z}')| \le M, \quad |\mathbf{z}'| \le R,$

3. $\partial D \cap \boldsymbol{\omega}(\mathbf{y}_0, R, M) = Q_{\mathbf{y}_0}\Gamma + \mathbf{y}_0$,
4. $D \cap \boldsymbol{\omega}(\mathbf{y}_0, R, M) = Q_{\mathbf{y}_0}\Omega + \mathbf{y}_0$.

Here we have used the notations:

$$\begin{aligned}
&\mathbf{z}' = (z_1, z_2), \quad B(R) \stackrel{\text{def}}{=} \{\mathbf{z}' \in \mathbb{R}^2 \mid |\mathbf{z}'| < R\},\\
&\boldsymbol{\omega}(\mathbf{y}_0, R, M) \stackrel{\text{def}}{=} Q_{\mathbf{y}_0} C(R, 2MR) + \mathbf{y}_0,\\
&C(R, H) \stackrel{\text{def}}{=} B(R) \times]-H, H[,\\
&\Gamma \stackrel{\text{def}}{=} \{\mathbf{z} \in C(R, 2MR) \mid \mathbf{z}_3 = \Psi_{\mathbf{y}_0}(\mathbf{z}')\},\\
&\Omega \stackrel{\text{def}}{=} \{\mathbf{z} \in C(R, 2MR) \mid \mathbf{z}_3 > \Psi_{\mathbf{y}_0}(\mathbf{z}')\}.
\end{aligned}$$

From the above definitions, it follows that

$$|\Psi_{\mathbf{y}_0}(\mathbf{z}')| \le \frac{1}{2} M R^2 \qquad \text{for} \qquad |\mathbf{z}'| \le R.$$

Now, let us introduce the set

$$D_\delta \stackrel{\text{def}}{=} \{\mathbf{y} \in D \mid \operatorname{dist}\{\mathbf{y}, \partial D\} > \delta\}.$$

Lemma A.1. *Suppose that we are given a function* $\mathbf{u}$ *such that*

$$\mathbf{u} \in W_2^2(D \backslash \overline{D_\delta}) \qquad \textit{and} \qquad \mathbf{u} = \mathbf{0} \quad \textit{outside of } D, \tag{A.1}$$

a matrix $Q \in SO(3)$*, and a vector* $\overline{\mathbf{x}} \in \mathbb{R}^3$*. We let* $\Delta = |Q - I| + |\overline{\mathbf{x}}|$ *and assume that* Δ *is small enough to insure the inclusion*

$$\widehat{D} \supset D_\delta,$$

where $\widehat{D} \stackrel{\text{def}}{=} QD + \overline{\mathbf{x}}$*. Then, there exists a positive constant* $c = c(M, A)$ *such that, for all sufficiently small* Δ*, we have the estimate*

$$\int_{\partial \hat{D}} |\mathbf{u}|^2 \, ds \le c\Delta^2 \left(\int_{\partial D} |\nabla \mathbf{u}|^2 \, ds + \Delta \int_{D \backslash \overline{D}_\delta} |\nabla^2 \mathbf{u}|^2 \, d\mathbf{y} \right), \tag{A.2}$$

where $A \stackrel{\text{def}}{=} \sup_{\mathbf{y} \in \partial D} |\mathbf{y}|$.

Proof. First, we note that the domain $\widehat{D}$ is of class C^2 with the same numbers R and M.

Let x_0 be an arbitrary point in $\partial \widehat{D}$. Then, $\mathbf{y}_0 = Q^*(\mathbf{x}_0 - \overline{\mathbf{x}}) \in \partial D$ and

$$\partial \widehat{D} \cap \widehat{\boldsymbol{\omega}}(\mathbf{x}_0, R, M) = QQ_{\mathbf{y}_0}\Gamma + \mathbf{x}_0 \quad \text{and} \quad \widehat{D} \cap \widehat{\boldsymbol{\omega}}(\mathbf{x}_0, R, M) = QQ_{\mathbf{y}_0}\Omega + \mathbf{x}_0,$$

where $\widehat{\boldsymbol{\omega}}(\mathbf{x}_0, R, M) \stackrel{\text{def}}{=} Q\boldsymbol{\omega}(\mathbf{y}_0, R, M) + \overline{\mathbf{x}}$. Next, we wish to obtain the equation of the boundary ∂D in terms of the local coordinates $\mathbf{z}$ connected with the point $\mathbf{x}_0$, i.e., $\mathbf{x} = QQ_{\mathbf{y}_0}\mathbf{z} + \mathbf{x}_0$, $\mathbf{z} \in C(R, 2MR)$. First, we set

