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OPTIMAL BOUNDARY CONTROL FOR THE EVOLUTIONARY NAVIER–STOKES SYSTEM: THE THREE-DIMENSIONAL CASE*

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Dedicated to the memory of Olga Aleksandrovna Ladyzhenskaya

Abstract. Optimal boundary control problems for the three-dimensional, evolutionary Navier–Stokes equations in the exterior of a bounded domain are studied. Control is effected through the Dirichlet boundary condition and is sought in a subset of the trace space of velocity fields with almost minimal possible regularity. The control objective is to minimize the drag functional. The existence of an optimal solution is proved. A strong form of an optimality system of equations is derived on the basis of regularity results established in this work for the adjoint Oseen equations with regular initial data which do not satisfy the compatibility conditions.

Key words. optimal control, Navier–Stokes equations, boundary value problem, drag reduction

AMS subject classifications. 76D05, 49J20, 49K20, 35K50

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1. Introduction.

This paper is devoted to the investigation of using boundary controls to minimize the drag about a three-dimensional body B moving at a constant velocity $\mathbf{v}_\infty$ in a fluid. The fluid flow is assumed to be governed by the time-dependent, viscous, incompressible Navier–Stokes system. The drag minimization problem is formulated as an optimal boundary control problem by introducing an appropriate drag functional and by choosing a suitable control space. The two-dimensional analogue of this problem was studied in [8], and this paper represents a continuation of that work in the three-dimensional case. Our aims are to prove the existence of an optimal solution and derive an optimality system of equations. In order to achieve these aims, we will need to define the correct mathematical formulation of the optimal control problem in question.

The type of optimal control problem we study has the following form:

$$\mathcal{J}(\mathbf{v}) \to \inf, \tag{1.1}$$

$$\mathrm{NS}(\mathbf{v}, p_1) = \mathbf{0}, \quad \mathbf{v}\big|_{t=0} = \mathbf{v}_0, \quad \mathbf{v}\big|_{|\mathbf{x}|\to\infty} = \mathbf{v}_\infty, \quad \mathbf{v}\big|_{(0,T)\times\partial B} = \mathbf{b}, \tag{1.2}$$

$$R(\mathbf{b}) \le M. \tag{1.3}$$

Here, $\mathbf{v}(t,\mathbf{x})$ and $p_1(t,\mathbf{x})$ for $t \in [0,T]$ and $\mathbf{x} \in \Omega \equiv \mathbb{R}^3 \setminus B$ are the vector-valued velocity field and scalar-valued pressure field, respectively, of the fluid flow surrounding the body B; we attach the coordinates to B, i.e., we treat B as fixed so that the fluid is in motion relative to B. Also, $\mathcal{J}(\mathbf{v})$ is the drag functional, $\mathrm{NS}(\mathbf{v}, p_1) = \mathbf{0}$ is

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the Navier–Stokes system, $\mathbf{v}_0$ is the initial condition, and $\mathbf{b}$ is the Dirichlet boundary condition which acts as the control in our problem.

It is clear that the correct physical setting of the optimal drag reduction problem must contain the constraint (1.3), where $R(\mathbf{b})$ is a norm-like functional for functions defined on the boundary $\Sigma = (0,T)\times\partial\Omega \equiv (0,T)\times\partial B$. Indeed, if (1.3) is not imposed or the constant M in that condition is too large, then, instead of drag reduction, the boundary control $\mathbf{b}$ can actually push the body B in the direction opposite to $\mathbf{v}_\infty$ (see the relevant discussions in [8]).

Note that choosing the appropriate form for $R(\mathbf{b})$ is not a trivial task, since choosing R to be a simple functional such as the $\mathbf{L}^2(\Sigma)$-norm is unsuccessful even in two-dimensional case (see [8]); of course, in three dimensions, it is impossible as well. The correct choice of $R(\mathbf{b})$ is strictly connected to choosing a proper boundary control space. That is, they must be chosen in such a way that the validity of the use of the Lagrange multiplier principle is guaranteed (see [2, 6]). This amounts to the requirement that the state space $\mathbf{W}$ must be sufficiently regular so that the derivative operator $\mathrm{NS}'(\widetilde{\mathbf{v}}) : \mathbf{W} \to \mathbf{F}$ is an epimorphism for some suitably chosen function space $\mathbf{F}$. It turns out that this requirement can be met if the solution to the Navier–Stokes equations is unique within the space $\mathbf{W}$. It is well known (see [5]) that if we restrict ourselves to Hilbert space settings, then $\mathbf{W}$ should have the following form to ensure the uniqueness property:

