Available online at www.sciencedirect.com
SCIENCE DIRECT®
Nonlinear Analysis 62 (2005) 19–40
Nonlinear Analysis
www.elsevier.com/locate/na

Modeling and analysis of the forced Fisher equation

M.D. Gunzburger*, L.S. Hou, W. Zhu
Department of Mathematics, Iowa State University, Ames, IA 50011-2064, USA

Received 3 November 2003; accepted 26 January 2005

Abstract

The Fisher equation with inhomogeneous forcing is considered in this paper. First, a forced Fisher equation and boundary conditions are derived. Then, the existence of a local solution for the forced equation with a homegeneous Dirichlet condition is proved by Galerkin's method. Next, a maximum principle is established and the existence of a global solution is obtained as a consequence of the maximum principle. Finally, generalizations of the results to cases of less regular forces are discussed.

1. Introduction

The unforced Fisher equation in one dimension

$$u_t - u_{xx} - \lambda u(1-u) = 0 \quad \text{in } (0,T)\times(0,1)$$

provides a simple model for gene selection/migration with u denoting the frequency of an advantageous gene and λ measuring the intensity of selection (see, e.g., [7]). This equation gives a deterministic approximation to a model for the spread of an advantageous gene in a population (see [8]). This equation is also referred to as the KPP equation after the pioneering paper of Kolmogorov et al. [12] concerning travelling wave fronts. This equation, along with the homogeneous boundary conditions

$$u(t,0) = u(t,1) = 0 \quad \text{for } t \in (0,T)$$

* Corresponding author. Tel.: +1 850 644 7060; fax: +1 850 644 0098.
E-mail addresses: gunzburg@iastate.edu (M.D. Gunzburger), hou@math.iastate.edu (L.S. Hou), wzhu@iastate.edu (W. Zhu).

0362-546X/$ - see front matter
doi:10.1016/j.na.2005.01.094

and the initial condition

$$u(0, x) = u_0(x) \quad \text{for } x \in (0, 1),$$

has been studied thoroughly in the literature. In particular, it is known that there exists a global solution (see, e.g., [11, p. 61, Exer. 9]), that the solution stays between 0 and 1 provided $0 \leqslant u_0(x) \leqslant 1$ (see, e.g., [5, p. 37; 3]), and the solution approaches either the trivial steady state 0 or a unique nontrivial, nonnegative steady state as $t \to \infty$ (see, e.g., [5, p. 51]). Similar results are available for the multidimensional case (see, e.g., [5,4]).

However, little seems to exist in the literature about Fisher equations with inhomogeneous forcing. In this paper, we consider the model derivation and analysis of the forced Fisher equation

$$u_t - \Delta u - \lambda u(1 - u) = (1 - u) f \quad \text{in } (0, T) \times \Omega \tag{1.1}$$

with homogeneous boundary conditions. Here, Ω is a bounded open set in $\mathbb{R}^n$, $n = 1, 2$, or 3, $[0, T]$ is a finite time interval, and the "force" f is a given function of $(t, \mathbf{x})$. Note that forcing in the Fisher equation enters not merely as an inhomogeneous source term, but in the more complicated way indicated in (1.1).

The plan of this paper is as follows. In Section 2 we derive the forced Fisher equation and boundary conditions in the context of the study of gene frequency distributions. In Section 3 we introduce function spaces, define a weak formulation and recall some results from ODE theories. In Section 4 we establish the existence of a local solution. In Section 5, we prove a maximum principle under suitable assumptions on the data f and u_0. In Section 6 we prove the existence and uniqueness of a global solution.

We briefly discuss some of the novelty and technical difficulties in this paper:

- *Model derivation*: The derivation of the forced Fisher equation and boundary conditions in Section 2 is our first contribution. Our derivation reveals that the general form of the equation should be cubic, rather than quadratic—the standard unforced Fisher equation is merely a special case.
- *Local existence*: In showing the local existence in Section 4, we define a weak formulation that is amenable to numerical approximations and we follow the main steps of Temam [25] concerning the existence of a local solution for the Navier–Stokes equations. Additional efforts are made in our paper to derive a uniform local existence interval for the sequence of Galerkin solutions. Other approaches in the literature may be used to prove the local and global existence of solutions for general semi-linear parabolic equations, but they either require higher regularity on f than what is assumed in this paper or give a solution in a weaker sense (see, e.g., [1,2,6,13,16,21–24,26].
- *Maximum principles*: The maximum principles we prove in Section 5 are weak versions corresponding to the weak formulation given in Section 4. A similar type of maximum principles can be found in [17] for a different system; however the techniques of Manley et al. [17] do not directly apply to our situation.
- *Global existence*: As a straightforward consequence of the maximum principles we may easily obtain a preliminary version of global existence. We also prove a generalized version of global existence that involves a less regular f. Such a generalized existence

result for less regular f will be useful in the formulation of optimal control problems, though control problems are beyond the scope of this work and will be studied elsewhere.

2. Derivation of a forced Fisher equation and boundary conditions

We follow closely the derivation given in [7] of the (unforced) Fisher equation to derive a forced Fisher equation. The derivation is given in one space dimension; however, the generalization to higher dimensions is obvious.

2.1. A forced Fisher equation

We assume that the population is diploid, with two available alleles, A_1 and A_2. The genotypes are denoted by A_iA_k, $i,k=1,2$, and their densities by ρ_{ik}. Note that the genotypes A_iA_k and A_kA_i are the same, but it is convenient to distinguish between them, keeping in mind that $\rho_{ik}=\rho_{ki}$. We then let $\rho=\sum_{i,k=1}^{2}\rho_{ik}$ denote the total density of the population, $p_{ik}=\rho_{ik}/\rho$ the frequency of the genotype A_iA_k in the population, and $p_i=\sum_{k=1}^{2}p_{ik}=(\sum_{k=1}^{2}\rho_{ik})/\rho$ the frequency of the allele A_i in the population. Then, we have that, for $i,k=1,2$,

$$\partial_t\rho_{ik}-\partial_x(\tfrac{1}{2}\partial_x(V\rho_{ik})-M\rho_{ik})=\rho[r_\mathrm{b}p_ip_k(1+s\eta_{ik})-r_\mathrm{d}p_{ik}(1+s\gamma_{ik})]+s\rho f_{ik}, \tag{2.1}$$

i.e., the rate of change of the density of genotype A_iA_k is affected by diffusion (the term containing V), drift (the term containing M), births (the term containing r_b), deaths (the term containing r_d), and artificially introduced genotypes (the term containing f_{ik}.) The case $s=0$ corresponds to no selection being present; small values of the parameter s allow for selection. In (2.1), η_{ik} and γ_{ik} may be functions of x, t, and all the ρ_{ik}; V, M, r_b, and r_d are given functions of ρ, x, and t.

One last assumption is needed to proceed with the derivation, and that is that there exists a carrying capacity $\rho_\mathrm{c}(x,t)$, i.e.,

$$r_\mathrm{b}(x,t,\rho)-r_\mathrm{d}(x,t,\rho)=\begin{cases}>0 & \text{for } \rho<\rho_\mathrm{c}(x,t),\\ =0 & \text{for } \rho=\rho_\mathrm{c}(x,t),\\ <0 & \text{for } \rho>\rho_\mathrm{c}(x,t).\end{cases} \tag{2.2}$$

The derivation (see, e.g., [7]) of the Fisher equation proceeds from (2.1) without the forcing term $s\rho f_{ik}$. We have added forcing under the following assumptions. The amount of forcing is small and proportional to the total density; hence, the factor s and ρ, respectively. The artificial addition of alleles is then determined by the f_{ik}, which are given functions of x and t.

Summing (2.1) over i and k, we obtain

$$\begin{aligned}&\partial_t\rho-\partial_x(\tfrac{1}{2}\partial_x(V\rho)-M\rho)\\ &\quad=\rho\left[r_\mathrm{b}-r_\mathrm{d}+sr_\mathrm{b}\sum_{i,k=1}^{2}p_ip_k\eta_{ik}-sr_\mathrm{d}\sum_{i,k=1}^{2}p_{ik}\gamma_{ik}\right]+s\rho f,\end{aligned} \tag{2.3}$$

where

$$f = \sum_{i,k=1}^{2} f_{ik}.$$

From (2.1) and (2.3), we have that

$$\begin{aligned}\partial_t p_{ik} &= \frac{1}{\rho}\,\partial_t \rho_{ik} - \frac{p_{ik}}{\rho}\,\partial_t \rho \\ &= \frac{1}{2\rho^2 V}\,\partial_x(V^2\rho^2\partial_x p_{ik}) - M\partial_x p_{ik} + r_{\mathrm{b}}(p_i p_k - p_{ik}) \\ &\quad + s(r_{\mathrm{b}}(p_i p_k \eta_{ik} - p_{ik}\overline{\eta}) - r_{\mathrm{d}} p_{ik}(\gamma_{ik} - \overline{\gamma}) + f_{ik} - p_{ik} f),\end{aligned} \tag{2.4}$$

where

$$\overline{\eta} = \sum_{k,\ell=1}^{2} p_k p_\ell \eta_{k\ell} \quad \text{and} \quad \overline{\gamma} = \sum_{k,\ell=1}^{2} p_{k\ell}\gamma_{k\ell}.$$