$$\psi(\mathbf{z}) = \Psi_{\mathbf{y}_0}(\mathbf{z}') - \mathbf{z}_3, \quad \mathbf{z} \in \overline{C}(R, 2MR); \quad \varphi(\mathbf{z}) = \psi(\widetilde{\mathbf{z}}(\mathbf{z})), \quad \mathbf{z} \in \overline{C}\Big(\frac{R}{2}, MR\Big),$$

where

$$\widetilde{\mathbf{z}}(\mathbf{z}) = Q^*_{\mathbf{y}_0} Q Q_{\mathbf{y}_0}\mathbf{z} + Q^*_{\mathbf{y}_0}(Q-I)\mathbf{y}_0 + Q^*_{\mathbf{y}_0}\overline{\mathbf{x}}. \tag{A.3}$$

We show that the function φ is well-defined. From (A.3) we easily obtain that

$$|\widetilde{\mathbf{z}} - \mathbf{z}| \le c_1(A)\Delta, \quad |\nabla_{\mathbf{z}}\widetilde{\mathbf{z}} - I| \le c_2\Delta. \tag{A.4}$$

Thus, according to (A.4), $\widetilde{\mathbf{z}}(\mathbf{z}) \in \overline{C}(R, 2MR)$ if $\mathbf{z} \in \overline{C}\left(\frac{R}{2}, MR\right)$ and Δ is sufficiently small. Now, assume that the points $\mathbf{z}$ and $\widetilde{\mathbf{z}}$ are connected by the relation (A.3). If $\widetilde{\mathbf{z}} \in \Gamma$, then

$$\mathbf{x} = QQ_{\mathbf{y}_0}\widetilde{\mathbf{z}} + \mathbf{x}_0 \in \partial\widehat{D}$$

and

$$\mathbf{x}' = Q^*(\mathbf{x} - \overline{\mathbf{x}}) \in \partial D, \quad \mathbf{x}' = QQ_{\mathbf{y}_0}\mathbf{z} + \mathbf{x}_0.$$

Thus, the equation of the boundary ∂D near point $\mathbf{x}_0$ has the form

$$\varphi(\mathbf{z}) = 0, \quad \mathbf{z} \in \overline{C}\Big(\frac{R}{2}, MR\Big).$$

Obviously, for sufficiently small Δ, the surface $\varphi(\mathbf{z}) = \mathbf{0}$ is still a graph of some function $\Phi \in C^2(\overline{B}(R/2))$, i.e.,

$$\varphi(\mathbf{z}) = 0 \quad \text{if and only if} \quad \mathbf{z}_3 = \Phi(\mathbf{z}'), \quad |\mathbf{z}'| < R/2.$$

Let

$$\widetilde{\Gamma} \overset{\text{def}}{=} \left\{\mathbf{z} \in C\Big(\frac{R}{2}, MR\Big) \mid \mathbf{z}_3 = \Phi(\mathbf{z}')\right\}, \ \widetilde{\Omega} \overset{\text{def}}{=} \left\{\mathbf{z} \in C\Big(\frac{R}{2}, MR\Big) \mid \mathbf{z}_3 > \Phi(\mathbf{z}')\right\}.$$

Clearly,

$$\partial D \cap \widehat{\boldsymbol{\omega}}(\mathbf{x}_0, R/2, M) = QQ_{\mathbf{y}_0}\widetilde{\Gamma} + \mathbf{x}_0, \quad D \cap \widehat{\boldsymbol{\omega}}(\mathbf{x}_0, R/2, M) = QQ_{\mathbf{y}_0}\widetilde{\Omega} + \mathbf{x}_0.$$

We also introduce the three sets

$$\Gamma_0 \overset{\text{def}}{=} \{\mathbf{z} \in \mathbb{R}^3 \mid \mathbf{z}_3 = \Psi_{\mathbf{y}_0}(\mathbf{z}'), |\mathbf{z}'| < R/2\}$$
$$\Gamma_{01} \overset{\text{def}}{=} \{\mathbf{z} \in \Gamma_0 \mid \Phi(\mathbf{z}') < \Psi_{\mathbf{y}_0}(\mathbf{z}')\}, \qquad \Gamma_{02} \overset{\text{def}}{=} \Gamma_0 \backslash \Gamma_{01}.$$