$$\mathcal{V}^{(s)}(Q) \equiv \{\mathbf{v} \in L^2(0,T;\mathbf{H}^s(\Omega)) : \partial_t \mathbf{v} \in L^2(0,T;\mathbf{H}^{s-2}(\Omega)),\ \operatorname{div}\mathbf{v} = 0\} \tag{1.4}$$

for $s \geq 3/2$, where $Q \equiv (0,T)\times\Omega$ and the function spaces used will be precisely defined in subsection 2.4. In other words, if the smoothness of functions from $\mathbf{W}$ is less than $\mathcal{V}^{(3/2)}(Q)$, then it is not clear how to prove surjectivity of the operator $\mathrm{NS}'(\widetilde{\mathbf{v}}) : \mathbf{W} \to \mathbf{F}$ or the solvability of the corresponding Oseen equations with such nonsmooth coefficients. To find a suitable norm R, we could simply take the norm of space of restrictions to Σ of a function belonging to the space $\mathcal{V}^{(3/2)}(Q)$, whose trace space was already characterized in [9]. However, to simplify the definition of R, we choose the weakest norm R which avoids fractional derivatives and whose natural domain of definition is a subspace of the trace space of $\mathcal{V}^{(3/2)}(Q)$ (see [10]). Precisely, we choose

$$R(\mathbf{b}) \equiv \|\mathbf{b}\|^2_{\mathbf{H}^1(\Sigma)} = \int_\Sigma \left(|\partial_t \mathbf{b}|^2 + |\nabla_\tau \mathbf{b}|^2 + |\mathbf{b}|^2\right) ds\,dt, \tag{1.5}$$

where $\mathbf{H}^1(\Sigma)$ is the Sobolev space of vector-valued functions defined on Σ possessing square integrable first derivatives. (The definition of surface gradient ∇_τ will be given below; see (2.11).)

Since the norm (1.5) we choose for the boundary data $\mathbf{b}$ is stronger than the norm for the space generated by restricting the space (1.4) to the boundary, we should choose the solution space $\mathbf{W}$ for the Navier–Stokes equations to be the one that corresponds to the boundary norm (1.5) instead of choosing $\mathbf{W}$ to be simply (1.4). The problem of characterizing such a space $\mathbf{W}$ was solved in [10].

Note that our aim is to investigate the case for which there are no restrictions on the magnitude $|\mathbf{v}_\infty|$ of the velocity $\mathbf{v}_\infty$ of the body B. It is precisely the case of large $|\mathbf{v}_\infty|$ that is most interesting in applications. To fulfill such investigation under general assumptions is not possible because of the absence of theorems regarding the existence of smooth solutions for the three-dimensional evolutionary Navier–Stokes equations with arbitrary data; only when the corresponding norms of the data are sufficiently small has the existence of a smooth solution been proved.

To circumvent this difficulty we consider the following concrete and physically reasonable situation. Let a body B move in a steady-state regime with velocity $\mathbf{v}_\infty$. Then, at the instant t_0 (say $t_0 = 0$), we switch on the control $\mathbf{b}$ on ∂B and solve the optimal drag reduction problem over the time interval $(0, T)$, where $T > 0$ is a given arbitrary number. In contrast to the evolutionary case, the existence of a smooth solution $\mathbf{v}_0$ for the steady-state three-dimensional Navier–Stokes equations with the adhesion condition on ∂B and an arbitrary data $\mathbf{v}_\infty$ at infinity has been proved. So, we can take this steady-state solution $\mathbf{v}_0$ as the initial condition in (1.2); this in turn will allow us to investigate optimal control problem (1.1)–(1.3) on the basis of local existence theorems of smooth solutions for the three-dimensional evolutionary Navier–Stokes equations in a neighborhood of $\mathbf{v}_0$. In addition, we will fulfill the locality condition by choosing a sufficiently small parameter M in (1.3). Of course, in such an approach M depends on $|\mathbf{v}_\infty|$:

$$M(|\mathbf{v}_\infty|) \to 0 \qquad \text{as} \qquad |\mathbf{v}_\infty| \to \infty. \tag{1.6}$$