Summing (2.4) over k we obtain

$$\begin{aligned}&\partial_t p_i - \frac{1}{2\rho^2 V}\,\partial_x(V^2\rho^2\partial_x p_i) + M\partial_x p_i \\ &\quad = s\left(r_{\mathrm{b}} p_i(\overline{\eta}_i - \overline{\eta}) - r_{\mathrm{d}}\left(\sum_{k=1}^{2} p_{ik}\gamma_{ik} - p_i\overline{\gamma}\right) + f_i - p_i f\right),\end{aligned} \tag{2.5}$$

where

$$\overline{\eta}_i = \sum_{k=1}^{2} p_k \eta_{ik} \quad \text{and} \quad f_i = \sum_{k=1}^{2} f_{ik}.$$

Now, let

$$\tau = s r_0 t, \quad \xi = x\sqrt{\frac{s r_0}{V_0}}, \quad \alpha_{\mathrm{b}} = \frac{r_{\mathrm{b}}}{r_0}, \quad \alpha_{\mathrm{d}} = \frac{r_{\mathrm{d}}}{r_0},$$

$$\Delta = \frac{V}{V_0}, \quad \mu = \frac{M}{\sqrt{s r_0 V_0}}, \quad F_{ik} = \frac{f_{ik}}{r_0}, \quad F_i = \frac{f_i}{r_0}, \quad F = \frac{f}{r_0}.$$

Then, (2.5) with $i = 1$, (2.4) with $i = 1$ and $k = 2$, and (2.3), respectively, become (where $p = p_1$)

$$\partial_\tau p - \frac{1}{2\rho^2\Delta}\,\partial_\xi(\rho^2\Delta^2\partial_\xi p) + \mu\partial_\xi p = h_1 + F_1 - pF, \tag{2.6}$$

$$\begin{aligned}&s\left(\partial_\tau p_{12} - \frac{1}{2\rho^2\Delta}\,\partial_\xi(\rho^2\Delta^2\partial_\xi p_{12}) + \mu\partial_\xi p_{12}\right) \\ &\quad = \alpha_{\mathrm{b}}(p_1 p_2 - p_{12}) + s h_2 + s(F_{12} - p_{12}F)\end{aligned} \tag{2.7}$$

and

$$s(\partial_\tau \rho - \tfrac{1}{2}\partial_\xi^2(\Delta\rho) + \partial_\xi(\mu\rho)) = \rho(\alpha_{\rm b} - \alpha_{\rm d}) + sh_3 + s\rho F, \tag{2.8}$$

where h_i, $i = 1, 2, 3$ are specific functions of ξ, τ, p_{ik}, and ρ. Setting $s = 0$ in (2.7) yields $p_{12} = p_1 p_2$; setting $s = 0$ in (2.8) yields $\alpha_{\rm b} = \alpha_{\rm d}$ so that $r_{\rm b} = r_{\rm d}$. Then, from (2.2), $\rho = \rho_{\rm c}$ and then (2.6) becomes

$$\partial_\tau p - \frac{1}{2\rho_{\rm c}^2 \Delta}\,\partial_\xi(\rho_{\rm c}^2 \Delta^2 \partial_\xi p) + \mu\partial_\xi p = h_1 + F_1 - pF,$$

where

$$\begin{aligned} h_1 &= \alpha_{\rm b} p \left(\sum_{k=1}^{2} p_k \eta_{1k} - \sum_{k,\ell=1}^{2} p_k p_\ell \eta_{kl} \right) - \alpha_{\rm d} \left(\sum_{k=1}^{2} p_{ik}\gamma_{ik} - p \sum_{k,\ell=1}^{2} p_{kl}\gamma_{kl} \right) \\ &= \alpha_{\rm b} p(1-p)(p(\omega_{11} - \omega_{21}) + (1-p)(\omega_{12} - \omega_{22})) \end{aligned}$$

and where

$$\omega_{ik} = \eta_{ik} - \gamma_{ik}, \quad i, k = 1, 2.$$

To summarize what has transpired so far, we have obtained the scalar equation

$$\begin{aligned} &\partial_\tau p - \frac{1}{2\rho_{\rm c}^2 \Delta}\,\partial_\xi(\rho_{\rm c}^2 \Delta^2 \partial_\xi p) + \mu\partial_\xi p \\ &\quad = \alpha_{\rm b} p(1-p)(p(\omega_{11} - \omega_{21}) + (1-p)(\omega_{12} - \omega_{22})) + F_1 - pF. \end{aligned} \tag{2.9}$$

Note that the nonlinearity is *cubic*.

To obtain the (unforced) Fisher equation from (2.9) (with $F_1 = F = 0$, of course) one has to also assume that

- all data ($\rho_{\rm c}$, $\alpha_{\rm b}$, Δ, ω_{ik}) are constant (independent of *x* and *t*);
- $\mu = 0$;
- $\omega_{12} + \omega_{21} = \omega_{11} + \omega_{22}$ (this choice gets rid of the cubic term);
- $y = \sqrt{(2/\Delta)}\xi$;
- $\sigma = \alpha_{\rm b}(\omega_{12} - \omega_{22})$.

Making the same assumptions in the forced case leads to the *forced Fisher equation*

$$\partial_\tau p - \partial_{yy} p = \sigma p(1-p) + F_1 - pF. \tag{2.10}$$

2.2. Boundary conditions

Suppose first that *Dirichlet boundary conditions* are prescribed on the densities, i.e.,

$$\rho_{ik} = g_{ik}, \quad i, k = 1, 2 \quad \text{on the boundary.}$$

Then, it follows that

$$p = p_1 = \sum_{k=1}^{2} p_{1k} = \frac{1}{\rho} \sum_{k=1}^{2} \rho_{1k} = \frac{\sum_{k=1}^{2} g_{1k}}{\sum_{\ell,k=1}^{2} g_{\ell k}} \quad \text{on the boundary.}$$

Homogeneous boundary conditions on the densities ρ_{ik} seem to be a problem since we then get the indeterminate form $p = 0/0$. On the other hand, we could impose data such that $\sum_{k=1}^{2} g_{1k} = 0$ but $\rho = \sum_{\ell,k=1}^{2} g_{\ell k} \neq 0$ on the boundary; in this case we obtain the boundary condition $p = 0$ on the boundary.

Suppose now that *Neumann boundary conditions* are prescribed on the densities, i.e.,

$$\partial_n \rho_{ik} = q_{ik}, \quad i, k = 1, 2 \quad \text{on the boundary,}$$

so that

$$\partial_n p_{ik} = \frac{1}{\rho} \partial_n \rho_{ik} - \frac{p_{ik}}{\rho} \partial_n \rho,$$

so that

$$\begin{aligned} \partial_n p = \partial_n p_1 = \sum_{k=1}^{2} \partial_n p_{1k} &= \frac{1}{\rho} \sum_{k=1}^{2} \partial_n \rho_{1k} - \frac{\sum_{k=1}^{2} p_{1k}}{\rho} \partial_n \rho \\ &= \frac{1}{\rho} \sum_{k=1}^{2} q_{1k} - \frac{p}{\rho} \sum_{\ell,k=1}^{2} q_{\ell k}. \end{aligned}$$

In general, ρ is not known on the boundary. However, we know that in the limit $s \to 0$ that $\rho = \rho_c$ so that we obtain the *Robin condition* for p

$$\partial_n p + \left(\frac{1}{\rho_c} \sum_{\ell,k=1}^{2} q_{\ell k} \right) p = \frac{1}{\rho_c} \sum_{k=1}^{n} q_{1k}.$$

Homogeneous Neumann boundary conditions for ρ_{ik} are permissible as long as $\rho_c \neq 0$ on the boundary; we then obtain that $\partial_n p = 0$ on the boundary.

3. Problem statement and preliminaries

In this paper, we consider the initial-boundary value problem with homogeneous Dirichlet boundary conditions for the special case of (2.10) in which $F_1 = F$. Note that this implies that $f_{21} + f_{22} = 0$. This can be realized, e.g., if no artificial additions of the allele A_2 is effected. Eq. (2.10) reduces to

$$\partial_\tau p - \partial_{yy} p = \sigma p(1 - p) + F_1(1 - p). \tag{3.1}$$

Evidently, other models can be derived by choosing different values for ω_{ik} and the other data.

The problem we wish to study is a multidimensional version of the forced Fisher equation (3.1) with a Dirichlet boundary condition (we use u in place of p), i.e.,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \lambda u(1-u) + f(t,\mathbf{x})(1-u) & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0,\mathbf{x})u_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{3.2}$$

where $Q_T = (0,T) \times \Omega$ and $\Sigma_T = (0,T) \times \partial\Omega$. In this section, we will introduce function spaces and introduce a weak formulation of (3.2). We will also quote some relevant ODE results.

3.1. Function spaces and problem statement

In the sequel, Ω is a bounded, open bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, with a smooth boundary $\partial\Omega$. $H^s(\Omega)$ for $s \in \mathbb{R}$ denotes the standard real Sobolev space of order s with its norm denoted by $\|\cdot\|_{H^s(\Omega)}$. We use the convention $H^0(\Omega) = L^2(\Omega)$.