By (A.1),

$$\int_{QQ_{\mathbf{y}_0}\Gamma_{02} + \mathbf{x}_0} |\mathbf{u}|^2 \, ds = 0. \tag{A.5}$$

Let us define the fucntion

$$\mathbf{U}(\mathbf{z}) \overset{\text{def}}{=} Q^*_{\mathbf{y}_0} Q^* \mathbf{u}(x), \quad \mathbf{x} = QQ_{\mathbf{y}_0}\mathbf{z} + \mathbf{x}_0.$$

For $\mathbf{z} \in \Gamma_{01}$, we have

$$|\mathbf{U}(\mathbf{z}', \Psi_{\mathbf{y}_0}(\mathbf{z}')) - \mathbf{U}(\mathbf{z}', \Phi(\mathbf{z}'))| = |\mathbf{U}(\mathbf{z}', \Psi_{\mathbf{y}_0}(\mathbf{z}'))|$$
$$\le (\Psi_{\mathbf{y}_0}(\mathbf{z}') - \Phi(\mathbf{z}')) \left(|\nabla \mathbf{U}(\mathbf{z}', \Phi(\mathbf{z}'))| + \int_{\Phi(\mathbf{z}')}^{\Psi_{\mathbf{y}_0}(\mathbf{z}')} |\nabla^2 \mathbf{U}(\mathbf{z}', \mathbf{z}_3)| \, d\mathbf{z}_3 \right)$$

and, therefore,

$$
\begin{aligned}
&|\mathbf{U}(\mathbf{z}', \Psi_{\mathbf{y}_0}(\mathbf{z}'))|^2 \\
&\qquad \le c_3 \Delta_0^2 \left(|\nabla \mathbf{U}(\mathbf{z}', \Phi(\mathbf{z}'))|^2 + \Delta_0 \int_{\Phi(\mathbf{z}')}^{\Psi_{\mathbf{y}_0}(\mathbf{z}')} |\nabla^2 \mathbf{U}(\mathbf{z}', \mathbf{z}_3)|^2 d\mathbf{z}_3 \right),
\end{aligned}
\tag{A.6}
$$

where

$$
\Delta_0 \stackrel{\text{def}}{=} \sup\{|\Psi_{\mathbf{y}_0}(\mathbf{z}') - \Phi(\mathbf{z}')| \ | \ \mathbf{z}' \in B(R/2)\}.
$$

Let $\widetilde{\mathbf{n}}$ be the unit outer normal vector to $\widetilde{\Gamma}$, $\mathbf{n}_0$ be the unit outer normal to Γ_0. We know that

$$
|\cos(\mathbf{n}_0, l_3)| \ge \frac{1}{\sqrt{1+M^2}}, \tag{A.7}
$$

where $l_3 = (0,0,1)$. Also, again, for sufficiently small Δ, we have

$$
|\cos(\widetilde{\mathbf{n}}, l_3)| \ge \frac{1}{2\sqrt{1+M^2}}. \tag{A.8}
$$

Using the rule for the differentiation of implicit functions, the representation (A.3), and the estimate (A.4), we establish the inequality

$$
\begin{aligned}
\Delta_0 &\le \frac{R}{2} \sup_{|\mathbf{z}'|<R/2} |\nabla \Psi_{\mathbf{y}_0}(\mathbf{z}') - \nabla \Phi(\mathbf{z}')| + |\Psi_{\mathbf{y}_0}(\mathbf{0}) - \Phi(\mathbf{0})| \\
&\le \frac{1}{2} \sup_{|\mathbf{z}'|<R/2} |\nabla \Psi_{\mathbf{y}_0}(\mathbf{z}') - \nabla \Phi(\mathbf{z}')| + |\Phi(\mathbf{0})| \le c_4(M, A)\Delta.
\end{aligned}
$$

Thus, by (A.5)–(A.8) and the definition of surface integral, we arrive at the local estimate

$$
\begin{aligned}
&\int_{\partial \hat{D} \cap \hat{\boldsymbol{\omega}}(\mathbf{x}_0, R/2, M)} |\mathbf{u}|^2 \, ds \\
&\le c_5(M, A)\Delta^2 \left(\int_{\partial D \cap \hat{\boldsymbol{\omega}}(\mathbf{x}_0, R/2, M)} |\nabla \mathbf{u}|^2 \, ds + \Delta \int_{D \cap \hat{\boldsymbol{\omega}}(\mathbf{x}_0, R/2, M)} |\nabla^2 \mathbf{u}|^2 \, d\mathbf{y} \right).
\end{aligned}
\tag{A.9}
$$