The plan described above will be realized mathematically in this paper. We point out one difficulty arising in this realization that is connected with the property $\mathbf{v}_0(x) - \mathbf{v}_\infty \notin \mathbf{L}^2(\Omega)$ for the steady-state solution. This property does not complicate very much our proof of the existence theorem for the control problem (1.1)–(1.3). However, the problem of constructing a weak solution for the optimality system becomes quite difficult. We will invoke a special form of the abstract Lagrange multiplier principle (see [6, Chap. 2, Thm. 1.6, Thm. 1.8]); to apply this result we are forced to introduce some special Orlicz spaces which are connected with the properties of a vector field $\mathbf{v}_0(x) \to \mathbf{v}_\infty$.

In section 2, we give a precise statement of the optimal control problems we consider. In section 3 and Appendix A, we establish regularity results for the Navier–Stokes system as well as for linearized Navier–Stokes systems and the adjoint linearized Navier–Stokes systems. Although several of these results are of interest in their own right, they are used in this paper as auxiliary results to help us prove the main results of this paper: the existence of optimal solutions and the derivation of weak and strong forms of an optimality system. Section 4 is devoted to the proof of the existence theorem for the control problems defined in section 2. In sections 5–7, we derive corresponding weak and strong forms of the optimality systems.

Finally, we emphasize that we consider this investigation to be a major, but not final, step towards a complete solution of the drag reduction problem. Even after this paper and [8, 9, 10], there remain a number of unresolved problems. For example, by virtue of (1.6) for sufficiently large $|\mathbf{v}_\infty|$, the size of the bound M in (1.3) may become too small to be of practical use in applications. The issue of how to increase M in this situation goes beyond the scope of this paper. Nevertheless, we consider the resolution of this issue quite realistic; indeed, as was shown in [7, 15], it is quite possible to solve nonlocal control problems in the space of smooth solutions for the three-dimensional evolutionary Navier–Stokes equations if the control function is supported on the whole boundary.

2. Formulation of the problem. In this section, we provide a precise statement of the optimal control problems treated in this paper.

2.1. The state system and the cost functional. As discussed in section 1, we consider the problem of using boundary controls to minimize the drag about a three-dimensional body B moving at a constant velocity $\mathbf{v}_\infty$ in a viscous fluid. In coordinates attached to the body B, this problem transforms into the drag reduction problem for a fixed body B surrounded by a fluid flow having velocity $\mathbf{v}_\infty$ at infinity. Mathematically, the fluid flow is described as follows. Let $B \subset \mathbb{R}^3$ be a bounded domain and let $\Omega \equiv \mathbb{R}^3 \backslash B$. Suppose that the boundary $\partial\Omega$ of Ω is of class C^∞ and is a connected surface. (We impose the last assumption only because it is reasonable from the physical point of view; the generalization of our results to the case of unconnected $\partial\Omega$ is a simple matter.) In the flow domain Ω, we consider the Navier–Stokes system

$$
\left\{
\begin{array}{l}
\partial_t \mathbf{v} - \Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p_1 = \mathbf{0} \quad \text{in } Q,\\
\operatorname{div} \mathbf{v} = 0 \quad \text{in } Q,\\
\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{for } \mathbf{x} \in \Omega,\\
\mathbf{v}|_\Sigma = \mathbf{b} \quad \text{with} \quad \displaystyle\int_{\partial\Omega} \mathbf{b}\cdot\mathbf{n}\, ds = 0 \quad \text{for } t \in (0,T),\\
\mathbf{v} \to \mathbf{v}_\infty \quad \text{as } |\mathbf{x}| \to \infty,
\end{array}
\right.
\tag{2.1}
$$

where $Q = (0,T)\times\Omega$, $\Sigma = (0,T)\times\partial\Omega$, $\mathbf{v}(t,\mathbf{x}) = \big(v_1(t,\mathbf{x}), v_2(t,\mathbf{x}), v_3(t,\mathbf{x})\big)$ for $t \in [0,T]$, $\mathbf{x}\in\Omega$ is the velocity field, and $\nabla p_1(t,\mathbf{x})$ is a pressure gradient.