For a $p \in [1,\infty]$, an open interval $(a,b) \subset \mathbb{R}$ and a Banach space B with the norm $\|\cdot\|_B$, we denote by $L^p(a,b;B)$ the set of measurable functions $u : (a,b) \to B$ such that $t \to \|u(t)\|_B$ belongs to $L^p(a,b)$. The norm on $L^p(a,b;B)$ is defined by

$$\|u\|_{L^p(a,b;B)} = \begin{cases} \left(\int_a^b \|u(t)\|_B^p \,\mathrm{d}t\right)^{1/p} & \text{if } p < \infty, \\ \operatorname{Ess} \sup\limits_{t\in(a,b)} \|u(t)\|_B & \text{if } p = \infty. \end{cases}$$

We denote by $C([a,b];B)$ the set of continuous functions $u : [a,b] \to B$ with the norm $\|u\|_{C([a,b];B)} = \max_{t\in[a,b]} \|u(t)\|_B$. We introduce

$$W(a,b) = \left\{ u | u \in L^2(a,b;H_0^1(\Omega)),\ \frac{\mathrm{d}u}{\mathrm{d}t} = u' \in L^2(a,b;H^{-1}(\Omega)) \right\},$$

where u' is taken in the sense of distribution. The norm on $W(a,b)$ is defined by

$$\|u\|_{W(a,b)} = (\|u\|^2_{L^2(a,b;H_0^1(\Omega))} + \|u'\|^2_{L^2(a,b;H^{-1}(\Omega))})^{1/2} \quad \forall u \in W(a,b).$$

The duality paring between a Banach space B and its dual will be denoted by $\langle \cdot,\cdot \rangle$. The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$, i.e., for $p, q \in L^2(\Omega)$, $(p,q) = \int_\Omega p\,q \,\mathrm{d}\Omega$. Also, C, $\tilde{C}$, $\tilde{\tilde{C}}$, C_1, C_2, etc. denote positive constants whose values change with context.

A solution for (3.2) is defined as a solution of the following weak formulation: given $y_0 \in L^2(\Omega)$ and $f \in L^\beta(0,T;L^\gamma(\Omega)$ where $\beta \in [1,\infty)$ and $\gamma > \beta'$ in $\mathbb{R}^2$ and $\gamma > 3\beta'/2$ in $\mathbb{R}^3$, find a $u \in W(0,T)$ such that

$$\langle u'(t), \phi \rangle + (\nabla u(t), \nabla \phi) = \langle \lambda u(t)(1-u(t)) + f(t)(1-u(t)), \phi \rangle \quad \forall \phi \in H_0^1(\Omega),\ \text{a.e. } t \in (0,T) \tag{3.3}$$

and

$$u(0,\mathbf{x}) = u_0(\mathbf{x}). \tag{3.4}$$

Note that the initial condition and the term $\langle fu, \phi\rangle$ make sense thanks to the following two lemmas.

Lemma 3.1. *Let V, H, V' be three Hilbert spaces such that $V \subset H = H' \subset V'$, where V' is the dual of V. If a function u satisfies that $u \in L^2(t_0, t_1; V)$ and $u' \in L^2(t_0, t_1; V')$, then u is almost everywhere equal to a function continuous from $[t_0, t_1]$ into H and we have the following equality, which holds in the scalar distributional sense on (t_0, t_1):*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_H^2 = 2\langle u', u\rangle.$$

Proof. See [25, p. 69]. □

Lemma 3.2. *Suppose $u \in W(0, T)$ and $f \in L^{\beta}(0, T; L^{\gamma}(\Omega))$ with $\beta \in [1, \infty)$ and γ satisfying*

$$\gamma \geqslant \frac{3\beta'}{2} \text{ when } \Omega \subset \mathbb{R}^3 \quad \text{and} \quad \gamma > \beta' \text{ when } \Omega \subset \mathbb{R}^2, \quad \text{where } \beta' = \beta/(\beta - 1).$$

Then $(fu) \in L^2(0, T; H^{-1}(\Omega))$.

Proof. See [15, p. 26]. □

3.2. Relevant ODE results

In this subsection, we recall some relevants results concerning the existence of and estimates for solutions of ODE systems.

The first lemma is about the existence of a maximal solution on a rectangular region.

Lemma 3.3. *Let real numbers $t_0, y_0, a > 0, b > 0$ be given and set*

$$R_0 = [t_0, t_0 + a][y_0 - b, y_0 + b].$$

Assume that $g \in C(R_0; \mathbb{R})$. Denote

$$M = \max_{(t,y)\in R_0} |g(t, y)| \quad \text{and} \quad \tau = \min\left\{a, \frac{b}{2M + b}\right\}.$$

Then the initial value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} = g(t, y), \qquad y(t_0) = u_0 \tag{3.5}$$

has a maximal solution $\overline{y}(t)$ on $[t_0, t_0 + \tau]$, i.e., every solution $y = y(t)$ of

$$y' = g(t, y) \quad \text{on } [t_0, t_0 + \tau], \qquad y(t_0) = u_0$$

satisfies $y(t) \leqslant \overline{u}(t)$ on $[t_0, t_0 + \tau]$.

Proof. See [10, p. 10, Theorem 2.1] and [10, p. 25, Lemma 2.1]. □

The second lemma is about the existence of a maximal solution on a strip.

Lemma 3.4. *Let* $t_0, y_0, a > 0$ *be given. Assume that* $g \in C([t_0, t_0+a] \times \mathbb{R}; \mathbb{R})$. *Then there exists a* $\tau > 0$ *such that* (3.5) *has a maximal solution* $\overline{u}(t)$ *on* $[t_0, t_0 + \tau]$, *i.e., every solution* $y = y(t)$ *of*

$$y' = g(t, y) \quad \text{on } [t_0, t_0 + \tau], \qquad y(t_0) \leqslant y_0$$

satisfies $y(t) \leqslant \overline{y}(t)$ *on* $[t_0, t_0 + \tau]$.

Proof. This is a direct consequence of the previous lemma (e.g., by fixing a finite rectangle on the strip). □

The third lemma is about the extension of solutions of a system of ODEs over a maximal interval of existence.

Lemma 3.5. *Assume that* $\mathbf{G} \in C([t_0, t_0 + a] \times \mathbb{R}^d; \mathbb{R}^d)$ *where* $d \geqslant 1$ *is an integer. Let* $\mathbf{y} = \mathbf{y}(t)$ *be a solution, on a right maximal interval J, of* $\mathbf{y}' = \mathbf{G}(t, \mathbf{y})$ *with a given* $\mathbf{y}(t_0)$. *Then either* $J = [t_0, t_0 + a]$ *or* $J = [t_0, t_0 + \delta)$, $\delta < a$, *and* $\|\mathbf{y}(t)\|_{\mathbb{R}^d} \to \infty$ *as* $t \to t_0 + \delta$.

Proof. See [10, p. 14, Corollary 3.1]. □

The next lemma is about an estimate, in terms of the maximal solution, for solutions of an integral inequality.

Lemma 3.6. *Assume that* $g \in C([t_0, t_0 + a] \times \mathbb{R}; \mathbb{R})$, $g(t, y)$ *is nondecreasing in u for each* $t \in [t_0, t_0 + a]$, *and a maximal solution* $\overline{y}(t)$ *of* (3.5) *exists on* $[t_0, t_0 + a]$. *Assume further that* $h \in C([t_0, t_0 + a]; \mathbb{R})$, $h(t_0) \leqslant u_0$, *and*

$$h(t) \leqslant h(t_0) + \int_{t_0}^{t} g(s, h(s))\,\mathrm{d}s \quad \forall t \in [t_0, t_0 + a).$$

Then

$$h(t) \leqslant \overline{u}(t) \quad \forall t \in [t_0, t_0 + a).$$

Proof. See [10, p. 29, Corollary 4.4]. □

4. Local existence

In this section, we study the existence of a local solution for the weak formulation (3.3) and (3.4). We first recall the following compact embedding result:

Lemma 4.1. *Let X_0, X, X_1 be three Banach Spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$ and $X_0 \hookrightarrow\hookrightarrow X$, let $1 < p, q < \infty$ and*

$$W = \left\{ v(t) \in L^p(t_0, t_1; X_0) : \frac{\mathrm{d}u(t)}{\mathrm{d}t} \in L^q(t_0, t_1; X_1) \right\}.$$

Then $W \hookrightarrow\hookrightarrow L^p(t_0, t_1; X)$.

Proof. See [14, Chapter 1, Theorem 5.1]. □

We are now in a position to state and prove the local existence results.