Now, the estimate (A.1) follows from (A.9) in the usual way. Lemma A.1 is proved. □

Now, let us introduce the set

$$
\mathcal{R}_d \stackrel{\text{def}}{=} \left\{ D \in \mathcal{R} \mid \operatorname{dist}\{\partial D, S_0\} > \frac{d}{2} \right\}.
$$

Lemma A.2. *There is a constant $c = c(D_0, S_0, d)$ such that, for any $D \in \mathcal{R}_d$, any $\mathbf{v} \in H^1(D)$, and any $\mathbf{g} \in L_2(D_+; \mathbb{R}^3)$, satisfying the equations*

$$
\begin{aligned}
-\Delta \mathbf{v} + \nabla p &= \mathbf{g} \quad \text{in } D \backslash \overline{S}_0, \\
\operatorname{div} \mathbf{v} &= 0 \quad \text{in } D \backslash \overline{S}_0,
\end{aligned}
\tag{A.10}
$$

the following estimate is valid:

$$\begin{aligned}\|\mathbf{v}\|^2_{L_2(D\setminus\overline{S}_0)} + \|\nabla\mathbf{v}\|^2_{L_2(D\setminus\overline{S}_0)} + \|\nabla^2\mathbf{v}\|^2_{L_2(D\setminus\overline{S}_0)} \\ \leq c\left(\|\mathbf{g}\|^2_{L_2(D\setminus\overline{S}_0)} + \|\mathbf{v}\|^2_{L_2(S_0)}\right).\end{aligned} \tag{A.11}$$

Proof. First, we introduce sets

$$S_0^{d/4} \stackrel{\text{def}}{=} \left\{\mathbf{y}\in\mathbb{R}^3 \mid \operatorname{dist}\{\mathbf{y},S_0\} < \frac{d}{4}\right\}, \qquad S_d \stackrel{\text{def}}{=} S_0^{d/4}\setminus\overline{S_0},$$

and fix a cut-off function $\varphi\in C_0^\infty(\mathbb{R}^3)$, satisfying condition (4.33).

We know that

$$\mathbf{v}(\mathbf{y}) = \mathbf{u}(\mathbf{y}), \quad \mathbf{y}\in S_0,$$

where

$$\mathbf{u}(\mathbf{y}) = \boldsymbol{\omega}\wedge\mathbf{y} + \mathbf{w}, \qquad \mathbf{y}\in\mathbb{R}^3,$$

for some $\boldsymbol{\omega}\in\mathbb{R}^3$ and for some $\mathbf{w}\in\mathbb{R}^3$.

Let us set $\overline{\mathbf{u}}_0 = \Lambda\mathbf{u}$. Then $\overline{\mathbf{u}}_0 = \mathbf{u}$ in S_0 and, by (A.10), we obtain

$$\int_{D\setminus\overline{S}_0} \nabla\mathbf{v} : \nabla(\mathbf{v}-\overline{\mathbf{u}}_0)\,d\mathbf{y} = \int_{D\setminus\overline{S}_0} \mathbf{g}\cdot(\mathbf{v}-\overline{\mathbf{u}}_0)\,d\mathbf{y}. \tag{A.12}$$

Recalling the definition of the number λ_+ (see (2.17)), we see that

$$\begin{aligned}\rho^f \int_{D\setminus\overline{S}_0} |\mathbf{v}-\overline{\mathbf{u}}_0|^2\,d\mathbf{y} &\leq \int_{D_+} \rho\,|\mathbf{v}-\overline{\mathbf{u}}_0|^2\,d\mathbf{y} \\ &\leq \frac{1}{\lambda_+}\int_{D_+} |\varepsilon(\mathbf{v}-\overline{\mathbf{u}}_0)|^2\,d\mathbf{y} \\ &= \frac{1}{2\lambda_+}\int_{D_+} |\nabla(\mathbf{v}-\overline{\mathbf{u}}_0)|^2\,d\mathbf{y} = \frac{1}{2\lambda_+}\int_{D\setminus\overline{S}_0} |\nabla(\mathbf{v}-\overline{\mathbf{u}}_0)|^2\,d\mathbf{y}\end{aligned} \tag{A.13}$$

and thus we easily obtain from (4.37), (4.38), (A.12), and (A.13)

$$\|\mathbf{v}\|^2_{L_2(D\setminus\overline{S}_0)} + \|\nabla\mathbf{v}\|^2_{L_2(D\setminus\overline{S}_0)} \leq c_1(S_0,d)\left(\|\mathbf{g}\|^2_{L_2(D\setminus\overline{S}_0)} + \|\mathbf{v}\|^2_{L_2(S_0)}\right). \tag{A.14}$$