The vector field $\mathbf{b}$ is defined on Σ and is the control available to effect optimization. We wish to minimize the work due to drag through a proper choice of $\mathbf{b}$. The work due to drag, or the drag functional, is defined by the formula

$$
\mathcal{W} = \int_0^T \int_{\partial\Omega} (\mathbf{v} - \mathbf{v}_\infty)\cdot \mathcal{T}\mathbf{n}\, ds\, dt,
$$

where $\mathcal{T} = -p_1 I + 2\mathcal{D}$ is the stress tensor, $\mathcal{D} = \mathcal{D}(\mathbf{v}) = (\nabla\mathbf{v} + \nabla\mathbf{v}^T)/2$ is the rate of deformation tensor, and $\mathbf{n}$ is the unit, outward-pointing normal along the boundary $\partial\Omega$. We can derive, just as in the two-dimensional case (see [8]), the following equivalent expression for $\mathcal{W}$:

$$
\begin{aligned}
\mathcal{J}(\mathbf{v}) = \int_0^T\int_\Omega \mathcal{D}(\mathbf{v}) : \mathcal{D}(\mathbf{v})\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathbf{v}-\mathbf{v}_\infty|^2 \mathbf{v}\cdot\mathbf{n}\, ds\, dt \\
+ \frac{1}{2}\int_\Omega |\mathbf{v}(T,\mathbf{x}) - \mathbf{v}_\infty|^2\, d\mathbf{x} - \frac{1}{2}\int_\Omega |\mathbf{v}_0 - \mathbf{v}_\infty|^2\, d\mathbf{x}.
\end{aligned}
\tag{2.2}
$$

The functional (2.2) is precisely the functional to be minimized through a proper choice of the boundary control $\mathbf{b}$ on Σ.

2.2. The initial condition. The correct choice for the initial data $\mathbf{v}_0(\mathbf{x})$ is an important issue since it is related to the physical context in which we formulate the optimal control problem and affects the mathematical proof of existence of optimal solutions. As was mentioned in section 1, we suppose that $\mathbf{v}_0(\mathbf{x})$ is the solution of the steady-state Navier–Stokes problem:

$$
\left\{
\begin{array}{l}
-\Delta \mathbf{v}_0 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0 + \nabla p_0 = \mathbf{0} \quad \text{in } \Omega,\\
\operatorname{div} \mathbf{v}_0 = 0 \quad \text{in } \Omega,\\
\mathbf{v}_0\big|_{\partial\Omega} = \mathbf{0},\\
\mathbf{v}_0 \to \mathbf{v}_\infty \quad \text{as } |\mathbf{x}| \to \infty.
\end{array}
\right.
\tag{2.3}
$$

The no-slip condition $\mathbf{v}_0\big|_{\partial\Omega} = \mathbf{0}$ is imposed purely for simplicity and is quite reasonable from a physical point of view. There is no difficulty in treating the case for which $\mathbf{v}_0$ satisfies an inhomogeneous boundary condition.

The following proposition asserts the existence of a solution of (2.3).

PROPOSITION 2.1. *There exists a solution* $(\mathbf{v}_0, p_0) \in [C^\infty(\overline{\Omega})]^4$ *to* (2.3) *satisfying*

$$\|\mathbf{v}_0 - \mathbf{v}_\infty\|_{\mathbf{L}^6(\Omega)} + \|\nabla \mathbf{v}_0\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{v}_0\|_{\mathbf{C}^2(\overline{\Omega})} \leq C|\mathbf{v}_\infty|. \tag{2.4}$$

A proof of this result can be found in, e.g., [11, 13]. Thus, throughout, we assume that the data $\mathbf{v}_0$ in the initial condition in (2.1) is a $\mathbf{C}^\infty(\overline{\Omega})$ solution of (2.3) satisfying (2.4).

