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On the Global Unique Solvability of Initial-Boundary Value Problems for the Coupled Modified Navier–Stokes and Maxwell Equations

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Abstract. The global unique solvability of the first initial-boundary value problem in a bounded, two or three-dimensional domain with fixed perfectly conducting boundaries is proved for the modified Navier–Stokes equations coupled with the Maxwell equations. The system gives a deterministic description of the dynamics for conducting, incompressible, homogeneous fluids. Improved results are proved for the periodic boundary condition case.

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1. Introduction

It is customary to describe the dynamics of homogeneous, incompressible, conducting fluids under the influence of body forces and applied currents by the system

$$\begin{cases} \mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} - \nu\Delta\mathbf{v} + \mu\mathbf{H}\times\operatorname{curl}\mathbf{H} + \nabla p = \mathbf{f}, \\ \operatorname{div}\mathbf{v} = 0, \\ \operatorname{curl}\mathbf{E} = -\mu\mathbf{H}_t, \\ \operatorname{curl}\mathbf{H} = \sigma(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}) + \mathbf{j}, \\ \operatorname{div}\mathbf{H} = 0\,; \end{cases} \tag{1.1}$$

see, e.g., [4, 13]. Here, $\mathbf{v}$ denotes the velocity field, p the pressure, $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields, respectively, $\mathbf{f}$ the known body force, $\mathbf{j}$ the known applied current, μ the magnetic permeability, σ the electric conductivity, and ν the kinematic viscosity; the density of the fluid is assumed to be equal to one. We consider the standard case for which μ, σ, and ν are all positive constants. The displacement current is proportional to $\mathbf{E}_t$ and has been assumed to be negligible.

In [11], three different initial-boundary value problems for the system (1.1) were considered. For those problems, results were derived concerning their unique solvability; these results were analogous to the results of A. Kiselev and O. Ladyzhenskaya [3] for the three-dimensional Navier–Stokes system and those of O. Ladyzhenskaya [6, 7] for the two-dimensional Navier–Stokes system. In particular, in [11], it was shown that the three problems are globally uniquely solvable in the two-dimensional case and, for the three-dimensional case, they are uniquely solvable for all $t \geq 0$ if some norms of the data are small enough and for $t \in [0, T)$, for some $T > 0$, if these norms are arbitrary.

On the other hand, for the *modified* Navier–Stokes equations suggested by O. Ladyzhenskaya, the principal initial-boundary value problems are uniquely solvable globally for any nonsingular data; see, e.g., [8, 9]. Therefore, it is of interest to clarify the situation with respect to unique, global solvability for the modified Navier–Stokes equations for conducting fluids. Thus, in this paper, we consider the modified Navier–Stokes equations coupled with the Maxwell equations and prove for them the global unique solvability of the initial-boundary value problem in an immovable vessel with perfectly conducting boundaries. In an analogous manner, the cases of nonhomogeneous boundary conditions and several other types of initial-boundary value problems for this coupled system can be studied.

The plan of the paper is as follows. In the rest of this section, we establish some notation that will be used throughout the paper and then provide a description of the problems we consider. In Section 2, several estimates are derived that are needed in Sections 3 and 4 where uniqueness and existence results, respectively, are derived.

1.1. Notation

In this section, we introduce the notation that will be used throughout the paper. Vector-valued functions will be denoted in bold-face, i.e., $\mathbf{u} = (u_1, u_2, \dots, u_\ell) \in \mathbb{R}^\ell$. Furthermore, $\mathbf{u} \cdot \mathbf{v} = \sum_{k=1}^{\ell} u_k v_k$ and $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$. Unless explicitly noted, we will use the summation convention for repeated indices, e.g., $\mathbf{u} \cdot \mathbf{v} = u_k v_k$. Points in Euclidean space $\mathbb{R}^\ell$ are denoted by $\mathbf{x} = (x_1, \dots, x_\ell)$ and spatial partial derivatives are denoted by $\phi_{,k}$ or $\partial_k \phi$, i.e., $\phi_{,k} = \partial_k \phi = \partial \phi / \partial x_k$. $\mathbb{M}^{\ell \times \ell}_{sym}$ denotes the Euclidean space of symmetric tensors $\varepsilon = (\varepsilon_{ij})$, where $\varepsilon_{ij} = \varepsilon_{ji}$, $i, j = 1, \dots, \ell$. For $\varepsilon, \kappa \in \mathbb{M}^{\ell \times \ell}_{sym}$, we define $\varepsilon : \kappa = \sum_{i,j=1}^{\ell} \varepsilon_{ij} \kappa_{ij} = \varepsilon_{ij} \kappa_{ij}$ and $|\varepsilon| = (\varepsilon : \varepsilon)^{1/2}$.

Let $\Omega \subset \mathbb{R}^\ell$ denote an open domain with boundary $\partial\Omega$; $|\Omega|$ denotes its volume. For $\mathbf{x} \in \partial\Omega$, $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes an outward unit normal to $\partial\Omega$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathbf{x})$ denotes unit vectors tangential to $\partial\Omega$.

The Lebesgue spaces are denoted by $L^m(\Omega)$ and have norms

$$\|\phi\|_{m,\Omega} = \left(\int_\Omega |\phi(\mathbf{x})|^m \, d\mathbf{x} \right)^{1/m} \qquad \text{for } m \in [1, \infty) \quad \text{and} \quad \|\phi\|_{\infty,\Omega} = \operatorname*{ess\,sup}_{\mathbf{x} \in \Omega} |\phi| \, .$$

The inner product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$, i.e., $(\phi, \psi) = \int_\Omega \phi \psi \, d\mathbf{x}$. Sobolev spaces are denoted by $W_m^k(\Omega)$; their associated norms are given by

$$\|\phi\|_{W_m^k(\Omega)} = \sum_{|i|=0}^{k} \left\| \frac{\partial^{|i|}\phi}{\partial x_1^{i_1}\partial x_2^{i_2}\cdots\partial x_\ell^{i_\ell}} \right\|_{m,\Omega},$$

where $i_1,\dots,i_\ell$ are non-negative integers and $|i| = \sum_{j=1}^{\ell} i_j$.

We will use the same notation for spaces of vector-valued functions and their associated norms. For example, $\mathbf{u} = (u_1,\dots,u_\ell) \in L^m(\Omega)$ implies that each component $u_j \in L^m(\Omega)$.

The set of all infinitely differentiable functions with compact support with respect to Ω is denoted by $\dot{C}^\infty(\Omega)$. We then introduce the set

$$\dot{\mathcal{J}}^\infty(\Omega) = \left\{ \mathbf{v} \in \dot{C}^\infty(\Omega) \,\middle|\, \operatorname{div}\mathbf{v} = \sum_{i=1}^{\ell} v_{i,i} = 0 \right\}$$

and the subspace of $L^2(\Omega)$

$$\mathcal{J}(\Omega) = \left\{ \mathbf{v} \in L^2(\Omega) \;\middle|\; \operatorname{div}\mathbf{v} = 0 \right\},$$

where $\operatorname{div}\mathbf{v} = 0$ is understood in the sense of distributions, i.e.,

$$\int_\Omega \mathbf{v}\cdot\nabla\phi\, d\mathbf{x} = 0 \quad \forall\, \phi \in \dot{C}^\infty(\Omega)\,.$$

Then, $\overset{\circ}{\mathcal{J}}(\Omega)$ is defined to be the closure of $\dot{\mathcal{J}}^\infty(\Omega)$ in the norm of $L^2(\Omega)$. Thus,

$$\overset{\circ}{\mathcal{J}}(\Omega) \subset \mathcal{J}(\Omega) \subset L^2(\Omega)\,.$$

We also define

$$\mathcal{J}_m^k(\Omega) = W_m^k(\Omega) \cap \mathcal{J}(\Omega)$$

and

$$\overset{\circ}{\mathcal{J}}{}_m^1(\Omega), \text{ the closure of } \dot{\mathcal{J}}^\infty(\Omega) \text{ in the norm of } W_m^1(\Omega).$$

The following subspaces of the spaces $\mathcal{J}_2^2(\Omega)$ and $\mathcal{J}_2^1(\Omega)$ will also be needed:

$$\widetilde{\mathcal{J}}_2^2(\Omega) = \left\{ \mathbf{v} \in \mathcal{J}_2^2(\Omega) \mid v_n|_{\partial\Omega} = (\mathbf{v}\cdot\mathbf{n})|_{\partial\Omega} = 0,\ (\operatorname{curl}\mathbf{v})_\tau|_{\partial\Omega} = (\operatorname{curl}\mathbf{v}\cdot\boldsymbol{\tau})|_{\partial\Omega} = \mathbf{0} \right\}$$

and

$$\widetilde{\mathcal{J}}_2^1(\Omega), \text{ the closure of } \widetilde{\mathcal{J}}_2^2(\Omega) \text{ in the norm of } W_2^1(\Omega).$$

For periodic functions with fundamental domain $\Omega = \{\mathbf{x} \mid 0 < x_i < 2\pi,\ i = 1,2,3\}$, we consider the spaces

$$\widehat{L}^2(\Omega) = \left\{ \phi \in L^2(\Omega) \;\middle|\; \overline{\phi} = \frac{1}{|\Omega|}\int_\Omega \phi(\mathbf{x})\, d\mathbf{x} = 0 \right\}$$

and

$$\widehat{W}_m^k(\Omega) = W_m^k(\Omega) \cap \widehat{L}^2(\Omega).$$

Note that for $\phi \in \widehat{W}^k_m(\Omega)$, all derivatives $\partial^i_x \phi$, $|i| < k$ are periodic functions. We also consider the spaces

$$\widehat{\mathcal{J}}_2(\Omega) = \mathcal{J}(\Omega) \cap \widehat{L}^2(\Omega) \quad \text{and} \quad \widehat{\mathcal{J}}^k_m(\Omega) = \mathcal{J}^k_m(\Omega) \cap \widehat{L}^2(\Omega).$$

Finally, C, C_1, etc. will denote constants whose value changes with context.