Theorem 4.1. *Assume that $f \in C(0, T; L^3(\Omega))$, $b_0 \in L^2(\Omega)$ and $t_0 \in [0, T)$. Then there exists a $t_1 = t_1(t_0, b_0) \in (t_0, T)$ and a $u \in W(t_0, t_1)$ satisfying*

$$\begin{aligned} &\langle u'(t), \phi\rangle + (\nabla u(t), \nabla \phi) \\ &\quad = \langle \lambda u(t)(1 - u(t)) + f(t)(1 - u(t)), \phi\rangle \quad \forall \phi \in H_0^1(\Omega),\ a.e.\ t \in (t_0, t_1) \end{aligned} \tag{4.1}$$

and

$$u(t_0) = b_0. \tag{4.2}$$

Proof. We divide the proof into three steps. In Step 1 Faedo–Galerkin approximations are defined. In Step 2 a priori estimates for u_m are derived. In Step 3 passage to limits, derive the regularity of u' and justify the initial condition.

Step 1: *Faedo–Galerkin approximations*: Since $H_0^1(\Omega)$ is separable, there exists a basis $\{w_i\}_{i=1}^{\infty}$ for $H_0^1(\Omega)$. For each m, we define an approximate solution u_m of (4.1) as follows:

$$u_m = \sum_{j=1}^{m} g_j^{(m)}(t) w_j \tag{4.3}$$

such that

$$\begin{aligned} &\langle u_m'(t), w_i\rangle + (\nabla u_m(t), \nabla w_i) \\ &\quad = \langle \lambda u_m(t)[1 - u_m(t)] + f(t)[1 - u_m(t)], w_i\rangle, \quad \text{a.e. } t \in (t_0, t_1),\ i = 1, 2, \ldots, m \end{aligned} \tag{4.4}$$

and

$$u_m(t_0) = b_0^{(m)}, \tag{4.5}$$

where $b_0^{(m)} = \sum_{j=1}^{m} \xi_j^{(m)} w_j$ is the $L^2(\Omega)$ projection of b_0 onto the span of $\{w_1, w_2, \ldots, w_m\}$. Properties of projection operators imply

$$\|b_0^{(m)}\|_{L^2(\Omega)} \leqslant \|b_0\|_{L^2(\Omega)}. \tag{4.6}$$

Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$ and $\{w_i\}$ is a basis for $H_0^1(\Omega)$, we easily deduce that

$$b_0^{(m)} \to b_0 \text{ in } L^2(\Omega) \quad \text{as } m \to \infty. \tag{4.7}$$

The system of nonlinear differential equations (4.4) and (4.5) can be rewritten as

$$\sum_{j=1}^{m}(w_j, w_i)\frac{\mathrm{d}}{\mathrm{d}t}g_j^{(m)}(t)+\sum_{j=1}^{m}(\nabla w_j, \nabla w_i)g_j^{(m)}(t)$$
$$=\lambda\left\langle\sum_{j=1}^{m}g_j^{(m)}(t)w_j-\sum_{j,k=1}^{m}g_j^{(m)}(t)g_k^{(m)}(t)w_jw_k, w_i\right\rangle$$
$$+\left\langle f(t)\left(1-\sum_{j=1}^{m}g_j^{(m)}(t)w_j\right), w_i\right\rangle, \quad i=1,\ldots,m \tag{4.8}$$

and

$$\sum_{j=1}^{m}(w_j, w_i)g_j^{(m)}(t_0)=(b_0, w_i), \quad i=1,\ldots,m. \tag{4.9}$$

The linear independency of $\{w_i\}_{i=1}^m$ as functions implies that the matrix with entries (w_j, w_i) is nonsingular, so that we may use the inverse of this matrix to reduce (4.8) and (4.9) to the following standard form of a system of ODEs:

$$\frac{\mathrm{d}}{\mathrm{d}t}g_i^{(m)}(t)+[f(t)-\lambda]g_i^{(m)}(t)+\sum_{j=1}^{m}\alpha_{ij}g_j^{(m)}(t)+\sum_{j,k=1}^{m}\beta_{ijk}\,g_j^{(m)}(t)g_k^{(m)}(t)$$
$$=\sum_{j=1}^{m}\gamma_{ij}\langle f(t), w_j\rangle, \quad i=1,\ldots,m \tag{4.10}$$

and

$$g_i^{(m)}(t_0)=\xi_i, \quad i=1,\ldots,m, \tag{4.11}$$

for $i=1,2,\ldots,m$, where $\alpha_{ij}, \beta_{ijk}, \gamma_{ij}, \xi_i \in \mathbb{R}$ and they depend on $\{w_i\}_{i=1}^m$. Note that the regularity assumption on f guarantees $\langle f(\cdot,t), w_j\rangle \in C([0,T];\mathbb{R})$. From standard theories of ordinary differential equations (see, e.g., [10] or [19]), the nonlinear differential system (4.10) and (4.11) has a solution defined on a maximal right interval $[t_0, \tau^{(m)}]$. Or equivalently, system (4.4) and (4.5) has a solution $u_m(t)$ defined on a maximal interval $[t_0, \tau^{(m)}]$.

Step 2: *A priori estimates for* $u_m(t)$: For $t \in [t_0, \tau^{(m)}]$, multiplying system (4.4) by $g_i^{(m)}(t)$, $i=1,\ldots,m$, and adding these equations up we obtain

$$\langle u_m'(t), u_m(t)\rangle+\|\nabla u_m(t)\|_{L^2(\Omega)}^2=\langle[\lambda u_m(t)+f(t)][1-u_m(t)], u_m(t)\rangle. \tag{4.12}$$

Thus, applying Young's inequality to the terms on the right-hand side we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_m(t)\|_{L^2(\Omega)}^2+\|\nabla u_m(t)\|_{L^2(\Omega)}^2$$
$$\leqslant\lambda\|u_m(t)\|_{L^2(\Omega)}^2+\lambda\|u_m(t)\|_{L^3(\Omega)}^3+\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^{3/2}(\Omega)}$$
$$+\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^2(\Omega)}\|u_m(t)\|_{L^6(\Omega)}. \tag{4.13}$$

The embedding $H_0^1(\Omega) \hookrightarrow L^{3/2}(\Omega)$ and Poincaré's inequality imply that

$$\begin{aligned}\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^{3/2}(\Omega)} &\leqslant C\,\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{H_0^1(\Omega)}\\ &\leqslant \tilde{C}\,\|f(t)\|_{L^3(\Omega)}\|\nabla u_m(t)\|_{L^2(\Omega)}\\ &\leqslant \frac{C_1}{\varepsilon}\,\|f(t)\|^2_{L^3(\Omega)} + \varepsilon\,\|\nabla u_m(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.14}$$

The embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$ and Poincaré's inequality imply

$$\begin{aligned}&\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^2(\Omega)}\|u_m(t)\|_{L^6(\Omega)}\\ &\quad\leqslant C\,\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^2(\Omega)}\|u_m(t)\|_{H_0^1(\Omega)}\\ &\quad\leqslant \tilde{C}\,\|f(t)\|_{L^3(\Omega)}\|u_m(t)\|_{L^2(\Omega)}\|\nabla u_m(t)\|_{L^2(\Omega)}\\ &\quad\leqslant \frac{C_2}{\varepsilon}\,\|f(t)\|^2_{L^3(\Omega)}\|u_m(t)\|^2_{L^2(\Omega)} + \varepsilon\,\|\nabla u_m(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.15}$$

Moreover, since $H^{1/2}(\Omega) \hookrightarrow L^3(\Omega)$, we may use the interpolation between $L^2(\Omega)$ and $H_0^1(\Omega)$ and Poincaré's and Young's inequalities to derive

$$\begin{aligned}\lambda\|u_m(t)\|^3_{L^3(\Omega)} &\leqslant \lambda(C\,\|u_m(t)\|_{H^{1/2}(\Omega)})^3 \leqslant \lambda(\tilde{C}\,\|u_m(t)\|^{1/2}_{L^2(\Omega)}\|u_m(t)\|^{1/2}_{H_0^1(\Omega)})^3\\ &\leqslant \lambda(\tilde{\tilde{C}}\|u_m(t)\|^{3/2}_{L^2(\Omega)}\|\nabla u_m(t)\|^{3/2}_{L^2(\Omega)})\\ &\leqslant \lambda\,\frac{C_3}{\varepsilon}\,\|u_m(t)\|^6_{L^2(\Omega)} + \varepsilon\|\nabla u_m(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.16}$$

Adding up (4.14)–(4.16) and then applying the resulting inequality with $\varepsilon = 1/6$ to (4.13), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\,\|u_m(t)\|^2_{L^2(\Omega)} + \|\nabla u_m(t)\|^2_{L^2(\Omega)}\\ &\quad\leqslant 2\lambda\|u_m(t)\|^2_{L^2(\Omega)} + 2C_1\|f(t)\|^2_{L^3(\Omega)} + 2C_2\|f(t)\|^2_{L^3(\Omega)}\|u_m(t)\|^2_{L^2(\Omega)}\\ &\qquad + 2\lambda C_3\|u_m(t)\|^6_{L^2(\Omega)}. \end{aligned} \tag{4.17}$$

Integrating (4.17) from t_0 to t, where $t \in (t_0, \tau^{(m)})$, we are led to

$$\begin{aligned}&\|u_m(t)\|^2_{L^2(\Omega)} - \|u_m(t_0)\|^2_{L^2(\Omega)} + \int_{t_0}^{t} \|\nabla u_m(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s\\ &\quad\leqslant \int_{t_0}^{t}[2\lambda\|u_m(s)\|^2_{L^2(\Omega)} + 2C_1\|f(s)\|^2_{L^3(\Omega)}]\,\mathrm{d}s\\ &\qquad + \int_{t_0}^{t}[2C_2\|f(s)\|^2_{L^3(\Omega)}\|u_m(s)\|^2_{L^2(\Omega)} + 2\lambda C_3\|u_m(s)\|^6_{L^2(\Omega)}]\,\mathrm{d}s, \end{aligned} \tag{4.18}$$

Setting $y_m(t) = \|u_m(t)\|^2_{L^2(\Omega)}$ and $\gamma(t) = \|f(t)\|^2_{L^3(\Omega)}$ we have

$$y_m(t) \leqslant y_m(t_0) + \int_{t_0}^{t} g(s, y_m(s))\,\mathrm{d}s \quad \forall t \in (t_0, \tau^{(m)}), \tag{4.19}$$

where

$$g(t, y) = 2\lambda C_3 y^3 + 2\lambda y + 2C_2\gamma(t)y + 2C_1\gamma(t).$$

It is easily verified that $g \in C([0, T] \times \mathbb{R}; \mathbb{R})$ and g is nondecreasing in y for each t.