2.3. Change of variables. We introduce the change of variables

$$\mathbf{w}(t, \mathbf{x}) = \mathbf{v}(t, \mathbf{x}) - \mathbf{v}_0(\mathbf{x}) \qquad \text{and} \qquad p(t, \mathbf{x}) = p_1(t, \mathbf{x}) - p_0(\mathbf{x}). \tag{2.5}$$

The unknown vector field $\mathbf{w}$ is more convenient to work with than $\mathbf{v}$ due to its zero initial condition and its vanishing values at infinity. After substitution of (2.5) into (2.2) we see that the last two integrals in (2.2) contain two terms $\mathbf{v}_0 - \mathbf{v}_\infty$ that do not belong to $\mathbf{L}^2(\Omega)$. But these terms annul each other. So the problem of minimizing the functional (2.2) subject to (2.1) and (1.3) is then recast as an optimization problem for $\mathbf{w}$ as follows: minimize the functional

$$\begin{aligned} \mathcal{J}(\mathbf{w}) = &\int_0^T \int_\Omega \mathcal{D}(\mathbf{w} + \mathbf{v}_0) : \mathcal{D}(\mathbf{w} + \mathbf{v}_0)\, d\mathbf{x}\, dt + \frac{1}{2}\int_0^T \int_{\partial\Omega} |\mathbf{w} - \mathbf{v}_\infty|^2 \mathbf{w} \cdot \mathbf{n}\, ds\, dt \\ &+ \frac{1}{2}\int_\Omega (|\mathbf{w}(T, \mathbf{x})|^2 + 2[(\mathbf{w}(T, x) \cdot (\mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty)])\, d\mathbf{x} \end{aligned} \tag{2.6}$$

subject to the constraints

$$\partial_t \mathbf{w} - \Delta \mathbf{w} + [(\mathbf{w} + \mathbf{v}_0) \cdot \nabla]\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{v}_0 + \nabla p = \mathbf{0} \quad \text{in } Q, \tag{2.7}$$

$$\operatorname{div} \mathbf{w} = 0 \quad \text{in } Q, \tag{2.8}$$

$$\mathbf{w}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \tag{2.9}$$

$$\mathbf{w} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.10}$$

$$R(\mathbf{w}) = \int_\Sigma \left(|\partial_t \mathbf{w}|^2 + |\nabla_\tau \mathbf{w}|^2 + |\mathbf{w}|^2\right) ds\, dt \leq M. \tag{2.11}$$

Recall that the surface gradient $\nabla_\tau \mathbf{w}\big|_\Sigma = (\nabla \mathbf{w})\big|_\Sigma - (\partial_n \mathbf{w})\big|_\Sigma$, where $\nabla \mathbf{w}$ is the usual gradient of $\mathbf{w}$ in $\mathbb{R}^3$ and $(\partial_n \mathbf{w})$ is the derivative of $\mathbf{w}$ with respect to the outward-pointing unit normal $\mathbf{n}$ on $\partial\Omega$.

In addition, we suppose that $\mathbf{w}$ satisfies the compatibility condition

$$\big(\mathbf{w}(t, \mathbf{x})|_\Sigma\big)|_{t=0} = \mathbf{0} \quad \text{in } \partial\Omega. \tag{2.12}$$

We omitted, from the system (2.7)–(2.11), the boundary condition and eliminated the unknown Dirichlet boundary control $\mathbf{b}$ in (2.1); instead, the Dirichlet control is expressed by $\mathbf{w}|_\Sigma$.

By virtue of (2.5) and $\mathbf{v}_0|_{\partial\Omega} = 0$, the functional $R(\mathbf{w})$ and the parameter M in (2.11) coincide with R and M in (1.3) and (1.5).

Note that functional (2.6) is well defined on $\mathbf{w}$ satisfying (2.7)–(2.11): this is shown in subsection 3.2.

2.4. Function spaces. In this subsection, we define function spaces for the state variables and the Dirichlet controls. The structure of the space where we look for the solution of the optimal control problem is quite complicated because it should be the space where solutions of the Navier–Stokes equations are unique, and besides it has to contain all solutions of the boundary value problem for the Navier–Stokes equations with arbitrary Dirichlet boundary conditions from $\mathbf{H}^1(\Sigma)$ (i.e., with not so smooth conditions) but with the right side belonging to the Lebeque space. Besides, we recall definitions of well-known Sobolev spaces.