1.2. Formulation of problems

We consider the system

$$\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \operatorname{div} \Xi(\mathbf{v}) + \mu \mathbf{H} \times \operatorname{curl} \mathbf{H} + \nabla p = \mathbf{f} \tag{1.2}$$

$$\operatorname{div} \mathbf{v} = 0 \tag{1.3}$$

$$\mu \mathbf{H}_t + \frac{1}{\sigma} \operatorname{curl} \operatorname{curl} \mathbf{H} + \mu(\mathbf{v} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{v}) = \frac{1}{\sigma} \operatorname{curl} \mathbf{j} \tag{1.4}$$

$$\operatorname{div} \mathbf{H} = 0 \tag{1.5}$$

with $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{H} = (H_1, H_2, H_3)$, and

$$\Xi(\mathbf{v}) = \left. \frac{\partial \mathcal{D}(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon = \varepsilon(\mathbf{v})},$$

where

$$\varepsilon(\mathbf{v}) = (\varepsilon_{ij}(\mathbf{v})), \quad \varepsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right), \quad v_{i,j} \equiv \frac{\partial v_i}{\partial x_j}$$

in a bounded domain $\Omega \subset \mathbb{R}^3$ supplemented by the initial data

$$\mathbf{v}|_{t=0} = \mathbf{v}^0 \quad \text{and} \quad \mathbf{H}|_{t=0} = \mathbf{H}^0 \tag{1.6}$$

and one of the following sets of boundary conditions:

$$\mathbf{v}|_{S_T} = \mathbf{0}, \quad H_n|_{S_T} = 0, \quad \text{and} \quad (\operatorname{curl} \mathbf{H})_\tau|_{S_T} = \mathbf{0}, \tag{1.7}$$

where $S_T = \partial\Omega \times [0, T]$, or

$$\mathbf{v},\ \mathbf{H},\ \text{and } p \text{ are periodic with respect to } x_k,\ k = 1, 2, 3. \tag{1.8}$$

Here $H_n = \mathbf{H} \cdot \mathbf{n}$ is the projection of $\mathbf{H}$ onto the outer normal $\mathbf{n}$ of $\partial\Omega$ and $\mathbf{u}_\tau$ is the projection of the vector $\mathbf{u}$ onto vectors tangential to $\partial\Omega$. In (1.2), (1.4), $\mu > 0$ denotes the constant magnetic permeability and $\sigma > 0$ the constant electric conductivity. In this paper, the global unique solvability of the problems (1.2)–(1.7) and (1.2)–(1.6) plus (1.8) is proved, in the three-dimensional case, under the assumption that $\delta \geq 1/4$. For two-dimensional domains Ω, the parameter δ can be any nonnegative number. Note that (1.4) is easily derived from the reduced Maxwell equations (the third and fourth equations in (1.1)) along with the identity $\operatorname{curl}(\mathbf{v} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{H}$.

We consider (1.2)–(1.5) in $Q_T = \Omega \times (0, T)$ with a fixed $T \in (0, \infty)$. The potential $\mathcal{D}(\cdot)$ is a smooth function having the properties:

a) $\mathcal{D} : \mathbb{M}^{3\times 3}_{sym} \to \mathbb{R}^1_+ = [0,\infty)$ and $\mathcal{D} \in C^2(\mathbb{M}^{3\times 3}_{sym})$;
b) $\nu_1 m(\varepsilon) \le \mathcal{D}(\varepsilon) \le \nu_2 m(\varepsilon), \quad$ where $\quad m(\varepsilon) = |\varepsilon|^2 + |\varepsilon|^{2+2\delta}$;
c) $\nu_3 m(\varepsilon) \le \dfrac{\partial \mathcal{D}(\varepsilon)}{\partial \varepsilon_{ij}} \varepsilon_{ij} \le \nu_4 m(\varepsilon)$;
d) $\nu_5(1+|\varepsilon|^{2\delta})|\kappa|^2 \le \dfrac{\partial^2 \mathcal{D}(\varepsilon)}{\partial \varepsilon_{ij}\varepsilon_{k\ell}} \kappa_{ij}\kappa_{k\ell} \le \nu_6(1+|\varepsilon|^{2\delta})|\kappa|^2$

with $\nu_k > 0$, $k = 1, 2, \ldots, 6$, constants and κ an arbitrary element of $\mathbb{M}^{3\times 3}_{sym}$. The functions $\mathbf{f}$, $\operatorname{curl} \mathbf{j} \in L^2(Q_T)$ (with $\operatorname{div} \mathbf{j} = 0$) are known and the functions $\mathbf{v}, p, \mathbf{H}$ have to be determined.

For the Navier–Stokes equations, we have $\Xi(\varepsilon) = \nu|\varepsilon|^2$ and $\operatorname{div} \Xi(\mathbf{v}) = \nu \Delta \mathbf{v}$. For this case, in [11] several initial-boundary value problems were considered, including (1.2)–(1.7); for these problems, solvability results analogous to those of [3, 6] were proved.

2. Estimates

In this section, we lay the groundwork for the uniqueness and existence results of Sections 3 and 4. In most instances, the estimates we derive hold for both the boundary value problem (1.2)–(1.7) and the periodic boundary condition problem (1.2)–(1.6) and (1.8). For the latter case, we can also prove some additional estimates; see Section 2.4.

2.1. An energy relation and corresponding estimates

Most of the estimates we prove hold not only for exact solutions of (1.2)–(1.8), but also for some special Galerkin approximations. Instead of the equations (1.2) and (1.4), we will use the integral identities

$$(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}, \boldsymbol{\eta}) + \left(\left.\frac{\partial \mathcal{D}(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})}, \varepsilon(\boldsymbol{\eta}) \right) - (\mu \mathbf{H}\cdot\nabla\mathbf{H}, \boldsymbol{\eta}) = (\mathbf{f}, \boldsymbol{\eta}) \tag{2.1}$$

for any $\boldsymbol{\eta} \in \overset{\circ}{\mathcal{J}}{}^1_{2+2\delta}(\Omega)$ and

$$(\mu \mathbf{H}_t, \boldsymbol{\zeta}) - \frac{1}{\sigma}(\Delta \mathbf{H}, \boldsymbol{\zeta}) + \mu(\mathbf{v}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{v}, \boldsymbol{\zeta}) = \frac{1}{\sigma}(\operatorname{curl}\mathbf{j}, \boldsymbol{\zeta}) \tag{2.2}$$

for any $\boldsymbol{\zeta} \in L^2(\Omega)$. It is easy to see that (2.1) follows from the inner product in $L^2(\Omega)$ of (1.2) and $\boldsymbol{\eta} \in \overset{\circ}{\mathcal{J}}{}^1_{2+2\delta}(\Omega)$ and that (2.2) follows from the inner product of (1.4) and $\boldsymbol{\zeta} \in L^2(\Omega)$, if we take into account (1.3), (1.5), either (1.7) or (1.8), and the identity $\mathbf{H} \times \operatorname{curl} \mathbf{H} = -\mathbf{H}\cdot\nabla\mathbf{H} + \frac{1}{2}\nabla|\mathbf{H}|^2$.

From the sum of (2.1) with $\boldsymbol{\eta} = \mathbf{v}$ and (2.2) with $\boldsymbol{\zeta} = \mathbf{H}$, we obtain the energy relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{2,\Omega}+\left(\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon},\varepsilon\right)\Big|_{\varepsilon=\varepsilon(\mathbf{v})}+\frac{\mu}{2}\frac{d}{dt}\|\mathbf{H}\|^2_{2,\Omega}+\frac{1}{\sigma}\|\operatorname{curl}\mathbf{H}\|^2_{2,\Omega}&\\ =(\mathbf{f},\mathbf{v})+\frac{1}{\sigma}(\operatorname{curl}\mathbf{j},\mathbf{H}).&\end{aligned}\tag{2.3}$$

To obtain this identity, we have used the fact that $\mathbf{v}$ and $\mathbf{H}$ are divergence free fields satisfying the boundary conditions (1.7) or (1.8). In particular, we have used the relation

$$(\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w})=-(\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v})$$

that holds for arbitrary elements $\mathbf{u},\mathbf{v},\mathbf{w}$ of $W^1_2(\Omega)$ satisfying the conditions

$$\operatorname{div}\mathbf{u}=0\quad\text{and}\quad\int_{\partial\Omega}u_kv_iw_in_k\,dS=0$$

and the equality

$$(\operatorname{curl}\mathbf{u},\mathbf{v})=(\mathbf{u},\operatorname{curl}\mathbf{v})$$

that holds if $\mathbf{u}_\tau|_{\partial\Omega}=\mathbf{0}$. The inequality

$$\nu_7\|\mathbf{H}_x\|^2_{2,\Omega}-\nu_8\|\mathbf{H}\|^2_{2,\Omega}\le\|\operatorname{curl}\mathbf{H}\|^2_{2,\Omega}\tag{2.4}$$

with $\nu_7>0$ holds for any solenoidal $\mathbf{H}$ satisfying the boundary condition $H_n|_{\partial\Omega}=0$; see, e.g., [1]. We can then use this inequality and the assumption (c) on $\mathcal{D}$ to conclude from (2.3) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{2,\Omega}+\nu_3\left(\|\varepsilon(\mathbf{v})\|^2_{2,\Omega}+\|\varepsilon(\mathbf{v})\|^{2+2\delta}_{2+2\delta,\Omega}\right)+\frac{\mu}{2}\frac{d}{dt}\|\mathbf{H}\|^2_{2,\Omega}+\frac{\nu_7}{\sigma}\|\mathbf{H}_x\|^2_{2,\Omega}&\\ \le\frac{\nu_8}{\sigma}\|\mathbf{H}\|^2_{2,\Omega}+\|\mathbf{f}\|_{2,\Omega}\|\mathbf{v}\|_{2,\Omega}+\frac{1}{\sigma}\|\operatorname{curl}\mathbf{j}\|_{2,\Omega}\|\mathbf{H}\|_{2,\Omega}.&\end{aligned}\tag{2.5}$$

We could have used weaker norms on $\mathbf{f}$ and $\operatorname{curl}\mathbf{j}$ in (2.5); however, in later estimates, we shall need the norms $\|\mathbf{f}\|_{2,Q_T}$ and $\|\operatorname{curl}\mathbf{j}\|_{2,Q_T}$.