Now, let $\overline{y}(t)$ be the maximal solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2\lambda C_3 y^3 + 2\lambda y + 2C_2\gamma(t)u + 2C_1\gamma(t) \tag{4.20}$$

with the initial value

$$y(t_0) = \|y(t_0)\|^2_{L^2(\Omega)} \tag{4.21}$$

which, according to Lemma 3.4, exists on an interval $J = [t_0, t_1]$ for some $t_1 \in (t_0, T]$ (note that system (4.20) and (4.21) and t_1 are independent of m, though it depends on the norm of f.) Inequality (4.6) implies $y_m(0) \leqslant \overline{y}(t_0)$. Hence by Lemma 3.6 with $a = \tau^{(m)} - t_0$, we have

$$\|u_m(t)\|^2_{L^2(\Omega)} = u_m(t) \leqslant \overline{u}(t) \leqslant \max_{t\in[t_0,\tau]} \overline{u}(t) \equiv C(t_1) \quad \forall t \in (t_0, \tau^{(m)}). \tag{4.22}$$

As $[t_0, \tau^{(m)})$ is the maximal interval of existence for (4.19), Lemma 3.5 and (4.22) implies that $\tau^{(m)} = t_1$ and the existence interval is $[t_0, t_1]$. Hence, using (4.22) again we have

$$\sup_{t\in[t_0,t_1]} \|u_m(t)\|^2_{L^2(\Omega)} \leqslant C(t_1) \quad \forall t \in [t_0, t_1] \quad \forall m = 1, 2, \ldots\ . \tag{4.23}$$

Thus, we have shown that $\exists t_1 \in (t_0, T]$ such that

$$\{u_m\}_{m=1}^{\infty} \text{ belongs to a bounded set of } L^\infty(t_0, t_1; L^2(\Omega)). \tag{4.24}$$

Using (4.18) again, we see that relations (4.6), (4.24) yield

$$\{u_m\}_{m=1}^{\infty} \text{ belongs to a bounded set of } L^2(t_0, t_1; H^1_0(\Omega)). \tag{4.25}$$

Step 3: *Passage to limits*: A priori estimates (4.24) and (4.25) allow us to draw a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) such that

$$u_m \rightharpoonup u \quad \text{weak* in } L^\infty(t_0, t_1; L^2(\Omega)) \tag{4.26}$$

and

$$u_m \rightharpoonup u \quad \text{weakly in } L^2(t_0, t_1; H^1_0(\Omega)) \tag{4.27}$$

for some $u \in L^\infty(t_0, t_1; L^2(\Omega)) \cap L^2(t_0, t_1; H^1_0(\Omega))$. Furthermore, Lemma 3.1 with $X_0 = H^1_0(\Omega)$, $X = L^2(\Omega)$ and $X_1 = H^{-1}(\Omega)$ implies that

$$u_m \to u \quad \text{strongly in } L^2(t_0, t_1; L^2(\Omega)). \tag{4.28}$$

Now, let $\psi(t) \in C^1([t_0, t_1]; \mathbb{R})$ with $\psi(t_1) = 0$ be given. Multiplying (4.4) by $\psi(t)$ and integrating by parts, we are led to

$$\begin{aligned} &-(b_0^{(m)}, w_i)\psi(t_0) - \int_{t_0}^{t_1} \langle u_m(t), \psi'(t) w_i \rangle \, \mathrm{d}t + \int_{t_0}^{t_1} (\nabla u_m(t), \psi(t) \nabla w_i) \, \mathrm{d}t \\ &\quad = \int_{t_0}^{t_1} \langle \lambda u_m(t)(1 - u_m(t)) + f(t)(1 - u_m(t)), \psi(t) w_i \rangle \, \mathrm{d}t. \end{aligned} \tag{4.29}$$

Relations (4.26), (4.27) and (4.7) imply that as $m \to \infty$,

$$\int_{t_0}^{t_1} (u_m(t), \psi'(t) w_i) \, \mathrm{d}t \longrightarrow \int_{t_0}^{t_1} (u(t), \psi'(t) w_i) \, \mathrm{d}t, \tag{4.30}$$

$$\int_{t_0}^{t_1} (\nabla u_m(t), \psi(t) \nabla w_i) \, \mathrm{d}t \longrightarrow \int_{t_0}^{t_1} (\nabla u(t), \psi(t) \nabla w_i) \, \mathrm{d}t, \tag{4.31}$$

$$\int_{t_0}^{t_1} \langle u_m(t), \psi(t) w_i \rangle \, \mathrm{d}t \longrightarrow \int_{t_0}^{t_1} \langle u(t), \psi(t) w_i \rangle \, \mathrm{d}t, \tag{4.32}$$

$$(b_0^{(m)}, w_i)\psi(t_0) \longrightarrow (b_0, w_i)\psi(t_0). \tag{4.33}$$

As for the nonlinear term, using the generalized Hölder's inequality and the facts that w_i is a spatial function and that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned} &\left| \int_{t_0}^{t_1} \langle u_m^2(t) - u^2(t), \psi(t) w_i \rangle \, \mathrm{d}t \right| \\ &\quad \leqslant \int_{t_0}^{t_1} |\psi(t)| \int_{\Omega} |u_m(t, \mathbf{x}) - u(t, \mathbf{x})| \cdot |u_m(t, \mathbf{x}) + u(t, \mathbf{x})| \cdot |w_i(\mathbf{x})| \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \\ &\quad \leqslant \int_{t_0}^{t_1} |\psi(t)| \, \|u_m(t) - u(t)\|_{L^2(\Omega)} \, \|u_m(t) + u(t)\|_{L^4(\Omega)} \, \|w_i\|_{L^4(\Omega)} \, \mathrm{d}t \\ &\quad \leqslant C \|\psi\|_{L^\infty(t_0, t_1)} \|w_i\|_{H_0^1(\Omega)} \int_{t_0}^{t_1} \|u_m(t) - u(t)\|_{L^2(\Omega)} \|u_m(t) + u(t)\|_{H_0^1(\Omega)} \, \mathrm{d}t \\ &\quad \leqslant C \|\psi\|_{L^\infty(t_0, t_1)} \|w_i\|_{H_0^1(\Omega)} \|u_m - u\|_{L^2(t_0, t_1; L^2(\Omega))} \|u_m + u\|_{L^2(t_0, t_1; H_0^1(\Omega))}. \end{aligned} \tag{4.34}$$

From (4.25), (4.28) and (4.34), we easily deduce that as $m \to \infty$,

$$\int_{t_0}^{t_1} \langle u_m^2(t), \psi(t) w_i \rangle \, \mathrm{d}t \longrightarrow \int_{t_0}^{t_1} \langle u^2(t), \psi(t) w_i \rangle \, \mathrm{d}t. \tag{4.35}$$

Similarly, we have

$$\begin{aligned}&\left|\int_{t_0}^{t_1}\langle f(t)(u_m(t)-u(t)),\psi(t)w_i\rangle\,\mathrm{d}t\right|\\&\quad\leqslant\int_{t_0}^{t_1}|\psi(t)|\,\|u_m(t)-u(t)\|_{L^2(\Omega)}\,\|f(t)\|_{L^3(\Omega)}\,\|w_i\|_{L^6(\Omega)}\,\mathrm{d}t\\&\quad\leqslant C\|\psi\|_{L^\infty(t_0,t_1)}\|w_i\|_{H_0^1(\Omega)}\|u_m-u\|_{L^2(t_0,t_1;L^2(\Omega))}\|f\|_{L^2(t_0,t_1;L^3(\Omega))},\end{aligned}$$

so that

$$\int_{t_0}^{t_1}\langle f(t)u_m(t),\psi(t)w_i\rangle\,\mathrm{d}t\to\int_{t_0}^{t_1}\langle f(t)u(t),\psi(t)w_i\rangle\,\mathrm{d}t\quad\text{as } m\to\infty.\tag{4.36}$$