Let us set

$$\overline{p} = p - \frac{1}{|S_d|}\int_{S_d} p\,d\mathbf{y}.$$

We know that

$$\int_{D\setminus\overline{S}_0} \nabla\mathbf{v} : \nabla\widetilde{\mathbf{u}}\,d\mathbf{y} = \int_{D\setminus\overline{S}_0} \overline{p}\operatorname{div}\widetilde{\mathbf{u}}\,d\mathbf{y} + \int_{D\setminus\overline{S}_0} \mathbf{g}\cdot\widetilde{\mathbf{u}}\,d\mathbf{y} \tag{A.15}$$

for any $\widetilde{\mathbf{u}}\in \overset{o}{W}{}_2^1(D\setminus\overline{S}_0;\mathbb{R}^3)$. One can find a function $\widehat{\mathbf{u}}$ (see [7]) such that

$$\widehat{\mathbf{u}}\in \overset{o}{W}{}_2^1(S_d;\mathbb{R}^3), \qquad \operatorname{div}\widehat{\mathbf{u}} = \overline{p} \ \text{ in } S_d,$$

and

$$\| \widehat{\mathbf{u}} \|^2_{L_2(S_d)} + \| \nabla \widehat{\mathbf{u}} \|^2_{L_2(S_d)} \le c_2(S_0, d) \| \overline{p} \|^2_{L_2(S_d)}. \tag{A.16}$$

If we extend $\widehat{\mathbf{u}}$ by zero to $D_+ \backslash S_d$, then one may insert $\widetilde{\mathbf{u}} = \widehat{\mathbf{u}}$ into (A.15). Then, using (A.14) and (A.16), we obtain

$$\| \overline{p} \|^2_{L_2(S_d)} \le c_3(S_0, d) \left(\| \mathbf{g} \|^2_{L_2(D \backslash \overline{S}_0)} + \| \mathbf{v} \|^2_{L_2(S_0)} \right). \tag{A.17}$$

Let us set

$$\psi = 1 - \varphi, \quad \mathbf{v}^{(1)} = \psi \mathbf{v}, \quad p_1 = \psi \overline{p}, \quad \mathbf{v}^{(2)} = \varphi \mathbf{v}, \quad p_2 = \varphi \overline{p}.$$

Obviously,

$$\mathbf{v}^{(1)} + \mathbf{v}^{(2)} \equiv \mathbf{v} \quad \text{and} \quad p_1 + p_2 \equiv \overline{p} \qquad \text{in } D \backslash \overline{S}_0.$$

For $\mathbf{v}^{(2)}$ and p_2, we have the nonhomogeneous Stokes system

$$-\Delta \mathbf{v}^{(2)} + \nabla p_2 = -2(\nabla \mathbf{v}) \nabla \varphi - \mathbf{v} \Delta \varphi + \overline{p} \nabla \varphi + \varphi \mathbf{g} \quad \text{in } S_d,$$

$$\operatorname{div} \mathbf{v}^{(2)} = \nabla \varphi \cdot \mathbf{v} \quad \text{in } S_d,$$

$$\mathbf{v}^{(2)} = \mathbf{0} \quad \text{on } \partial S_0^{d/4}, \qquad \mathbf{v}^{(2)} = \mathbf{u} \quad \text{on } \partial S_0.$$

For the solution of this system, the following estimate is known (see [6])

$$\| \nabla^2 \mathbf{v}^{(2)} \|^2_{L_2(S_d)} \le c_4(S_0, d) \Big(\| -2(\nabla \mathbf{v}) \nabla \varphi - \mathbf{v} \Delta \varphi + \overline{p} \nabla \varphi + \varphi \mathbf{g} \|^2_{L_2(S_d)} + \| \nabla (\nabla \varphi \cdot \mathbf{v}) \|^2_{L_2(S_d)} + |\mathbf{w}|^2 + |\boldsymbol{\omega}|^2 \Big).$$