According to the Korn inequalities,

$$C(q)\|\mathbf{v}_x\|_{q,\Omega}\le\|\varepsilon(\mathbf{v})\|_{q,\Omega}\quad\forall\,q\in(1,\infty)$$

for some $C(q)>0$ and for any $\mathbf{v}\in\overset{\circ}{\mathcal{J}}{}^1_q(\Omega)$. Thus, from (2.5), we obtain the inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2_{2,\Omega}+\nu_3\left(C_1\|\mathbf{v}_x\|^2_{2,\Omega}+C_2\|\mathbf{v}_x\|^{2+2\delta}_{2+2\delta,\Omega}\right)+\frac{\mu}{2}\frac{d}{dt}\|\mathbf{H}\|^2_{2,\Omega}+\frac{\nu_7}{\sigma}\|\mathbf{H}_x\|^2_{2,\Omega}&\\ \le\frac{\nu_8}{\sigma}\|\mathbf{H}\|^2_{2,\Omega}+\|\mathbf{f}\|_{2,\Omega}\|\mathbf{v}\|_{2,\Omega}+\frac{1}{\sigma}\|\operatorname{curl}\mathbf{j}\|_{2,\Omega}\|\mathbf{H}\|_{2,\Omega}&\end{aligned}$$

from which the estimate

$$\begin{aligned}\max_{t\in[0,T]}\|\mathbf{v}(t)\|^2_{2,\Omega}+\|\mathbf{v}_x\|^2_{2,Q_T}+\|\mathbf{v}_x\|^{2+2\delta}_{2+2\delta,Q_T}+\max_{t\in[0,T]}\|\mathbf{H}(t)\|^2_{2,\Omega}+\|\mathbf{H}_x\|^2_{2,Q_T}&\\ \le\Phi(T,\|\mathbf{v}^0\|_{2,\Omega},\|\mathbf{H}^0\|_{2,\Omega},\|\mathbf{f}\|_{2,Q_T},\|\operatorname{curl}\mathbf{j}\|_{2,Q_T})&\end{aligned}\tag{2.6}$$

is derived by a well-known procedure. Here, Φ is a continuous function of the indicated arguments that can be easily calculated.

2.2. Some additional estimates for H

First, we estimate the norm $\|\cdot\|_{2,\Omega}$ of $\mathbf{H}_x$, using the identity (2.2) with $\boldsymbol{\zeta} = -\Delta\mathbf{H}$ and the estimate (2.6). Choosing $\boldsymbol{\zeta} = -\Delta\mathbf{H}$, (2.2) can be transformed in the following way:

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}\|\operatorname{curl}\mathbf{H}\|_{2,\Omega}^2 + \frac{1}{\sigma}\|\Delta\mathbf{H}\|_{2,\Omega}^2 &= \mu(\mathbf{v}\cdot\nabla\mathbf{H}-\mathbf{H}\cdot\nabla\mathbf{v},\Delta\mathbf{H}) - \frac{1}{\sigma}(\operatorname{curl}\mathbf{j},\Delta\mathbf{H})\\ &\le -\mu(v_{k,j}H_{i,k},H_{i,j}) + \frac{1}{2\sigma}\|\Delta\mathbf{H}\|_{2,\Omega}^2\\ &\quad + C(\|\mathbf{H}\cdot\nabla\mathbf{v}\|_{2,\Omega}^2 + \|\operatorname{curl}\mathbf{j}\|_{2,\Omega}^2).\end{aligned} \tag{2.7}$$

Now we will majorize the terms in the right-hand side of (2.7) using the Hölder inequality and the imbedding inequalities

$$\|\mathbf{u}\|_{m,\Omega} \le C(m,r)\|\mathbf{u}_x\|_{r,\Omega} + C_1(m,r)\|\mathbf{u}\|_{2,\Omega} \tag{2.8}$$

with

$$\begin{cases} m \le \dfrac{3r}{3-r} & \text{for } r\in[1,3)\\ \text{for any } m<\infty & \text{for } r=3\\ m=\infty & \text{for } r>3.\end{cases}$$

Here, $C_1(m,r)=0$ if $\mathbf{u}|_{\partial\Omega}=\mathbf{0}$ or $\int_\Omega \mathbf{u}\,d\mathbf{x}=\mathbf{0}$. We will also apply the multiplicative inequalities

$$\|\mathbf{u}\|_{q,\Omega} \le C(q)\|\mathbf{u}_x\|_{2,\Omega}^{\alpha}\|\mathbf{u}\|_{2,\Omega}^{1-\alpha} + C_1(q)\|\mathbf{u}\|_{2,\Omega}^2 \tag{2.9}$$

with

$$\alpha = 3\left(\frac{1}{2}-\frac{1}{q}\right)\in[0,1], \quad m\in[2,6].$$

If $\mathbf{u}|_{\partial\Omega}=\mathbf{0}$ or $\int_\Omega \mathbf{u}\,d\mathbf{x}=\mathbf{0}$, then $C_1(q)=0$. For the estimation of $\mathcal{Y}_1 = \mu|(v_{k,j}H_{i,k},H_{i,j})|$, we first apply the Hölder inequality with powers $q=2+2\delta$ and $q'=(2+2\delta)/(1+2\delta)$; after that, we apply the multiplicative inequality and in the end a Young's inequality. In more detail,

$$\begin{aligned}\mathcal{Y}_1 &\le C\int_\Omega |\mathbf{v}_x|\,|\mathbf{H}_x|^2\,d\mathbf{x} \le C\|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\mathbf{H}_x\|^2_{\frac{4(1+\delta)}{1+2\delta},\Omega}\\ &\le C_1\|\mathbf{v}_x\|_{2+2\delta,\Omega}\left(\|\mathbf{H}_x\|_{2,\Omega}^{2(1-\alpha)}\|\mathbf{H}_{xx}\|_{2,\Omega}^{2\alpha} + \|\mathbf{H}_x\|_{2,\Omega}^2\right)\\ &\le \epsilon\|\mathbf{H}_{xx}\|_{2,\Omega}^2 + C(\epsilon)\|\mathbf{v}_x\|_{2+2\delta,\Omega}^{\frac{1}{1-\alpha}}\|\mathbf{H}_x\|_{2,\Omega}^2 + C_1\|\mathbf{v}_x\|_{2+2\delta,\Omega}\|\mathbf{H}_x\|_{2,\Omega}^2\end{aligned} \tag{2.10}$$

for any $\epsilon\in(0,1]$ and $\alpha=\frac{3}{4(1+\delta)}$.

For the estimation of $\mathcal{Y}_2 = \|\mathbf{H}\cdot\nabla\mathbf{v}\|_{2,\Omega}^2$, we apply the Hölder inequality with exponents $q=1+\delta$ and $q'=(1+\delta)/\delta$, after that the inequality (2.8) with $m=2(1+\delta)/\delta$ and $r=6(1+\delta)/(2+5\delta)$, the multiplicative inequality (2.9) with

$q = 6(1+\delta)/(2+5\delta)$, and, in the end, Young's inequality. In more detail,

$$\begin{aligned}
\mathcal{Y}_2 &\leq \|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\|\mathbf{H}\|^2_{\frac{2+2\delta}{\delta},\Omega} \leq C\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\left(\|\mathbf{H}_x\|^2_{\frac{6+6\delta}{2+5\delta},\Omega} + \|\mathbf{H}\|^2_{2,\Omega}\right) \\
&\leq C_1\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\|\mathbf{H}_x\|^{2(1-\alpha_1)}_{2,\Omega}\|\mathbf{H}_{xx}\|^{2\alpha_1}_{2,\Omega} \\
&\qquad + C_1\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\left(\|\mathbf{H}_x\|^2_{2,\Omega} + \|\mathbf{H}\|^2_{2,\Omega}\right) \qquad (2.11) \\
&\leq \epsilon_1\|\mathbf{H}_{xx}\|^2_{2,\Omega} + C(\epsilon_1)\|\mathbf{v}_x\|^{\frac{2}{1-\alpha_1}}_{2+2\delta,\Omega}\|\mathbf{H}_x\|^2_{2,\Omega} \\
&\qquad + C_2\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\left(\|\mathbf{H}_x\|^2_{2,\Omega} + \|\mathbf{H}\|^2_{2,\Omega}\right)
\end{aligned}$$

for any $\epsilon_1 \in (0,1]$ and $\alpha_1 = \frac{1-2\delta}{2+2\delta}$. The inequalities (2.10) and (2.11) will be useful to us if

$$\frac{1}{1-\alpha} \leq 2+2\delta \quad \text{and} \quad \frac{2}{1-\alpha_1} \leq 2+2\delta.$$

These inequalities hold if we require that

$$\delta \geq \frac{1}{4}. \qquad (2.12)$$

In the sequel, we assume that (2.12) holds.

We will also need the inequality

$$\|\mathbf{H}_{xx}\|_{2,\Omega} \leq C_1(\Omega)\|\Delta\mathbf{H}\|_{2,\Omega} + C_2(\Omega)\|\mathbf{H}\|_{2,\Omega} \qquad (2.13)$$

that holds for any solenoidal vector field $\mathbf{H}$ satisfying the boundary conditions $H_n|_{\partial\Omega} = 0$ and $(\operatorname{curl}\mathbf{H})_\tau|_{\partial\Omega} = \mathbf{0}$ (or satisfying periodic boundary conditions) which can be proved by the method given by O. Ladyzhenskaya in 1950 for the same inequality for scalar functions satisfying one of the classical boundary conditions; see, e.g., [5, 10].