Relations (4.30), (4.31), (4.33), (4.35) and (4.36) allow us to pass to the limits in (4.4) to obtain

$$\begin{aligned}&-(b_0,w_i)\psi(t_0)-\int_{t_0}^{t_1}\langle u(t),\psi'(t)w_i\rangle\,\mathrm{d}t+\int_{t_0}^{t_1}(\nabla u(t),\psi(t)\nabla w_i)\,\mathrm{d}t\\&\quad=\int_{t_0}^{t_1}\langle\lambda u(t)(1-u(t))+f(t)(1-u(t)),\psi(t)w_i\rangle\,\mathrm{d}t\end{aligned}\tag{4.37}$$

for each $i=1,2,\ldots$. Using the linearity in w_i of (4.37) and the fact that $\{w_i\}$ is total in $H_0^1(\Omega)$ we deduce that

$$\begin{aligned}&-(b_0,\phi)\psi(t_0)-\int_{t_0}^{t_1}\langle u(t),\psi'(t)\phi\rangle\,\mathrm{d}t+\int_{t_0}^{t_1}(\nabla u(t),\psi(t)\nabla\phi)\,\mathrm{d}t\\&\quad=\int_{t_0}^{t_1}\langle\lambda u(t)(1-u(t))+f(t)(1-u(t)),\psi(t)\phi\rangle\,\mathrm{d}t\quad\forall\phi\in H_0^1(\Omega).\end{aligned}\tag{4.38}$$

In particular, (4.38) holds for all $\psi\in\mathscr{D}(0,T)$ so that u satisfies

$$\begin{aligned}&\langle u'(t),\phi\rangle+(\nabla u(t),\nabla\phi)\\&\quad=\langle u(t)(1-u(t))+f(t)(1-u(t)),\phi\rangle\quad\forall\phi\in H_0^1(\Omega).\end{aligned}\tag{4.39}$$

in the sense of distributions (in time). By [25, p. 250, Lemma 1.1] we have that

$$u'\in L^2(0,T;H^{-1}(\Omega)).$$

Finally, it remains to prove that u satisfies the initial condition $u(t_0)=b_0$. To this end, we multiply (4.39) by $\psi(t)$ and integrate by parts. This leads us to

$$\begin{aligned}&-(u_{t_0},\phi)\psi(t_0)-\int_{t_0}^{t_1}\langle u(t),\psi'(t)\phi\rangle\,\mathrm{d}t+\int_{t_0}^{t_1}(\nabla u(t),\psi(t)\nabla\phi)\,\mathrm{d}t\\&\quad=\int_{t_0}^{t_1}\langle\lambda u(t)(1-u(t))+f(t)(1-u(t)),\psi(t)\phi\rangle\,\mathrm{d}t\quad\forall\phi\in H_0^1(\Omega).\end{aligned}\tag{4.40}$$

A comparison of (4.38) and (4.40) yields

$$(b_0 - u(t_0), \phi)\psi(t_0) = 0 \quad \forall \phi \in H_0^1(\Omega).$$

Upon choosing a ψ with $\psi(t_0) = 1$, we have

$$(b_0 - u(t_0), \phi) = 0 \quad \forall \phi \in H_0^1(\Omega),$$

i.e. $b_0 = u(t_0)$ a.e. Ω. This completes the proof of Theorem 4.1. □

5. Maximum principles

Let u be a local solution, i.e., a solution of (4.1) and (4.2), that was guaranteed to exist on an interval (t_0, t_1) in the previous section. We are going to show that under suitable assumptions on f and u_0, the solution u satisfies $0 \leqslant u \leqslant 1$. This result is needed to ensure the solution to be physically meanful—recall that y represents the gene frequency. This result will also allow us to show the global existence in the next section.

We first recall the well-known Gronwall Lemma:

Lemma 5.1. *Assume that w, z are nonnegative, continuous functions on $[a, b]$; $K \geqslant 0$ is a constant; and*

$$z(t) \leqslant K + \int_a^t z(s)w(s)\,\mathrm{d}s \quad \forall t \in [a, b].$$

Then

$$z(t) \leqslant K \exp\left\{\int_a^t z(s)w(s)\,\mathrm{d}s\right\} \quad \forall t \in [a, b],$$

in particular, if $K = 0$, then $z(t) \equiv 0$.

Proof. See, e.g., [11, p. 24]. □

Now we prove the nonnegativity of the local solution u when f and u_0 are nonnegative.

Theorem 5.1. *Assume that $f \in C(0, T; L^3(\Omega))$, $f \geqslant 0$ almost everywhere on Q_T, $b_0 \in L^2(\Omega)$ and $b_0(\mathbf{x}) \geqslant 0$ almost everywhere in Ω. If $u(t, \mathbf{x})$ is a solution of* (4.1) and (4.2) *where $t_0 \in [0, T)$ and $t_1 \in (t_0, T)$, then $u \geqslant 0$ almost everywhere in $(t_0, t_1) \times \Omega$.*

Proof. For almost every $t \in [t_0, t_1]$, we set $\phi(\mathbf{x}) = u_-(t, \mathbf{x}) = \max\{-u(t, \mathbf{x}), 0\}$ in (4.1) and integrate in t to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_-(t)\|^2_{L^2(\Omega)} + \|\nabla u_-(t)\|^2_{L^2(\Omega)} \\
&\quad = \int_\Omega \lambda[u_-(t)]^2[1 + u_-(t)]\,\mathrm{d}\mathbf{x} - \int_\Omega u_-(t) f(t)[1 + u_-(t)]\,\mathrm{d}\mathbf{x} \\
&\quad \leqslant \int_\Omega \lambda[u_-(t)]^2[1 + u_-(t)]\,\mathrm{d}\mathbf{x} \leqslant \lambda(\|u_-(t)\|^2_{L^2(\Omega)} + \|u_-(t)\|^3_{L^3(\Omega)}).
\end{aligned}$$

As in (4.16), we have

$$\lambda\|u_-(t)\|^3_{L^3(\Omega)} \leqslant C\lambda\,\|u_-(t)\|^6_{L^2(\Omega)} + \tfrac{1}{2}\|\nabla u_-(t)\|^2_{L^2(\Omega)}. \tag{5.1}$$

Substituting of (5.1) into (5.1) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u_-(t)\|^2_{L^2(\Omega)} + \|\nabla u_-(t)\|^2_{L^2(\Omega)} \leqslant 2\lambda\|u_-(t)\|^2_{L^2(\Omega)} + 2C\lambda\|u_-(t)\|^6_{L^2(\Omega)},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_-(t)\|^2_{L^2(\Omega)} \leqslant 2\lambda\|u_-(t)\|^2_{L^2(\Omega)} + 2C\lambda\|u_-(t)\|^6_{L^2(\Omega)}.$$

Integration from t_0 to t leads us to

$$\|u_-(t)\|^2_{L^2(\Omega)} \leqslant \|u_-(t_0)\|^2_{L^2(\Omega)} + \int_{t_0}^{t} 2\lambda\,(1 + C\|u_-(s)\|^4_{L^2(\Omega)})\|u_-(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s, \tag{5.2}$$

where $t \in [t_0, t_1]$. The identity

$$u_- = \max\{-u, 0\} = \frac{|u| - u}{2}$$

and the result $u \in C([t_0, t_1]; L^2(\Omega))$ imply

$$u_- \in C([t_0, t_1]; L^2(\Omega)).$$

Setting $K = \|u_-(t_0)\|^2_{L^2(\Omega)} = 0$, $z = \|u_-(s)\|^2_{L^2(\Omega)}$ and $w = 2\lambda\,(1 + C\|u_-(s)\|^4_{L^2(\Omega)})$ we obtain from (5.2) that

$$z(t) \leqslant K + \int_a^t z(s)w(s)\,\mathrm{d}s \quad \forall t \in [t_0, t_1].$$

Hence, Lemma 5.1 implies $z(t) \equiv 0$ on $[t_0, t_1]$, i.e.,

$$\|u_-(t)\|^2_{L^2(\Omega)} \equiv 0$$

which implies $u_-(t, \mathbf{x}) = 0$ almost everywhere in $(t_0, t_1) \times \Omega$. □

Next we show that the local solution u is bounded above if the initial value is bounded above.

Theorem 5.2. *Suppose that the assumptions of Theorem* 5.1 *hold. Assume further that* $b_0(\mathbf{x}) \leqslant 1$ *almost everywhere in* Ω. *Then the local solution* $u(t, \mathbf{x})$ *of* (4.1) *and* (4.2) *satisfies* $u \leqslant 1$ *almost everywhere in* $(t_0, t_1) \times \Omega$.