Taking into account the estimates (A.14) and (A.17), we obtain

$$\| \nabla^2 \mathbf{v}^{(2)} \|^2_{L_2(D \backslash \overline{S}_0)} \le c_5(S_0, d) \left(\| \mathbf{g} \|^2_{L_2(D \backslash \overline{S}_0)} + \| \mathbf{v} \|^2_{L_2(S_0)} \right). \tag{A.18}$$

On the other hand, the functions $\mathbf{v}^{(1)}$ and p_1 can be regarded as a solution of the Stokes system

$$-\Delta \mathbf{v}^{(1)} + \nabla p_1 = -2(\nabla \mathbf{v}) \nabla \psi - \mathbf{v} \Delta \psi + \overline{p} \nabla \psi + \psi \mathbf{g} \quad \text{in } D,$$

$$\operatorname{div} \mathbf{v}^{(1)} = \nabla \psi \cdot \mathbf{v} \quad \text{in } D,$$

$$\mathbf{v}^{(1)} = \mathbf{0} \quad \text{on } \partial D.$$

Note that $\mathbf{v}^{(1)}$, p_1, and ψ are equal to zero in S_0. Since D is obtained from D_0 only by translations and rotations, one can state that

$$\begin{aligned} \| \nabla^2 \mathbf{v}^{(1)} \|^2_{L_2(D \backslash \overline{S}_0)} &= \| \nabla^2 \mathbf{v}^{(1)} \|^2_{L_2(D)} \\ &\le c_6(D_0) \Big(\| -2(\nabla \mathbf{v}) \nabla \psi - \mathbf{v} \Delta \psi + \overline{p} \nabla \psi \\ &\qquad + \psi \mathbf{g} \|^2_{L_2(S_d)} + \| \nabla (\nabla \psi \cdot \mathbf{v}) \|^2_{L_2(S_d)} \Big) \le \text{(see (A.14), (A.17))} \\ &\le c_7(D_0, S_0, d) \left(\| \mathbf{g} \|^2_{L_2(D \backslash \overline{S}_0)} + \| \mathbf{v} \|^2_{L_2(S_0)} \right). \end{aligned} \tag{A.19}$$

Combining (A.14), (A.18), and (A.19), we complete the proof of (A.11) and Lemma A.2. □

Lemma A.3. *Suppose that we are given a function v with the following properties:*

$$\begin{aligned} &\mathbf{v} \in C\left([0,T]; L_2(D_+;\mathbb{R}^3)\right) \\ &\mathbf{v}(\cdot,t) \in \mathcal{H}(D(t)) \text{ for all } t \in [0,T], \end{aligned} \tag{A.20}$$

where

$$\begin{gathered} D(t) = \left\{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = Q^*(t)(\mathbf{x} - \overline{\mathbf{x}}(t)),\, \mathbf{x} \in D_0\right\} \in \mathcal{R},\ t \in [0,T], \\ Q \in W^1_\infty(0,T;\mathbb{M}^{3\times 3}), \quad \overline{\mathbf{x}} \in W^1_\infty(0,T;\mathbb{R}^3), \\ Q(t) \in SO(3), \quad \operatorname{dist}\{\partial D(t), S_0\} \geq \frac{d}{2},\ t \in [0,T]. \end{gathered} \tag{A.21}$$

Let $\mathbf{u}(\cdot,t) = P_t\mathbf{v}(\cdot,t)$*, where* $P_t \stackrel{\text{def}}{=} P(D(t))$*. Then*

$$\mathbf{u} \in L_\infty\left(0,T; L_2(D_+;\mathbb{R}^3)\right). \tag{A.22}$$

Proof. For given Q and $\overline{\mathbf{x}}$, we define functions $\boldsymbol{\omega}$ and $\mathbf{w}$ in $L_\infty(0,T;\mathbb{R}^3)$ by the relations

$$\dot{\overline{\mathbf{x}}}(t) = Q(t)\mathbf{w}(t), \quad Q^*(t)\dot{Q}(t)\mathbf{z} = \boldsymbol{\omega}(t) \wedge \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^3, \quad t \in [0,T].$$

Then we approximate $\boldsymbol{\omega}$ and $\mathbf{w}$ by functions

$$\boldsymbol{\omega}^N(t) = \frac{1}{\Delta}\int_{t_{i-1}}^{t_i} \boldsymbol{\omega}(\tau)\,d\tau, \quad \mathbf{w}^N(t) = \frac{1}{\Delta}\int_{t_{i-1}}^{t_i} \mathbf{w}(\tau)\,d\tau,$$

$$t \in]t_{i-1}, t_i], \quad t_i - t_{i-1} = \Delta = \frac{T}{N}, \quad i = 1,2,\dots,N. \tag{A.23}$$