Using the inequalities (2.10), (2.11), and (2.13) with sufficiently small ϵ and ϵ_1, we conclude from (2.7) that

$$\begin{aligned}
&\frac{d}{dt}\|\operatorname{curl}\mathbf{H}\|^2_{2,\Omega} + \|\mathbf{H}_{xx}\|^2_{2,\Omega} \\
&\quad \leq C\left(\|\mathbf{v}_x\|^{\frac{1}{1-\alpha}}_{2+2\delta,\Omega} + \|\mathbf{v}_x\|_{2+2\delta,\Omega} + \|\mathbf{v}_x\|^{\frac{2}{1-\alpha_1}}_{2+2\delta,\Omega} + \|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\right)\|\mathbf{H}_x\|^2_{2,\Omega} \qquad (2.14) \\
&\qquad + C_1\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\|\mathbf{H}\|^2_{2,\Omega} + C_2\|\operatorname{curl}\mathbf{j}\|^2_{2,\Omega}.
\end{aligned}$$

The result of integrating (2.14) over t and (2.4) yield the relation

$$\begin{aligned}
&\nu_7\|\mathbf{H}_x(t)\|^2_{2,\Omega} + \|\mathbf{H}_{xx}\|^2_{2,Q_t} \leq \\
&\quad \|\operatorname{curl}\mathbf{H}^0\|^2_{2,\Omega} + C\left(\int_0^t F(\tau)\|\mathbf{H}_x(\tau)\|^2_{2,\Omega}\,d\tau + \int_0^t F_1(\tau)\,d\tau + \|\operatorname{curl}\mathbf{j}\|^2_{2,Q_T}\right),
\end{aligned}$$

where $Q_t = \Omega \times (0,t)$,

$$F(t) = \|\mathbf{v}_x(t)\|^{\frac{1}{1-\alpha}}_{2+2\delta,\Omega} + \|\mathbf{v}_x(t)\|^2_{2+2\delta,\Omega} + \|\mathbf{v}_x(t)\|^{\frac{2}{1-\alpha_1}}_{2+2\delta,\Omega} + \|\mathbf{v}_x(t)\|^2_{2+2\delta,\Omega},$$

and

$$F_1(t) = \big(1 + \|\mathbf{v}_x(t)\|^2_{2+2\delta}\big)\, \|\mathbf{H}(t)\|^2_{2,\Omega}.$$

For $\delta \geq 1/4$, the estimate (2.6) guarantees majorants for $\int_0^T F(t)dt$ and $\int_0^T F_1(t)dt$. Now, we apply an integral variant of Gronwall's inequality to obtain the second estimate

$$\max_{t\in[0,T]} \|\mathbf{H}_x(t)\|_{2,\Omega} + \|\mathbf{H}_{xx}\|_{2,Q_T} \leq \Phi_1(T, \|\mathbf{H}_x^0\|_{2,\Omega}) \tag{2.15}$$

for another continuous function Φ_1 that also depends on the arguments of Φ from (2.6).

It is easy to see that the identity (2.2) with $\boldsymbol{\zeta} = \mathbf{H}_t$ and (2.15) guarantee the estimate

$$\|\mathbf{H}_t\|_{2,Q_T} \leq \Phi_2(T, \|\mathbf{H}_x^0\|_{2,\Omega})$$

analogous to (2.15).

2.3. An estimate for $\mathbf{v}_t$

In (2.1), let $\boldsymbol{\eta} = \mathbf{v}_t$ (this is allowable since $\operatorname{div} \mathbf{v}_t = 0$ and $\mathbf{v}_t|_{S_T} = \mathbf{0}$) and then transform the result into the relation

$$\|\mathbf{v}_t\|^2_{2,\Omega} + \frac{d}{dt}\int_\Omega \mathcal{D}\big(\varepsilon(\mathbf{v}(t))\big)\, d\mathbf{x} = -(\mathbf{v}\cdot\nabla\mathbf{v} - \mu\mathbf{H}\cdot\nabla\mathbf{H}, \mathbf{v}_t) + (\mathbf{f}, \mathbf{v}_t).$$

From this we obtain the inequality

$$\frac{1}{2}\|\mathbf{v}_t\|^2_{2,\Omega} + \frac{d}{dt}\int_\Omega \mathcal{D}\big(\varepsilon(\mathbf{v}(t))\big)\, d\mathbf{x} \leq C\Big(\|\, |\mathbf{v}|\, |\mathbf{v}_x|\, \|^2_{2,\Omega} + \|\, |\mathbf{H}|\, |\mathbf{H}_x|\, \|^2_{2,\Omega}\Big) + 2\|\mathbf{f}\|^2_{2,\Omega}.$$

Now, we integrate over t and use the hypotheses (b) on $\mathcal{D}$ to obtain

$$\begin{aligned} &\|\mathbf{v}_t\|^2_{2,Q_T} + 2\nu_1\left(\|\mathbf{v}_x\|^2_{2,\Omega} + \|\mathbf{v}_x\|^{2+2\delta}_{2+2\delta,\Omega}\right) \\ &\qquad \leq 2\nu_2\left(\|\mathbf{v}_x^0\|^2_{2,\Omega} + \|\mathbf{v}_x^0\|^{2+2\delta}_{2+2\delta,\Omega}\right) + \mathcal{Y}_3 + \mathcal{Y}_4 + 4\|\mathbf{f}\|^2_{2,Q_T}, \end{aligned} \tag{2.16}$$

where

$$\mathcal{Y}_3 = 2C\|\, |\mathbf{v}|\, |\mathbf{v}_x|\, \|^2_{2,Q_T} \quad \text{and} \quad \mathcal{Y}_4 = 2C\|\, |\mathbf{H}|\, |\mathbf{H}_x|\, \|^2_{2,Q_T}. \tag{2.17}$$

For the estimation of $\mathcal{Y}_3$, we use the Hölder inequality with powers $q = 1+\delta$ and $q' = (1+\delta)/\delta$ and after that the inequality (2.8) with $m = (2+2\delta)/\delta$ and $r = 2+2\delta$. Thus, we obtain

$$\mathcal{Y}_3 \leq 2C\int_0^t \|\mathbf{v}\|^2_{\frac{2+2\delta}{\delta},\Omega}\|\mathbf{v}_x\|^2_{2+2\delta,\Omega}\, d\tau \leq C_1\int_0^t \|\mathbf{v}_x\|^4_{2+2\delta,\Omega}\, d\tau \quad \text{if } \delta \geq \tfrac{1}{5}. \tag{2.18}$$

For the estimation of $\mathcal{Y}_4$, we use the Hölder inequality with powers $q = 3$ and $q' = 3/2$, the inequality (2.8) with $m = 6$ and $r = 2$, and after that the inequality

(2.9) with $q = 3$. In greater detail,

$$\begin{aligned}
\mathcal{Y}_4 &\le 2C \int_0^t \|\mathbf{H}\|_{6,\Omega}^2 \|\mathbf{H}_x\|_{3,\Omega}^2 \, d\tau \\
&\le C_1 \int_0^t \big(\|\mathbf{H}_x\|_{2,\Omega}^2 + \|\mathbf{H}\|_{2,\Omega}^2\big)\big(\|\mathbf{H}_x\|_{2,\Omega}\|\mathbf{H}_{xx}\|_{2,\Omega} + \|\mathbf{H}_x\|_{2,\Omega}^2\big)\, d\tau \\
&\le C_1 \max_{\tau\in[0,t]} \Big(\|\mathbf{H}_x(\tau)\|_{2,\Omega}^3 + \|\mathbf{H}(\tau)\|_{2,\Omega}^2 \|\mathbf{H}_x(\tau)\|_{2,\Omega}\Big) \int_0^t \|\mathbf{H}_{xx}\| \, d\tau \\
&\quad + C_1 \int_0^t \big(\|\mathbf{H}_x(\tau)\|_{2,\Omega}^4 + \|\mathbf{H}(\tau)\|_{2,\Omega}^2 \|\mathbf{H}_x(\tau)\|_{2,\Omega}^2\big)\, d\tau \equiv F_3(t).
\end{aligned} \tag{2.19}$$

We know a majorant for $F_3(t)$ (see (2.15)) and so we can calculate a majorant for

$$\|\mathbf{v}_x(t)\|_{2,\Omega}^2 + \|\mathbf{v}_x(t)\|_{2+2\delta,\Omega}^{2+2\delta} \equiv z(t)$$

using (2.16)–(2.19) in the following way:

$$\begin{aligned}
z(t) &\le C_2\big(\|\mathbf{v}_x^0\|_{2,\Omega}^2 + \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}^{2+2\delta} + \mathcal{Y}_3 + \mathcal{Y}_4 + \|\mathbf{f}\|_{2,Q_T}^2\big) \\
&\le C_2\Bigg(\|\mathbf{v}_x^0\|_{2,\Omega}^2 + \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}^{2+2\delta} \\
&\qquad + C_1 \int_0^t \|\mathbf{v}_x(\tau)\|_{2+2+\delta,\Omega}^{2-2\delta} z(\tau)\, d\tau + F_3(t) + \|\mathbf{f}\|_{2,Q_T}^2\Bigg) \\
&\equiv \int_0^t \psi_1(\tau) z(\tau)\, d\tau + \psi_2(\tau),
\end{aligned} \tag{2.20}$$

where

$$\psi_1(t) = C_1 C_2 \|\mathbf{v}_x(t)\|_{2+2+\delta,\Omega}^{2-2\delta}$$

and

$$\psi_2(t) = C_2\big(\|\mathbf{v}_x^0\|_{2,\Omega}^2 + \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}^{2+2\delta} + F_3(t) + \|\mathbf{f}\|_{2,Q_T}^2\big).$$

Since we have majorants for $\int_0^T \psi_k(t)dt$, $k = 1, 2$, a majorant for $z(t)$ on $t \in [0, T]$ can be determined from (2.20). This majorant and (2.16) then yield that

$$\|\mathbf{v}_t\|_{2,Q_T} + \max_{t\in[0,T]} \big(\|\mathbf{v}_x(t)\|_{2+2\delta,\Omega} + \|\mathbf{v}_x(t)\|_{2,\Omega}\big) \le \Phi_3(T, \|\mathbf{v}_x^0\|_{2+2\delta,\Omega}, \|\mathbf{v}_x^0\|_{2,\Omega})$$

with a continuous function Φ_3 which depends also on the information about the known functions $\mathbf{v}^0$, $\mathbf{H}^0$, $\mathbf{f}$, and $\operatorname{curl}\mathbf{j}$ used before.