Proof. For almost every $t \in [t_0, t_1]$ we set $\phi(\mathbf{x}) = (u-1)_+(t, \mathbf{x}) \equiv \max\{u(t, \mathbf{x}) - 1, 0\}$ in (4.1) and integrate in t to obtain

$$\begin{aligned}\langle u'(t), (u-1)_+\rangle &+ \int_\Omega \nabla u \cdot \nabla (u-1)_+ \,\mathrm{d}\mathbf{x} \\ &= \int_\Omega \lambda u(1-u)(u-1)_+ \,\mathrm{d}\mathbf{x} + \int_\Omega f(t,\mathbf{x})(1-u)(u-1)_+ \,\mathrm{d}\mathbf{x}.\end{aligned}$$

The last equation can be written as

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u-1)_+\|^2_{L^2(\Omega)} &+ \|\nabla(u-1)_+\|^2_{L^2(\Omega)} \\ &= -\int_\Omega \lambda u[(u-1)_+]^2 \,\mathrm{d}\mathbf{x} - \int_\Omega f(t,\mathbf{x})[(u-1)_+]^2 \,\mathrm{d}\mathbf{x} \leqslant 0.\end{aligned}$$

Hence $(u-1)_+(t, \mathbf{x}) = 0$ almost everywhere in $(t_0, t_1) \times \Omega$. □

Remark 5.1. We point out that the proof of Theorems 5.1 and 5.2 do not apply to the finite dimensional Galerkin solution u_m studied in the previous section. The reason is that $[u_m]_-$ or $[u_m - 1]_+$ does not necessarily lie in the span of $\{w_1, w_2, \ldots, w_m\}$; as a result, we cannot set the test function to be $[u_m]_-$ or $[u_m - 1]_+$ in (4.4).

6. Global existence

From the local existence result (Theorem 4.1) and the maximum principles (Theorems 5.1 and 5.2), we concluded that under suitable assumptions on $u_0(\mathbf{x})$ and $f(t, \mathbf{x})$, a solution u for (4.1) and (4.2) exists on some interval $[t_0, t_1]$ and $0 \leqslant u \leqslant 1$ almost everywhere in $(t_0, t_1) \times \Omega$. Based on these we expect a global solution to exists on any finite time interval $[0, T]$. Indeed, we will prove the global existence and uniqueness in this section. While the first global existence theorem below is an easy consequence of the local existence theorem and maximum principles, the second global existence theorem requires additional effort.

The following theorem gives the global existence and uniqueness of the solution.

Theorem 6.1. *Let $T \in (0, \infty)$ be given. Assume that $f \in C(0, T; L^3(\Omega))$, $f(t, \mathbf{x}) \geqslant 0$ a.e. $(t, \mathbf{x}) \in (0, T) \times \Omega$ and $0 \leqslant u_0(\mathbf{x}) \leqslant 1$ a.e. $\mathbf{x} \in \Omega$. Then there exists a unique function $u \in W(0, T)$ such that u is a solution of* (3.3) *and* (3.4) *and $0 \leqslant u \leqslant 1$ a.e. in $(0, T) \times \Omega$.*

Proof. By Theorem 4.1, there is a $t_1 > 0$ such that there exists a function $u \in W(0, t_1)$ as a solution of (3.3) and (3.4) on $[0, t_1]$. By Theorems 5.1 and 5.2, we also have $0 \leqslant u(t, \mathbf{x}) \leqslant 1$ a.e. $(t, \mathbf{x}) \in (0, t_1) \times \Omega$. We let

$$\begin{aligned}\bar{t} = \sup\{\tilde{t} :\ &\text{there exists a } u \in W(0, \tilde{t}) \text{ such that (4.1) and (4.2) hold} \\ &\text{for a.e. } t \in (0, \tilde{t})\}\end{aligned}$$

Then $\bar{t}$ must equals T. For otherwise, Theorem 4.1 would allow us to continue the solution beyond $\bar{t}$ and this would contradict the maximality assumption of $\bar{t}$ (here we used the easily verifiable fact that if $u \in W(0, \bar{t})$ and $u \in W(\bar{t}, \bar{t}+\delta)$ for some $\delta > 0$, then $u \in W(0, \bar{t}+\delta)$.)

To prove the uniqueness of the solution, we suppose u_1 and u_2 are two solutions of (3.3) and (3.4) and set $z = u_1 - u_2$. Then z satisfies

$$\begin{aligned}\langle z'(t), \phi\rangle + (\nabla z(t), \nabla\phi) = (\lambda z(t), \phi) - \lambda(\lambda z(t)(u_1 + u_2), \phi)\\ - (f(t)z(t), \phi) \quad \forall\phi \in H_0^1(\Omega),\ \text{a.e. } t \in (0, T)\end{aligned} \tag{6.1}$$

and

$$z(0, \mathbf{x}) = 0. \tag{6.2}$$

Theorems 5.1and 5.2 imply that $0 \leqslant u_1(t, \mathbf{x}) + u_2(t, \mathbf{x}) \leqslant 2$ almost everywhere in Q_T. Setting $\phi = z(t, \cdot)$ in (6.1) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2_{L^2(\Omega)} + \|\nabla z(t)\|^2_{L^2(\Omega)} \leqslant \|z(t)\|^2_{L^2(\Omega)}, \quad \text{a.e. } t \in [0, T],$$

so that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2_{L^2(\Omega)} \leqslant \|z(t)\|^2_{L^2(\Omega)}, \quad \text{a.e. } t \in [0, T]. \tag{6.3}$$

Thus, using Gronwall's inequality and (6.2) we obtain

$$\|z(t)\|^2_{L^2(\Omega)} = 0, \quad \text{a.e. } t \in [0, T]$$

From which we conclude $z = 0$ a.e. in $(0, T) \times \Omega$. This completes the proof. □

Below, we prove the global existence when $f \in L^2(0, T; L^2(\Omega))$. We first establish a new *a priori* estimate.

Lemma 6.1. *Assume that* $f \in C(0, T; L^2(\Omega))$, $f(t, \mathbf{x}) \geqslant 0$ *for almost every* $(t, \mathbf{x}) \in (0, T) \times \Omega$, *and* $0 \leqslant b_0(\mathbf{x}) \leqslant 1$ *a.e.* $\mathbf{x} \in \Omega$. *Let u be a solution of* (3.3) *and* (3.4). *Then* $\forall t \in [0, T]$,

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H_0^1(\Omega))} + \|u'\|^2_{L^2(0,T;H^{-1}(\Omega))}\\ &\quad \leqslant \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T + C_2\|f\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{6.4}$$

Proof. By Theorems 5.1 and 5.2, we have $0 \leqslant u(t, \mathbf{x}) \leqslant 1$ almost everywhere in $(0, T) \times \Omega$. Thus, for a.e. $t \in [0, T]$, upon setting $\phi(\mathbf{x}) = u(t, \mathbf{x})$ in (3.3) we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)}\\ &\quad = \int_\Omega \lambda\,[u(t, \mathbf{x})]^2[1 - u(t, \mathbf{x})]\,\mathrm{d}\mathbf{x} + \int_\Omega f(t, \mathbf{x})u(t, \mathbf{x})[1 - u(t, \mathbf{x})]\,\mathrm{d}\mathbf{x}\\ &\quad \leqslant \int_\Omega \lambda\,\mathrm{d}\mathbf{x} + \int f(t, \mathbf{x})\,\mathrm{d}\mathbf{x} \leqslant \lambda C_1(\lambda, \Omega) + C_2(\Omega)\|f(t)\|^2_{L^2(\Omega)},\end{aligned} \tag{6.5}$$

where $C_1, C_2 > 0$ are constants. Integration from 0 to t yields

$$\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H_0^1(\Omega))} \leqslant \|u(t_0)\|^2_{L^2(\Omega)} + C_1 T + C_2\|f\|^2_{L^2(0,T;L^2(\Omega))}. \tag{6.6}$$

From Eq. (3.3) we obtain

$$\begin{aligned}\langle u'(t), \phi\rangle &= \int_\Omega \nabla u(t,\mathbf{x})\cdot\nabla\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&\quad + \lambda\int_\Omega u(t,\mathbf{x})[1-u(t,\mathbf{x})]\phi(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega f(t,\mathbf{x})u(t,\mathbf{x})[1-u(t,\mathbf{x})]\,\mathrm{d}\mathbf{x}\\&\leqslant \|u(t)\|_{H_0^1(\Omega)}\|\phi\|_{H_0^1(\Omega)} + \lambda C\int_\Omega|\phi(\mathbf{x})|\,\mathrm{d}\mathbf{x} + C\int_\Omega f(t,\mathbf{x})\phi(\mathbf{x})\,\mathrm{d}\mathbf{x}\\&\leqslant \|u(t)\|_{H_0^1(\Omega)}\|\phi\|_{H_0^1(\Omega)} + C(\lambda,\Omega)\|\phi\|_{L^2(\Omega)} + C\|f(t)\|_{L^2(\Omega)}\|\phi\|_{L^2(\Omega)}\end{aligned}$$

for all $\phi\in H_0^1(\Omega)$ and a.e. $t\in(0,T)$. Upon taking the supremum over $\phi\in H_0^1(\Omega)$ we obtain

$$\|u'(t)\|_{H^{-1}(\Omega)}\leqslant C\|u(t)\|_{H_0^1(\Omega)} + C(\lambda,\Omega) + C\|f(t)\|_{L^2(\Omega)} \tag{6.7}$$

for a.e. $t\in(0,T)$. Combining (6.6) and (6.7) we obtain (6.4). □

Theorem 6.2. *Let $T\in(0,\infty)$ be given. Assume that $f\in L^2(0,T;L^2(\Omega))$, $f(t,\mathbf{x})\geqslant 0$ a.e. $(t,\mathbf{x})\in(0,T)\times\Omega$ and $0\leqslant u_0(\mathbf{x})\leqslant 1$ a.e. $\mathbf{x}\in\Omega$. Then there exists a unique function $u\in W(0,T)$ such that u is a solution of* (3.3) *and* (3.4), $0\leqslant u\leqslant 1$ *a.e. in $(0,T)\times\Omega$, and*

$$\begin{aligned}&\|u(t)\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(0,T;H_0^1(\Omega))} + \|u'\|^2_{L^2(0,T;H^{-1}(\Omega))}\\&\quad\leqslant \|u(t_0)\|^2_{L^2(\Omega)} + C_1T + C_2\|f\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{6.8}$$