Next, we define the functions $Q^N \in W^1_\infty(0,T;\mathbb{M}^{3\times 3})$ and $\overline{\mathbf{x}}^N \in W^1_\infty(0,T;\mathbb{R}^3)$, solving the following initial value problems

$$\begin{aligned} &(Q^N)^*(t)\dot{Q}^N(t)\mathbf{z} = \boldsymbol{\omega}^N(t) \wedge \mathbf{z}, \quad \mathbf{z} \in \mathbb{R}^3, \quad t \in [0,T], \\ &Q^N(t) \in SO(3), \quad t \in [0,T], \quad Q^N(0) = Q(0); \\ &\dot{\overline{\mathbf{x}}}^N(t) = Q^N(t)\mathbf{w}^N(t), \quad t \in [0,T], \quad \overline{\mathbf{x}}^N(0) = \overline{\mathbf{x}}(0). \end{aligned} \tag{A.24}$$

Now, we let

$$\begin{aligned} &D^N(t) \stackrel{\text{def}}{=} \left\{\mathbf{y} \in \mathbb{R}^3 \mid \mathbf{y} = (Q^N)^*(t)(\mathbf{x} - \overline{\mathbf{x}}^N(t)),\, \mathbf{x} \in D_0\right\}, \quad t \in [0,T], \\ &\mathbf{v}^{(k)}(\cdot) \stackrel{\text{def}}{=} \mathbf{v}(\cdot,t_k), \quad \mathbf{u}^{(k)} \stackrel{\text{def}}{=} P(D^N(t_k))\mathbf{v}^{(k)}, \quad k = 0,1,\dots,N. \end{aligned} \tag{A.25}$$

The last relation in (A.25) is equivalent to the identity

$$\int\limits_{D^N(t_k)} (\mathbf{v}^{(k)} - \mathbf{u}^{(k)}) \cdot \widetilde{\mathbf{u}}\, d\mathbf{y} = 0, \quad \widetilde{\mathbf{u}} \in H(D^N(t_k)), \quad k = 0, 1, \dots, N. \tag{A.26}$$

Then we introduce

$$\mathbf{v}^N(\cdot, t) = \mathbf{v}^{(k)}(\cdot), \quad \mathbf{u}^N(\cdot, t) = \mathbf{u}^{(k)}(\cdot) \tag{A.27}$$

for $t \in]t_{i-1}, t_i], \quad i = 1, 2, \cdots, N$.

It follows from (A.23), (A.24) and (A.27) that

$$\begin{aligned}
&\mathbf{w}^N \overset{*}{\rightharpoonup} \mathbf{w} \quad \text{in} \quad L_\infty(0, T; \mathbb{R}^3),\\
&\boldsymbol{\omega}^N \overset{*}{\rightharpoonup} \boldsymbol{\omega} \quad \text{in} \quad L_\infty(0, T; \mathbb{R}^3),\\
&\overline{\mathbf{x}}^N \overset{*}{\rightharpoonup} \overline{\mathbf{x}} \quad \text{in} \quad W^1_\infty(0, T; \mathbb{R}^3),\\
&\overline{\mathbf{x}}^N \to \overline{\mathbf{x}} \quad \text{in} \quad C([0, T]; \mathbb{R}^3),\\
&Q^N \overset{*}{\rightharpoonup} Q \quad \text{in} \quad W^1_\infty(0, T; \mathbb{M}^{3\times 3}),\\
&Q^N \to Q \quad \text{in} \quad C([0, T]; \mathbb{M}^{3\times 3}),\\
&\mathbf{v}^N \to \mathbf{v} \quad \text{in} \quad L_\infty(0, T; L_2(D_+; \mathbb{R}^3)),\\
&\mathbf{u}^N \overset{*}{\rightharpoonup} \widehat{\mathbf{u}} \quad \text{in} \quad L_\infty(0, T; L_2(D_+; \mathbb{R}^3)),\\
&\widehat{\mathbf{u}}(\cdot, t) \in H(D(t)) \quad \text{for} \quad \text{a.a.} \quad t \in [0, T].
\end{aligned} \tag{A.28}$$