2.4. Estimates for $\mathbf{v}_{xx}$ and ∇p for the periodic boundary conditions case

All previous estimates were proved for the boundary conditions (1.7) in any bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. However, we cannot prove estimates for $\mathbf{v}_{xx}$ and p for these boundary conditions. In fact, they have not been derived even for the stationary equations

$$\begin{aligned} -\operatorname{div}\left(\left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})}\right)+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p &= \boldsymbol{\psi} \\ \operatorname{div}\mathbf{v} &= 0. \end{aligned} \tag{2.21}$$

For such systems (as it is known since 1966, where such type of equations were proposed by O. Ladyzhenskaya), it is possible to derive estimates for $\mathbf{v}_{xx}$ for periodic boundary conditions using the relation

$$\left(\left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})},\varepsilon(\boldsymbol{\eta})\right)+(\mathbf{v}\cdot\nabla\mathbf{v},\boldsymbol{\eta})=(\boldsymbol{\psi},\boldsymbol{\eta}) \tag{2.22}$$

which holds for arbitrary periodic smooth solenoidal vector fields $\boldsymbol{\eta}$. It is obvious that solutions to (2.21) satisfy (2.22). Setting $\boldsymbol{\eta}=-\Delta\mathbf{v}$ and performing integration by parts results in

$$\left.\left(\partial_{x_k}\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon_{ij}},\partial_{x_k}\varepsilon_{ij}\right)\right|_{\varepsilon=\varepsilon(\mathbf{v})}-(\mathbf{v}\cdot\nabla\mathbf{v},\Delta\mathbf{v})=-(\boldsymbol{\psi},\Delta\mathbf{v}). \tag{2.23}$$

According to our hypothesis d) on $\mathcal{D}$ (see Section 1.2), it follows from (2.23) that

$$\begin{aligned} \nu_5\int_\Omega\left(1+|\varepsilon(\mathbf{v})|^{2\delta}\right)|\nabla\varepsilon(\mathbf{v})|^2\,d\mathbf{x} &\le (\mathbf{v}\cdot\nabla\mathbf{v},\Delta\mathbf{v})-(\boldsymbol{\psi},\Delta\mathbf{v}) \\ &= -\int_\Omega v_{k,\ell}v_{i,k}v_{i,\ell}\,d\mathbf{x}-(\boldsymbol{\psi},\Delta\mathbf{v}) \\ &\le \|\mathbf{v}_x\|_{3,\Omega}^3+\|\boldsymbol{\psi}\|_{2,\Omega}\,\|\Delta\mathbf{v}\|_{2,\Omega} \\ &\le C\|\mathbf{v}_{xx}\|_{2,\Omega}^{3/2}\|\mathbf{v}_x\|_{2,\Omega}^{3/2}+\|\boldsymbol{\psi}\|_{2,\Omega}\,\|\Delta\mathbf{v}\|_{2,\Omega} \\ &\le \epsilon_1\|\mathbf{v}_{xx}\|_{2,\Omega}^2+C(\epsilon_1)\|\mathbf{v}_x\|_{2,\Omega}^6+\epsilon_1\|\mathbf{v}_{xx}\|_{2,\Omega}^2+C(\epsilon_1)\|\boldsymbol{\psi}\|_{2,\Omega}^2 \end{aligned} \tag{2.24}$$

with arbitrary $\epsilon_1\in(0,1]$. From (2.24) for sufficiently small ϵ_1, we derive the estimate

$$\|\mathbf{v}_{xx}\|_{2,\Omega}^2+\int_\Omega|\varepsilon(\mathbf{v})|^{2\delta}|\nabla\varepsilon(\mathbf{v})|^2\,d\mathbf{x}\le C\left(\|\mathbf{v}_x\|_{2,\Omega}^6+\|\boldsymbol{\psi}\|_{2,\Omega}^2\right). \tag{2.25}$$

We can apply (2.25) to the nonstationary case. The identity (2.1) for any $t\in[0,T]$ can be considered as the identity (2.22) with $\boldsymbol{\psi}=(\mathbf{f}-\mathbf{v}_t+\mu\mathbf{H}\cdot\nabla\mathbf{H})(t)$. Substituting this $\boldsymbol{\psi}$ in (2.25) and integrating the result over $t\in(0,T]$ yields the estimate

$$\int_{Q_T}\left(|\mathbf{v}_{xx}|^2+|\varepsilon(\mathbf{v})|^{2\delta}|\nabla\varepsilon(\mathbf{v})|^2\right)d\mathbf{x}dt\le C \tag{2.26}$$

with C under our control.

Now we can determine an estimate for the pressure from the system (1.2). For its gradient ∇p we have the estimate

$$\begin{aligned}\|\nabla p\|_{s,\Omega} &\le \left\| -\mathbf{v}_t + \operatorname{div}\left(\left.\frac{\partial \mathcal{D}}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})} \right) - \mathbf{v}\cdot\nabla\mathbf{v} + \mu\mathbf{H}\cdot\nabla\mathbf{H} + \mathbf{f} \right\|_{s,\Omega} \\ &\le \|\mathbf{v}_t\|_{s,\Omega} + \|\mathbf{v}\cdot\nabla\mathbf{v}\|_{s,\Omega} + \mu\|\mathbf{H}\cdot\nabla\mathbf{H}\|_{s,\Omega} + \|\mathbf{f}\|_{s,\Omega} + \mathcal{Y}_5 = \mathcal{Y}_6(t)\end{aligned}$$

where

$$\begin{aligned}\mathcal{Y}_5(t) = \left\| \operatorname{div}\left(\left.\frac{\partial \mathcal{D}}{\partial \varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v})} \right) \right\|_{s,\Omega} &\le C\|\mathbf{v}_{xx}\|_{s,\Omega} + C\left(\int_\Omega |\varepsilon|^{2\delta s}|\nabla\varepsilon|^s \right)^{1/s} \\ &\le C\|\mathbf{v}_{xx}\|_{s,\Omega} + C\|\,|\varepsilon|^\delta\,|\nabla\varepsilon|\,\|_{2,\Omega}\|\varepsilon\|^{\delta}_{\frac{2\delta s}{2-s}}.\end{aligned}$$

For $s = (2+2\delta)/(1+2\delta)$ the power $(2\delta s)/(2-s)$ is equal to $2+2\delta$ and therefore our previous estimates give a bound for $\int_0^T \mathcal{Y}_5(t)^2 dt$. We also have bounds for the time-integrals over $(0,T)$ of the second power of the other terms in $\mathcal{Y}_6$; all this guarantees the estimate

$$\|\nabla p\|^2_{s,2,Q_T} = \int_0^T \|\nabla p(t)\|^2_{s,\Omega}\,dt \le C \tag{2.27}$$

with $s = (2+2\delta)/(1+2\delta)$ and C a constant under our control.

3. A uniqueness theorem

In this section, we will prove a theorem of uniqueness that is similar to the uniqueness theorem for the Navier–Stokes equations. The theorem is based on the inequality

$$\|\mathbf{u}\|_{q,r,Q_T} \le \beta|\mathbf{u}|_{Q_T}, \quad \frac{3}{q} + \frac{2}{r} \ge \frac{3}{2}, \quad q \in [2,6] \tag{3.1}$$

stated by O. Ladyzhenskaya [9] for any function $\mathbf{u}$ with finite norm

$$|\mathbf{u}|_{Q_T} \equiv \operatorname*{ess\,sup}_{t\in[0,T]} \|\mathbf{u}(t)\|_{2,\Omega} + \|\mathbf{u}_x\|_{2,Q_T}$$

in $Q_T = \Omega \times (0,T)$ with $\Omega \subset \mathbb{R}^3$, $T < \infty$ (see, e.g., [7, 12]). We will also use the following consequence of (3.1):

$$\|\,|\mathbf{u}|\,|\mathbf{v}|\,\|_{2,Q_T} \le \beta\|\mathbf{u}\|_{q,r,Q_T}|\mathbf{v}|_{Q_T} \tag{3.2}$$

with arbitrary positive q, r satisfying the condition

$$\frac{3}{q} + \frac{2}{r} \le 1.$$

Theorem 3.1. *The initial-boundary value problems for the system* (1.2)–(1.6) *with boundary conditions* (1.7) (*homogeneous or nonhomogeneous*) *or* (1.8) *can have at most one solution in the class of generalized solutions with finite norms*

$$|\mathbf{v}|_{Q_T}, \quad \|\mathbf{v}_x\|_{2+2\delta,Q_T}, \quad \delta \geq \frac{1}{4}, \quad \|\mathbf{v}_t\|_{2,Q_T} \tag{3.3}$$

and

$$|\mathbf{H}|_{Q_T}, \quad \|\mathbf{H}_t\|_{2,Q_T}, \quad \|\mathbf{H}\|_{q,r,Q_T} \tag{3.4}$$

with powers q, r *satisfying the conditions*

$$\frac{3}{q} + \frac{2}{r} \leq 1, \quad q > 3. \tag{3.5}$$

Remark. Note that the conditions in (3.3) imply the inequality

$$\|\mathbf{v}\|_{q_1,r_1,Q_T} \leq C\|\mathbf{v}_x\|_{2+2\delta,Q_T} \tag{3.6}$$

with q_1 and r_1 again satisfying condition (3.5).

Proof. Our generalized solutions satisfy (1.2) and (1.4) in the form of the identities

$$\begin{aligned} \int_0^T \left((\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}, \boldsymbol{\eta}) + \left(\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\mathbf{v})}, \varepsilon(\boldsymbol{\eta}) \right) + \mu\left(\mathbf{H}, \mathbf{H}\cdot\nabla\boldsymbol{\eta}\right) \right) dt \\ = \int_0^T (\mathbf{f}, \boldsymbol{\eta})\, dt \end{aligned} \tag{3.7}$$

for any $\boldsymbol{\eta}$ having the properties

$$\operatorname{div}\boldsymbol{\eta} = 0, \quad \boldsymbol{\eta}|_{S_T} = 0, \quad |\boldsymbol{\eta}|_{Q_T}, \|\boldsymbol{\eta}_x\|_{2+2\delta,Q_T} < \infty, \quad \delta \geq \frac{1}{4} \tag{3.8}$$

and

$$\begin{aligned} \int_0^T \left(\mu(\mathbf{H}_t, \boldsymbol{\zeta}) + \frac{1}{\sigma}(\operatorname{curl}\mathbf{H}, \operatorname{curl}\boldsymbol{\zeta}) + \mu(\mathbf{v}\cdot\nabla\mathbf{H} - \mathbf{H}\cdot\nabla\mathbf{v}, \boldsymbol{\zeta}) \right) dt \\ = \frac{1}{\sigma}\int_0^T (\operatorname{curl}\mathbf{j}, \boldsymbol{\zeta})\, dt \end{aligned} \tag{3.9}$$

for any $\boldsymbol{\zeta}$ having the property

$$|\boldsymbol{\zeta}|_{Q_T} < \infty. \tag{3.10}$$

It is easy to determine that all integrals occurring in (3.7) and (3.9) are finite under our hypotheses for $\mathbf{v}$, $\boldsymbol{\eta}$, $\mathbf{H}$, and $\boldsymbol{\zeta}$. For example,