Proof. The denseness of $C(0,T;L^3(\Omega))$ in $L^2(0,T;L^2(\Omega))$ implies there is a sequence $\{f_m\}_{m=1}^\infty\subset C(0,T;L^3(\Omega))$ such that $f_m\to f$ in $L^2(0,T;L^2(\Omega))$, $f_m(t,\mathbf{x})\geqslant 0$ a.e. $(t,\mathbf{x})\in Q_T$ and $\|f-f_m\|_{L^2(0,T;L^2(\Omega))}\leqslant 1$ (for instance, f_m can be defined as the convoluation of *f* with a sequence of nonnegative modifying functions $\phi_m(\mathbf{x})$).

For each *m*, Theorem 6.1 implies the existence of a $u_m\in W(0,T)$ such that

$$\begin{aligned}&\langle u_m'(t),\phi\rangle + (\nabla u_m(t),\nabla\phi)\\&\quad = \lambda(u_m(t)(1-u_m),\phi) + (f_m(t)[1-u_m(t)],\phi)\quad \forall\phi\in H_0^1(\Omega),\ \text{a.e. } t\in(0,T)\end{aligned} \tag{6.9}$$

and

$$u_m(0,\mathbf{x}) = u_0. \tag{6.10}$$

Moreover, we have that $0\leqslant u_n(\mathbf{x},t)\leqslant 1$ a.e. $(t,\mathbf{x})\in(0,T)\times\Omega$. By Lemma 6.1, we have that,

$$\begin{aligned}&\|u_m\|^2_{L^2(0,T;H_0^1(\Omega))} + \|u_m'\|^2_{L^2(0,T;H_0^1(\Omega))}\\&\quad\leqslant \|u_0\|^2_{L^2(\Omega)} + C_1T + C_2\|f_m\|^2_{L^2(0,T;L^2(\Omega))}\\&\quad\leqslant \|u_0\|^2_{L^2(\Omega)} + C_1T + C_2\|f\|^2_{L^2(0,T;L^2(\Omega))} + C_3.\end{aligned} \tag{6.11}$$

Estimate (6.11) allows us to extract a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) such that

$$u_m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{6.12}$$

$$u'_m \rightharpoonup u' \quad \text{weakly in } L^\infty(0,T;H^{-1}(\Omega)) \tag{6.13}$$

and

$$u_m \to u \quad \text{strongly in } L^2(0,T;L^2(\Omega)). \tag{6.14}$$

We also note that for every $\psi(t) \in C^1([0,T];\mathbb{R})$ with $\psi(T)=0$ and $\forall \phi \in C_0^1(\Omega)$,

$$\begin{aligned}
&\left|\int_0^T \langle f(t,\cdot)u(t,\cdot) - f_m(t,\cdot)u_m(t,\cdot), \psi(t)\phi(\cdot)\rangle \,\mathrm{d}t\right| \\
&\quad \leqslant \int_0^T |\psi(t)|\,\|\phi\|_{L^\infty(\overline{\Omega})}\|f_m(t)\|_{L^2(\Omega)}\|u_m(t)-u(t)\|_{L^2(\Omega)}\,\mathrm{d}t \\
&\qquad + \int_0^T |\psi(t)|\,\|f_n(t)-f(t)\|_{L^2(\Omega)}\|u\|_{L^\infty(Q_T)}\|\phi\|_{L^2(\Omega)}\,\mathrm{d}t \\
&\quad \leqslant C\|\psi\|_{L^\infty(0,T)}\|\phi\|_{L^\infty(\Omega)}\|f_m\|_{L^2(0,T;L^2(\Omega))}\|u_n-u\|_{L^2(0,T;L^2(\Omega))} \\
&\qquad + C\|\psi\|_{L^\infty(0,T)}\|f_m - f\|_{L^2(0,T;L^2(\Omega))}\|\phi\|_{H_0^1(\Omega)},
\end{aligned}$$

so that

$$\int_0^T \langle f_m(t,\cdot)u_m(t,\cdot), \psi(t)\phi(\cdot)\rangle\,\mathrm{d}t \to \int_0^T \langle f(t,\cdot)u(t,\cdot), \psi(t)\phi(\cdot)\rangle\,\mathrm{d}t \tag{6.15}$$

as $m \to \infty$, for all $\phi \in C_0^1(\overline{\Omega})$ and all $\psi \in C^1[0,T]$ with $\psi(T)=0$. By repeating the arguments used in the proof of Theorem 4.1 (Step 3) we may pass to the limits in Eq. (6.9) to show

$$\begin{aligned}
&\langle u'(t),\phi\rangle + (\nabla u(t),\nabla\phi) \\
&\quad = \lambda(u(t)(1-u),\phi) + (f(t)[1-u(t)],\phi) \quad \forall \phi \in C_0^1(\Omega),\ \text{a.e. } t\in(0,T).
\end{aligned}$$

Using the denseness of $C_0^1(\Omega)$ in $H_0^1(\Omega)$ we have

$$\begin{aligned}
&\langle u'(t),\phi\rangle + (\nabla u(t),\nabla\phi) \\
&\quad = \lambda(u(t)(1-u),\phi) + (f(t)[1-u(t)],\phi) \quad \forall \phi \in H_0^1(\Omega),\ \text{a.e. } t\in(0,T).
\end{aligned} \tag{6.16}$$

The estimate (6.8) follows directly from (6.11).

Theorems 5.1and 5.2 imply $0\leqslant u\leqslant 1$ a.e. in $(0,T)\times\Omega$.

Similar to the proof of Theorem 6.1, we may prove the uniqueness of the solution. This completes the proof. □

References

[1] H. Amann, Existence and regularity for semilinear parabolic evolution equations, Ann. Scuola Norm. Sup. Pisa Ser. IV XI (1984) 593–676.

[2] H. Amann, Global existence for semilinear parabolic systems, J. Reine Angew. Math. 360 (1985) 47–83.
[3] D.G. Aronson, H.F. Weinberger, in: A. Dold, B. Eckmann (Eds.), Partial Differential Equations and Related Topics, Lecture Notes in Mathematics, vol. 446, Springer, New York, 1975.
[4] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, Adv. Math. 30 (1) (1978) 33–76.
[5] N.F. Britton, Reaction–Diffusion Equations and Their Applications to Biology, Academic Press, New York, 1986.
[6] T. Cazenave, A. Haraux, An Introduction to Semilinear Evolution Equations, Clarendon Press, Oxford, 1998.
[7] P.C. Fife, Mathematical Aspects of Reacting and Diffusing Systems, Lecture Notes in Biomathematics, vol. 28, Springer, Berlin, New York, 1979.
[8] R.A. Fisher, The advance of advantageous genes, Ann. Eugenics 7 (1937) 335–369.
[10] P. Hartman, Ordinary Differential Equations, Wiley, New York, 1964.
[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer, New York, 1981.
[12] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bulletin Université d'Etat à Moscou (Bjul. Moskowskogo Gos. Univ.), Série Internationale, Section A, vol. 1, 1937, pp. 1–26.
[13] A. Leung, Systems of Nonlinear Partial Differential Equations, Kluwer, Boston, 1989.
[14] J.-L. Lions, Quelques methodes de resolution des problemes aux limites nonlineaires, Dunod-Gauthier-Villars, Paris, 1969.
[15] J.-L. Lions, Control of Distributed Singular Systems, Bordas, Paris, 1983.
[16] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkháuser, Basel, Boston, Berlin, 1995.
[17] O. Manley, M. Marion, R. Temam, Equations of combustion in the presence of complex chemistry, Indiana Univ. Math. J. 42 (3) (1993) 941–967.
[19] R. Miller, Ordinary Differential Equations, Academic Press, New York, London, 1969.
[21] C.V. Pao, Nonlinear Parabolic and Elliptic Equations, Plenum Press, New York, 1992.
[22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York, 1983.
[23] F. Rothe, in: A. Dold, B. Eckmann (Eds.), Global Solutions of Reaction–Diffusion Systems, Lecture Notes in Mathematics, vol. 1072, Springer, New York, 1984.
[24] J. Smoller, Shock Waves and Reaction–Diffusion Equations, second ed., Springer, Berlin, 1994.
[25] R. Temam, Navier–Stokes Equations, North-Holland, Amsterdam, 1985.
[26] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin, 1988.