Now, our goal is to prove that, in fact,

$$\widehat{\mathbf{u}} = \mathbf{u}. \tag{A.29}$$

By (A.21), a positive number δ_0 exists such that

$$D_\delta(t) \overset{\text{def}}{=} \Big\{ \mathbf{y} \in D(t) \mid \operatorname{dist}\{\mathbf{y}, \partial D(t)\} > \delta \Big\} \ni \overline{S}_0$$

for all $t \in [0, T]$ and for all $\delta \in]0, \delta_0]$. Clearly, that, for any $\delta \in]0, \delta_0]$, there is a number $N_0(\delta)$ such that, for all $t \in [0, T]$ and for all $N > N_0(\delta)$,

$$D_{\frac{\delta}{2}}(t) \subset D^N(t).$$

Next, we fix numbers $\delta \in]0, \delta_0[$, $t_0 \in]0, T[$ and consider a positive number γ and a cut-off function ψ^γ, satisfying the following conditions

$$\psi^\gamma \in C_0^\infty(0, T), \quad 0 \le \psi^\gamma \le 1, \quad [t_0 - 2\gamma, t_0 + 2\gamma] \subset]0, T[,$$
$$\psi^\gamma \equiv 1 \quad \text{in} \quad [t_0 - \gamma, t_0 + \gamma],$$
$$\psi^\gamma \equiv 0 \quad \text{out} \quad \text{of} \quad [t_0 - 2\gamma, t_0 + 2\gamma].$$

A positive number $\gamma_0(\delta, t_0)$ exists such that

$$D_\delta(t_0) \subset D_{\frac{\delta}{2}}(t)$$

for all $t \in [t_0 - 2\gamma, t_0 + 2\gamma]$ and for all $\gamma \in]0, \gamma_0]$. In turn, this implies that

$$\operatorname{spt} \mathbf{U}^\gamma(\cdot, t) \subset D_{\frac{\delta}{2}}(t)$$

for all $t \in [0, T]$ and for all $\gamma \in]0, \gamma_0]$, where

$$\mathbf{U}^\gamma(y, t) = \psi^\gamma(t)\widetilde{\mathbf{u}}(y), \qquad \widetilde{\mathbf{u}} \in C_\infty(D_\delta(t_0)).$$

We define the approximation of $\mathbf{U}^\gamma$ as

$$\mathbf{U}^{\gamma,N}(\cdot, t) = \mathbf{U}^\gamma(\cdot, t_i), \qquad t \in]t_{i-1}, t_i]$$

for $= 1, 2, \ldots, N$. We know that

$$\mathbf{U}^{\gamma,N} \to \mathbf{U}^\gamma \quad \text{in} \quad L_\infty(0, T; L_2(D_+; \mathbb{R}^3))$$

and

$$\mathbf{U}^\gamma(\cdot, t_k) \subset H(D_{\frac{\delta}{2}}(t_k)) \subset H(D^N(t_k)).$$

Thus, by (A.26), we have

$$\int\limits_{D^N(t_k)} (\mathbf{v}^{(k)}(\mathbf{y}) - \mathbf{u}^{(k)}(\mathbf{y})) \cdot \mathbf{U}^\gamma(\mathbf{y}, t_k)\, d\mathbf{y} = 0$$

and, therefore,

$$\int\limits_0^T \int\limits_{D_+} (\mathbf{v}^N - \mathbf{u}^N) \cdot \mathbf{U}^{\gamma,N}\, d\mathbf{y}\, dt = 0.$$

Taking the limit as $N \to +\infty$ in the last identity, we obtain

$$\int\limits_0^T \int\limits_{D_+} (\mathbf{v} - \widehat{\mathbf{u}}) \cdot \psi^\gamma \widetilde{\mathbf{u}}\, d\mathbf{y}\, dt = 0.$$

for all $\gamma \in]0, \gamma_0]$. Let t_0 be a Lebesque point of the function $t \mapsto \widehat{\mathbf{u}}(\cdot, t)$. Then one can see that

$$\int\limits_{D_+} (\mathbf{v}(\mathbf{y}, t_0) - \widehat{\mathbf{u}}(\mathbf{y}, t_0)) \cdot \widetilde{\mathbf{u}}(\mathbf{y})\, d\mathbf{y} = 0$$

for all functions $\widetilde{\mathbf{u}} \in C_\infty(D_\delta)$. By the arbitrariness of $\delta \in]0, \delta_0[$ and the definition of the space $H(D(t_0))$, we obtain that $\widehat{\mathbf{u}}(\cdot, t_0) = \mathbf{u}(\cdot, t_0)$. Lemma A.3 is proved. □

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