$$\begin{aligned} \left| \int_0^T \left(\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon(\mathbf{v})}, \varepsilon(\boldsymbol{\eta}) \right) dt \right| &\leq C\int_{Q_T} \left(|\varepsilon(\mathbf{v})| + |\varepsilon(\mathbf{v})|^{1+2\delta}\right)|\varepsilon(\boldsymbol{\eta})|\, d\mathbf{x}dt \\ &\leq C\|\varepsilon(\mathbf{v})\|_{2,Q_T}\|\varepsilon(\boldsymbol{\eta})\|_{2,Q_T} + C\|\varepsilon(\mathbf{v})\|_{2+2\delta,Q_T}^{1+2\delta}\|\varepsilon(\boldsymbol{\eta})\|_{2+2\delta,Q_T} < \infty \end{aligned}$$

and

$$\left|\int_0^T (\mathbf{H}, \mathbf{H}\cdot\nabla\boldsymbol{\eta})\,dt\right| \le \textstyle\int_{Q_T} |\mathbf{H}|^2|\boldsymbol{\eta}_x|\,d\mathbf{x}dt \le \|\mathbf{H}\|^2_{\frac{4(1+\delta)}{1+2\delta},Q_T}\|\boldsymbol{\eta}_x\|_{2+2\delta,Q_T}$$
$$\le C|\mathbf{H}|^2_{Q_T}\|\boldsymbol{\eta}_x\|_{2+2\delta,Q_T} < \infty$$

because $q = r = 4(1+\delta)/(1+2\sigma)$ satisfy the condition (3.1) for $\delta \ge \frac{1}{4}$. Using (3.2) and (3.6), we obtain

$$\left|\int_0^T (\mathbf{v}\cdot\nabla\mathbf{H}, \boldsymbol{\zeta})\,dt\right| \le \|\mathbf{H}_x\|_{2,Q_T}\|\,|\mathbf{v}|\,|\boldsymbol{\zeta}|\,\|_{2,Q_T} \le \|\mathbf{H}_x\|_{2,Q_T}\|\mathbf{v}\|_{q,2,Q_T}|\boldsymbol{\zeta}|_{Q_T}$$
$$\le C\|\mathbf{H}_x\|_{2,Q_T}\|\mathbf{v}_x\|_{2+2\delta,Q_T}|\boldsymbol{\zeta}|_{Q_T} < \infty.$$

Now assume that the given problem has two generalized solutions $\mathbf{v}_1$, p_1, $\mathbf{H}_1$ and $\mathbf{v}_2$, p_2, $\mathbf{H}_2$ of our class. For their differences $\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2$, $q = p_1 - p_2$, and $\mathbf{B} = \mathbf{H}_1 - \mathbf{H}_2$, we have the identities

$$\begin{aligned}\int_0^T \Big((\mathbf{u}_t + \mathbf{v}_1\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{v}_2, \boldsymbol{\eta}) + (A, \varepsilon(\boldsymbol{\eta}))&\\ +\mu(\mathbf{B}, \mathbf{H}_1\cdot\nabla\boldsymbol{\eta}) + \mu(\mathbf{H}_2, \mathbf{B}\cdot\nabla\boldsymbol{\eta})\Big)dt = 0&\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\int_0^T \Big(\mu(\mathbf{B}_t, \boldsymbol{\zeta}) + \frac{1}{\sigma}(\operatorname{curl}\mathbf{B}, \operatorname{curl}\boldsymbol{\zeta})&\\ +\mu(\mathbf{v}_1\cdot\nabla\mathbf{B} - \mathbf{B}\cdot\nabla\mathbf{v}_2 - \mathbf{H}_1\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{H}_2, \boldsymbol{\zeta})\Big)dt = 0&\end{aligned} \tag{3.12}$$

which follow from (3.7) and (3.9). Here,

$$\begin{aligned}A = \left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v}_1)} - \left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon(\mathbf{v}_2)} &= \int_0^1 \frac{d}{d\tau}\left.\frac{\partial\mathcal{D}(\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=\varepsilon_\tau} d\tau\\ &= \int_0^1 \left.\frac{\partial^2\mathcal{D}(\varepsilon)}{\partial\varepsilon\partial\varepsilon}\right|_{\varepsilon=\varepsilon_\tau} d\tau : \varepsilon(\mathbf{u})\end{aligned}$$

with $\varepsilon_\tau = \tau\varepsilon(\mathbf{v}_1) + (1-\tau)\varepsilon(\mathbf{v}_2)$. By virtue of our hypothesis on $\mathcal{D}$, we have

$$(A, \varepsilon(\mathbf{u})) \ge \nu_5\|\varepsilon(\mathbf{u})\|^2_{2,\Omega} \ge \widetilde{\nu}_5\|\mathbf{u}_x\|^2_{2,\Omega}, \quad \widetilde{\nu}_5 > 0. \tag{3.13}$$

Now, in (3.11), we set

$$\boldsymbol{\eta}(\mathbf{x}, \tau) = \begin{cases}\mathbf{u}(\mathbf{x}, \tau) & \text{for } \tau \le t\\ 0 & \text{for } \tau > t\end{cases}$$

and in (3.12), we set

$$\boldsymbol{\zeta}(\mathbf{x}, \tau) = \begin{cases}\mathbf{B}(\mathbf{x}, \tau) & \text{for } \tau \le t\\ 0 & \text{for } \tau > t\end{cases}$$

for arbitrary $t \le T$ and transform them into the relations

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}(t)\|_{2,\Omega}^2 &+ \int_0^t (A, \varepsilon(\mathbf{u}))\, d\tau \\ &= \int_0^t \Big((\mathbf{v}_2, \mathbf{u}\cdot\nabla\mathbf{u}) - \mu(\mathbf{B}, \mathbf{H}_1\cdot\nabla\mathbf{u}) - \mu(\mathbf{H}_2, \mathbf{B}\cdot\nabla\mathbf{u})\Big) dt\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\frac{\mu}{2}\|\mathbf{B}(t)\|_{2,\Omega}^2 &+ \int_0^t \frac{1}{\sigma}\|\operatorname{curl}\mathbf{B}\|_{2,\Omega}^2 \\ &= \mu\int_0^t \Big((\mathbf{v}_2, \mathbf{B}\cdot\nabla\mathbf{B}) + (\mathbf{H}_1\cdot\nabla\mathbf{u}, \mathbf{B}) - (\mathbf{H}_2, \mathbf{u}\cdot\nabla\mathbf{B})\Big) dt,\end{aligned} \tag{3.15}$$

respectively. Now we majorize the right-hand sides of (3.14) and (3.15) using (3.2) and our hypotheses (3.3)–(3.6). In detail,

$$\begin{aligned}\|\mathbf{u}(t)\|_{2,\Omega}^2 + \int_0^t \|\mathbf{u}_x\|_{2,\Omega}^2 dt &\le C\Big(\|\,|\mathbf{v}_2|\,|\mathbf{u}|\,\|_{2,Q_t} + \|\,|\mathbf{B}|\,|\mathbf{H}_1|\,\|_{2,Q_t} \\ &\qquad + \|\,|\mathbf{H}_2|\,|\mathbf{B}|\,\|_{2,Q_t}\Big)\|\mathbf{u}_x\|_{2,Q_t} \\ &\le C_1 F(t)\left(|\mathbf{u}|_{Q_t}^2 + |\mathbf{B}|_{Q_t}^2\right)\end{aligned} \tag{3.16}$$

with

$$F(t) = \|\mathbf{v}_2\|_{q_1,r_1,Q_t} + \|\mathbf{H}_1\|_{q,r,Q_t} + \|\mathbf{H}_2\|_{q,r,Q_t}.$$

Analogously, from (3.15), we have

$$\|\mathbf{H}(t)\|_{2,\Omega}^2 + \int_0^t \|\mathbf{H}_x\|_{2,\Omega}^2\, dt \le C_2 F(t)\left(|\mathbf{u}|_{Q_t}^2 + |\mathbf{B}|_{Q_t}^2\right). \tag{3.17}$$

From (3.16) and (3.17), we have that

$$y(t) \le C_3 F(t) y(t) \quad \text{where} \quad y(t) = |\mathbf{u}|_{Q_t}^2 + |\mathbf{B}|_{Q_t}^2. \tag{3.18}$$

As $r, r_1 < \infty$, $F(t) \to 0$ when $t \to 0$, and therefore from (3.18) it follows that $y(t) \equiv 0$ for $t \in [0, t_1]$, where t_1 is such that $C_3 F(t_1) < 1$. If $t_1 < T$, we take (3.18) for $t \in [t_1, t_2]$ and the conclusion that $y(t) \equiv 0$ on this interval only if $C_3 \int_{t_1}^{t_2} F(t)\, dt < 1$. So, after some steps, we prove that $y(t) \equiv 0$ for $t \in [0, T]$ and thus, $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{H}_1 = \mathbf{H}_2$. Finally, the conclusion $p_1 = p_2$ can be derived by the same method as that for the Navier–Stokes equations. Thus, Theorem 3.1 is proved. □

The solutions that we will find in Section 4 have finite norms which were estimated in Sections 2.1–2.3. For $\mathbf{v}$, they are the same as in (3.3). For $\mathbf{H}$, we have estimated $|\mathbf{H}|_{Q_T}$, $\|\mathbf{H}_t\|_{2,Q_T}$, $\sup_{t\in(0,T)}\|\mathbf{H}_x(t)\|_{2,\Omega}$, and $\|\mathbf{H}_{xx}\|_{2,Q_T}$. Their majorants determine a majorant for $\|\mathbf{H}\|_{q,r,Q_T}$ with q, r satisfying the conditions (3.5) because of

$$\|\mathbf{H}(t)\|_{6,\Omega} \le C\|\mathbf{H}_x(t)\|_{2,\Omega},$$

implying $\|\mathbf{H}\|_{6,r,\Omega} < \infty$ with any $r < \infty$.

4. Existence results

With the aid of the a priori estimates stated in Sections 2.1–2.4 and the uniqueness theorem from Section 3, we can prove results for the global unique solvability of the problems (1.2)–(1.8). For this purpose, we use Galerkin approximations with a special choice of basis functions.

4.1. The boundary value case

For the case of the boundary condition (1.7), we use the eigenfunctions $\{\boldsymbol{\phi}^{(k)}(x)\}_{k=1}^{\infty}$ of the Stokes operator, i.e., $\boldsymbol{\phi}^{(k)}$ satisfies

$$\begin{cases} -\Delta\boldsymbol{\phi}^{(k)} + \nabla p^{(k)} &= \lambda_k \boldsymbol{\phi}^{(k)} \\ \operatorname{div}\boldsymbol{\phi}^{(k)} &= 0 \\ \boldsymbol{\phi}^{(k)}|_{\partial\Omega} &= 0, \end{cases}$$

and the eigenfunctions $\{\boldsymbol{\psi}^{(k)}(x)\}_{k=1}^{\infty}$ of the problem

$$\begin{cases} \operatorname{curl}\operatorname{curl}\boldsymbol{\psi}^{(k)} + \mu_0\boldsymbol{\psi}^{(k)} &= \mu_k\boldsymbol{\psi}^{(k)}, \\ \operatorname{div}\boldsymbol{\psi}^{(k)} &= 0 \\ (\boldsymbol{\psi}^{(k)}\cdot\mathbf{n})|_{\partial\Omega} &= 0, \quad (\operatorname{curl}\boldsymbol{\psi}^{(k)})_\tau|_{\partial\Omega} = 0. \end{cases}$$

For a simply connected domain Ω, we can take $\mu_0 = 0$. These sets of eigenfunctions can be orthonormalized with respect to $L^2(\Omega)$. The functions $\{\boldsymbol{\phi}^{(k)}\}_{k=1}^{\infty}$ form a basis in the spaces $\overset{\circ}{\mathcal{J}}(\Omega)$, $\overset{\circ}{\mathcal{J}}{}^1_2(\Omega)$, and $\overset{\circ}{\mathcal{J}}{}^1_2(\Omega)\cap W_2^2(\Omega)$ (see [7]). We order the eigenvalues in the usual way: $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_k$ with $\lambda_k\to\infty$ as $k\to\infty$. The functions $\{\boldsymbol{\psi}^{(k)}\}_{k=1}^{\infty}$ form a basis in $\overset{\circ}{\mathcal{J}}(\Omega)$, $\overset{\circ}{\mathcal{J}}{}^1_{2,n}(\Omega)$ and $\widetilde{\mathcal{J}}{}^2_2(\Omega)$. We order the eigenvalues as $\mu_0\le\mu_1\le\mu_2\le\cdots$, μ_k with $\mu_k\to\infty$ as $k\to\infty$. This is proved by the same arguments as for the Laplace operator with the Dirichlet boundary conditions (see, e.g. [5, 10]).

We choose Galerkin approximations $\mathbf{v}^{(m)}$ and $\mathbf{H}^{(m)}$, $m=1,2\ldots$, of the form

$$\mathbf{v}^{(m)}(\mathbf{x},t)=\sum_{k=1}^{m} a_k^{(m)}(t)\boldsymbol{\phi}^{(k)}(\mathbf{x}) \quad\text{and}\quad \mathbf{H}^{(m)}(\mathbf{x},t)=\sum_{k=1}^{m} b_k^{(m)}(t)\boldsymbol{\psi}^{(k)}(\mathbf{x}),$$

where the coefficients $a_k^{(m)}$ and $b_k^{(m)}$ are determined from the systems

$$\begin{aligned} &\frac{d}{dt}\left(\mathbf{v}^{(m)}(t),\boldsymbol{\phi}^{(k)}\right)+\left(\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t),\boldsymbol{\phi}^{(k)}\right)\\ &\qquad\qquad+\left(\frac{\partial\mathcal{D}}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})},\varepsilon(\boldsymbol{\phi}^{(k)})\right)+\mu\left(\mathbf{H}^{(k)}(t),\mathbf{H}^{(k)}(t)\cdot\nabla\boldsymbol{\phi}^{(k)}\right)\\ &\quad=\left(\mathbf{f}(t),\boldsymbol{\phi}^{(k)}\right),\quad k=1,2,\ldots,m, \end{aligned} \tag{4.1}$$

and

$$\mu\frac{d}{dt}\left(\mathbf{H}^{(m)}(t),\boldsymbol{\psi}^{(k)}\right)+\frac{1}{\sigma}\left(\operatorname{curl}\mathbf{H}^{(m)}(t),\,\operatorname{curl}\boldsymbol{\psi}^{(k)}\right)$$
$$+\mu\left(\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{H}^{(m)}(t)-\mathbf{H}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t),\boldsymbol{\psi}^{(k)}\right) \tag{4.2}$$
$$=\frac{1}{\sigma}\left(\operatorname{curl}\mathbf{j}(t),\boldsymbol{\psi}^{(k)}\right),\quad k=1,2,\dots,m,$$

along with the initial data

$$\mathbf{v}^{(m)}|_{t=0}=\sum_{k=1}^{m}\big(\mathbf{v}^0,\boldsymbol{\phi}^{(k)}\big)\boldsymbol{\phi}^{(k)}\quad\text{and}\quad\mathbf{H}^{(m)}|_{t=0}=\sum_{k=1}^{m}\big(\mathbf{H}^0,\boldsymbol{\psi}^{(k)}\big)\boldsymbol{\psi}^{(k)}. \tag{4.3}$$

Equations (4.1) and (4.2) form a nonlinear system of ordinary differential equations for the coefficients $a_k^{(m)}$ and $b_k^{(m)}$, $k=1,2,\dots,m$, on $t\in[0,T]$. The system has a unique solution on $[0,T]$ satisfying (4.3). This fact follows from the results of Section 2.1 which guarantee the estimates

$$\max_{t\in[0,T]}\|\mathbf{v}^{(m)}(t)\|_{2,\Omega}^2=\max_{t\in[0,T]}\sum_{k=1}^{m}\left(a_k^{(m)}(t)\right)^2\le C \tag{4.4}$$

and

$$\max_{t\in[0,T]}\|\mathbf{H}^{(m)}(t)\|_{2,\Omega}^2=\max_{t\in[0,T]}\sum_{k=1}^{m}\left(b_k^{(m)}(t)\right)^2\le C \tag{4.5}$$

with upper bounds C that do not depend on m. Moreover, we have for $\mathbf{v}^{(m)}$ and $\mathbf{H}^{(m)}$ the same estimates for all norms that we have considered in the previous sections for the exact solutions $\mathbf{v}$ and $\mathbf{H}$ because we have used for the latter pair instead of (1.2) and (1.4), the corresponding integral identities (2.1) and (2.2). More precisely, we have used (2.1) only for $\boldsymbol{\eta}=\mathbf{v}$ or $\boldsymbol{\eta}=\mathbf{v}_t$, and (2.2) for $\boldsymbol{\zeta}=\mathbf{H}$, $\boldsymbol{\zeta}=\Delta\mathbf{H}$, or $\boldsymbol{\zeta}=\mathbf{H}_t$. Just the same relations for $\mathbf{v}^{(m)}$ and $\mathbf{H}^{(m)}$ follow from (4.1) and (4.2). Namely, from (4.1) we can derive

$$\left(\mathbf{v}_t^{(m)}(t)+\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t),\mathbf{v}^{(m)}(t)\right)+\left(\frac{\partial\mathcal{D}}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})},\varepsilon(\mathbf{v}^{(m)}(t)\right)$$
$$-\mu\big(\mathbf{H}^{(m)}(t)\cdot\nabla\mathbf{H}^{(m)}(t),\mathbf{v}^{(m)}(t)\big)=\big(\mathbf{f}(t),\mathbf{v}^{(m)}(t)\big)$$

and

$$\left(\mathbf{v}_t^{(m)}(t)+\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t),\mathbf{v}_t^{(m)}(t)\right)+\left(\frac{\partial\mathcal{D}}{\partial\varepsilon}\Big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})},\varepsilon(\mathbf{v}_t^{(m)}(t)\right)$$
$$-\mu\left(\mathbf{H}^{(m)}(t)\cdot\nabla\mathbf{H}^{(m)}(t),\mathbf{v}_t^{(m)}(t)\right)=\left(\mathbf{f}(t),\mathbf{v}_t^{(m)}(t)\right),$$

which hold for all $t \in [0,T]$. For $\mathbf{H}^{(m)}$, (4.2) implies the relations

$$\begin{aligned}\mu\left(\mathbf{H}_t^{(m)}(t), \mathbf{H}^{(m)}(t)\right) &+ \frac{1}{\sigma}\left(\operatorname{curl}\mathbf{H}^{(m)}(t),\, \operatorname{curl}\mathbf{H}^{(m)}(t)\right)\\ &+\mu\left(\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{H}^{(m)}(t) - \mathbf{H}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t), \mathbf{H}^{(m)}(t)\right)\\ = \frac{1}{\sigma}&\left(\operatorname{curl}\mathbf{j}(t), \mathbf{H}^{(m)}(t)\right)\end{aligned}$$

and

$$\begin{aligned}-\mu\left(\mathbf{H}_t^{(m)}(t), \Delta\mathbf{H}^{(m)}(t)\right) &- \frac{1}{\sigma}\left(\operatorname{curl}\mathbf{H}^{(m)}(t),\, \operatorname{curl}\Delta\mathbf{H}^{(m)}(t)\right)\\ &-\mu\left(\mathbf{v}^{(m)}(t)\cdot\nabla\mathbf{H}^{(m)}(t) - \mathbf{H}^{(m)}(t)\cdot\nabla\mathbf{v}^{(m)}(t), \Delta\mathbf{H}^{(m)}(t)\right) \qquad (4.6)\\ = -\frac{1}{\sigma}&\left(\operatorname{curl}\mathbf{j}(t), \Delta\mathbf{H}^{(m)}(t)\right)\end{aligned}$$

and the analogous relation of the form (4.6) in which $\Delta\mathbf{H}^{(m)}(t)$ is replaced by $\mathbf{H}_t^{(m)}(t)$.

These relations guarantee the estimates (4.4)–(4.5) and

$$\max_{t\in[0,T]} \|\mathbf{v}_x^{(m)}(t)\|_{2+2\delta,\Omega}, \quad \|\mathbf{v}_t^{(m)}\|_{2,Q_T} \le C,$$

$$\max_{t\in[0,T]} \|\mathbf{H}_x^{(m)}(t)\|_{2,\Omega}, \quad \|\mathbf{H}_{xx}^{(m)}\|_{2,Q_T}, \quad \|\mathbf{H}_t^{(m)}\|_{2,Q_T} \le C,$$

with constants C that depend only on some numerical characteristics of Ω and the norms

$$\|\mathbf{v}_x^0\|_{2,\Omega}, \quad \|\mathbf{H}^0\|_{2,\Omega} + \|\mathbf{H}_x^0\|_{2,\Omega}, \quad \|\mathbf{f}\|_{2,Q_T}, \quad \|\operatorname{curl}\mathbf{j}\|_{2,Q_T},$$

and

$$\sup_{m\in\mathbb{Z}} \|\mathbf{v}_x^{(m)}(\cdot,0)\|_{2+2\delta,\Omega} = C_1.$$

We can guarantee $C_1 < \infty$ provided

$$\mathbf{v}^0 \in W_2^2(\Omega) \cap \overset{\circ}{\mathcal{J}}{}_2^1(\Omega) \quad \text{and} \quad \delta \in [\frac{1}{4}, 2].$$

Now we can pass to the limit as $m \to \infty$ and obtain $\mathbf{v}$ and $\mathbf{H}$ as limits of $\mathbf{v}^{(m)}$ and $\mathbf{H}^{(m)}$, respectively. These functions $\mathbf{v}$ and $\mathbf{H}$ will satisfy the integral identities (3.7) and (3.9) with the functions $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ described in (3.8) and (3.10), respectively. The limit procedure is easy for all terms appearing in (3.7) and (3.9) except for the term containing $\frac{\partial\mathcal{D}}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})}$ if we use test functions $\boldsymbol{\eta}(\mathbf{x},t) = \sum_{k=1}^{\ell} c^{(k)}(t)\boldsymbol{\phi}^{(k)}(\mathbf{x})$ and $\boldsymbol{\zeta}(\mathbf{x},t) = \sum_{k=1}^{\ell} d^{(k)}(t)\boldsymbol{\psi}^{(k)}(\mathbf{x})$ with smooth arbitrary functions $c^{(k)}$, $d^{(k)}$ and a fixed number $\ell < \infty$. (This then implies the relationships (3.7) and (3.9) with any admissible $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$.)

The limit procedure for the term containing $\frac{\partial\mathcal{D}}{\partial\varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})}$ can be done as in [12] (Sec. 6, Th. 6.7) for parabolic equations with monotonic principal part. In our case, we have the additional information about the uniform boundedness of $\|\mathbf{v}_t^{(m)}\|_{2,Q_T}$ and $\|\mathbf{H}_t^{(m)}\|_{2,Q_T}$ and, for $\frac{\partial \mathcal{D}}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v})}$, the strong monotonicity (3.13). These facts facilitate the proof that $\int_0^t (\frac{\partial \mathcal{D}}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v}^{(m)})}, \boldsymbol{\eta})dt \to \int_0^T (\frac{\partial \mathcal{D}}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v})}, \boldsymbol{\eta})dt$ where $\mathbf{v} = \lim_{m\to\infty} \mathbf{v}^{(m)}$. Consequently, we obtain the following existence theorem.

Theorem 4.1. *Suppose that* Ω *is a bounded domain in* $\mathbb{R}^3$ *with* $\partial\Omega \subset C^2$ *and let* $Q_T = \Omega \times (0,T)$. *Suppose that* $\mathbf{f},\ \operatorname{curl}\mathbf{j} \in L^2(Q_T)$, $\operatorname{div}\mathbf{j} = 0$, $\mathbf{v}^0 \in W_2^2(\Omega) \cap \overset{\circ}{\mathcal{J}}{}_2^1(\Omega)$, *and* $\mathbf{H}^0 \in \widetilde{\mathcal{J}}{}_2^1(\Omega)$. *Then, the problem* (1.2)–(1.7) *with* $\delta \in [\frac{1}{4}, 2]$ *has a unique generalized solution* $\mathbf{v}, \mathbf{H}$. *Moreover, the generalized solution has the properties*

$$|\mathbf{v}|_{Q_T}, \quad \|\mathbf{v}_t\|_{2,Q_T}, \quad \max_{t\in[0,T]} \|\mathbf{v}_x(t)\|_{2+2\delta,\Omega} < \infty$$

and

$$|\mathbf{H}|_{Q_T}, \quad \max_{t\in[0,t]} \|\mathbf{H}_x(t)\|, \quad \|\mathbf{H}_t\|_{2,Q_T}, \quad \|\mathbf{H}_{xx}\|_{2,Q_T} < \infty.$$

The solution satisfies the systems (1.2) *and* (1.4) *in the form of the identities* (3.7) *and* (3.9), *respectively, where* $\boldsymbol{\eta}$ *and* $\boldsymbol{\zeta}$ *are described in* (3.8) *and* (3.10), *respectively. The other equations in the system* (1.2)–(1.7) *are satisfied in the usual way.*

Having in hand $\mathbf{v}$ and $\mathbf{H}$, we can find the pressure p from the system (1.2) which we can rewrite in the form

$$\nabla p(\mathbf{x},t) = \frac{\partial \mathbf{F}^j(\mathbf{x},t)}{\partial x_j} + \mathbf{F}(\mathbf{x},t) \tag{4.7}$$

with $\mathbf{F} = \mathbf{f} - \mathbf{v}_t - \mathbf{v}\cdot\nabla\mathbf{v} - \mu\mathbf{H} \times \operatorname{curl}\mathbf{H}$ and $\mathbf{F}^j = (F_1^j, F_2^j, F_3^j)$, $\mathbf{F}^j = \frac{\partial \mathcal{D}}{\partial \varepsilon}\big|_{\varepsilon=\varepsilon(\mathbf{v})}$. We know that $\mathbf{F}^j(\cdot,t) \in L^s(\Omega)$, $s = (2+2\delta)/(1+2\delta)$, and $\mathbf{F}(\cdot,t) \in L^2(\Omega)$ for almost all $t \in [0,T]$ and (4.7) is understood in the sense of distributions. From this it follows (see, e.g., [2]) that $p(\cdot,t)$ belongs to $L^s(\Omega)$ and

$$\|p(\cdot,t) - \overline{p}(t)\|_{s,\Omega} \le C \sum_{j=1}^{3} \|\mathbf{F}^j(\cdot,t)\|_{s,\Omega} + C\|\mathbf{F}(\cdot,t)\|_{2,\Omega} \quad \text{for } t \in [0,T],$$

where $\overline{p}(t) = \frac{1}{|\Omega|}\int_\Omega p(\mathbf{x},t)\,dx$. Therefore

$$p - \overline{p} \in L_{s,2}(Q_T), \qquad s = \frac{2+2\delta}{1+2\delta}.$$

4.2. The periodic boundary condition case

For the periodic boundary conditions (1.8) we have the additional estimates proved in Section 2.4. They guarantee, for the solution $\mathbf{v}$ and p, the existence of $\mathbf{v}_{xx}$ and ∇p with the finite norms appearing in (2.26) and (2.27).

For determining $\mathbf{v}$ and $\mathbf{H}$, we can use the Galerkin approximations with basis $\{\boldsymbol{\phi}^{(k)}(\mathbf{x})\}_{k=1}^{\infty}$ in spaces which we describe below. Instead of $L^2(\Omega)$, we now use the space $\widehat{L}^2(\Omega)$ and instead of $W_2^k(\Omega)$, we now use the space $\widehat{W}_2^k(\Omega)$. For divergence free fields, we need the spaces $\widehat{\mathcal{J}}_2(\Omega)$ and $\widehat{\mathcal{J}}_2^k(\Omega)$. For the basis functions $\boldsymbol{\phi}^{(k)}$, we take the solutions of the special problem

$$\begin{aligned} -\Delta\boldsymbol{\phi}^{(k)} &= \lambda_k\boldsymbol{\phi}^{(k)} \\ \operatorname{div}\boldsymbol{\phi}^{(k)} &= 0 \\ \boldsymbol{\phi}^{(k)} &\in \widehat{W}_2^2(\Omega), \quad k = 1, 2, \dots . \end{aligned}$$

They can be chosen so that

$$(\boldsymbol{\phi}^{(k)}, \boldsymbol{\phi}^{(\ell)}) = \delta_k^{\ell}.$$

The Galerkin approximations $\mathbf{v}^{(m)}$ and $\mathbf{H}^{(m)}$ have the form

$$\mathbf{v}^{(m)}(\mathbf{x}, t) = \sum_{k=1}^{m} a^{(k)}(t)\boldsymbol{\phi}^{(k)}(\mathbf{x})$$

and

$$\mathbf{H}^{(m)}(\mathbf{x}, t) = \sum_{k=1}^{m} b^{(k)}(t)\boldsymbol{\phi}^{(k)}(\mathbf{x}),$$

respectively. We find their coefficients from relations analogous to (4.1)–(4.3) with $\boldsymbol{\psi}^{(k)}$ replaced with $\boldsymbol{\phi}^{(k)}$. By arguments similar to those described above, we obtain the following theorem.

Theorem 4.2. *The problem* (1.2)–(1.6) *in a cube* Ω *along with the periodic boundary conditions* (1.8) *and with* $\delta \in [\frac{1}{4}, 2]$, $\mathbf{f} \in L^2\big((0,T), \widehat{L}^2(\Omega)\big)$, $\operatorname{curl}\mathbf{j} \in L^2\big((0,T), \widehat{L}^2(\Omega)\big)$, $\operatorname{div}\mathbf{j} = 0$, $\mathbf{u}^0 \in \widehat{\mathcal{J}}_2^2(\Omega)$, *and* $\mathbf{H}^0 \in \widehat{\mathcal{J}}_2^1(\Omega)$ *has a unique solution* $\mathbf{v} \in L^2\big((0,T), \widehat{\mathcal{J}}_2(\Omega)\big)$ *and* $\mathbf{H} \in L^2\big((0,T), \widehat{\mathcal{J}}_2(\Omega)\big)$ *which has the same smoothness as that given in Theorem 4.1 for the solution of the boundary value problem. Moreover,* $\mathbf{v}$ *has derivative* $\mathbf{v}_{xx}$ *with finite norm* (2.26) *and* $\nabla p \in L_{s,2}(Q_T)$ *with* $s = (2 + 2\delta)/(1 + 2\delta)$.

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