The Sobolev spaces $H^k(\Omega)$ with k a nonnegative integer are defined by

$$H^k(\Omega) = \big\{ u \in L^2(\Omega) \ : \ \|u\|^2_{H^k(\Omega)} \equiv \sum_{|\alpha| \le k} \int_\Omega |D^\alpha u(\mathbf{x})|^2 \, d\mathbf{x} < \infty \big\},$$

where $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index (α_j being a nonnegative integer), $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, and $D^\alpha = \partial^{|\alpha|}/(\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3})$. The Sobolev space $H^s(\Omega)$ for an arbitrary $s > 0$ is defined through the interpolation of the spaces $H^k(\Omega)$ for integer k; see [14]. By definition, $H^s_0(\Omega)$, $s > 0$, is the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. The space $H^{-s}(\Omega)$, $s > 0$, is defined as the dual space of $H^s_0(\Omega)$, i.e., $H^{-s}(\Omega) = (H^s_0(\Omega))'$, with the norm

$$\|f\|_{H^{-s}(\Omega)} = \sup_{\phi \in H^s_0(\Omega),\, \phi \ne 0} \frac{\langle f, \phi \rangle}{\|\phi\|_{H^s_0(\Omega)}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-s}(\Omega)$ and $H^s_0(\Omega)$ generated by the scalar product in $L^2(\Omega)$. Sobolev spaces on $\partial\Omega$ are denoted by $H^r(\partial\Omega)$ and are defined with the help of partition of unity techniques; for details, see, e.g., [14]. Vector-valued spaces (including vector-valued Sobolev spaces) are denoted by boldface letters, e.g., $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^3$ for all $s \in \mathbb{R}$, $\mathbf{H}^s_0(\Omega) = [H^s_0(\Omega)]^3$, and $\mathbf{H}^r(\partial\Omega) = [H^r(\partial\Omega)]^3$. For $s \ge -1$, we define the divergence-free spaces

$$\mathbf{V}^s(\Omega) \equiv \big\{ \mathbf{v} \in \mathbf{H}^s(\Omega) \ : \ \operatorname{div} \mathbf{v} = 0 \big\}. \tag{2.13}$$

Also, we define

$$\mathbf{V}^0_0(\Omega) = \big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) \ : \ \operatorname{div} \mathbf{v} = 0, \ (\mathbf{v} \cdot \mathbf{n})|_{\partial\Omega} = 0 \big\} \tag{2.14}$$

equipped with the $\mathbf{L}^2(\Omega)$ norm where both equalities are understood in the sense of distributions.

For $s \ge 0$, we introduce the spaces of functions depending on both spatial and temporal variables:

$$H^{1,s}(Q) = \big\{ y(t, \mathbf{x}) \in L^2(0, T; H^{s+1}(\Omega)) \ : \ \partial_t y \in L^2(0, T; H^s(\Omega)) \big\}$$

where $Q = (0, T) \times \Omega$ and

$$H^{1,s}(\Sigma) = \big\{ y(t, \mathbf{x}) \in L^2(0, T; H^{s+1}(\partial\Omega)) \ : \ \partial_t y \in L^2(0, T; H^s(\partial\Omega)) \big\},$$

where $\Sigma = (0, T) \times \partial\Omega$. Analogously, we introduce the following spaces of solenoidal vector fields defined on Q:

$$\mathbf{V}^{1,s}(Q) = L^2(0, T; \mathbf{V}^{s+1}(\Omega)) \cap H^1(0, T; \mathbf{V}^s(\Omega)). \tag{2.15}$$

Recall that Ω is the exterior of a bounded domain $B \subset \mathbb{R}^3$. Let $\rho > 0$ be a fixed number satisfying

$$\partial\Omega \subset \{\mathbf{x} \in \mathbb{R}^3 \ : \ |\mathbf{x}| < \rho\}. \tag{2.16}$$

For arbitrary $k \geq 0$, we set

$$\Omega_{\rho+k} = \Omega \cap \{\mathbf{x} \in \mathbb{R}^3 \ : \ |\mathbf{x}| < \rho + k\} \quad \text{and} \quad Q_{\rho+k} = (0,T) \times \Omega_{\rho+k}. \tag{2.17}$$

Let X_1 and X_2 be two Hilbert spaces. Then, the direct sum $X_1 + X_2 = \{x = x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$ is also a Hilbert space with norm defined by

$$\|x\|^2_{X_1+X_2} = \inf_{x=x_1+x_2, x_1\in X_1, x_2 \in X_2} \Big(\|x_1\|^2_{X_1} + \|x_2\|^2_{X_2} \Big). \tag{2.18}$$

Evidently, (2.18) defines a Hilbert norm. We now define the space $\mathbf{V}^{1,1/2}_\rho(Q)$, which was introduced in [10]. Let $\rho > 0$ be fixed and satisfy (2.16). Then

$$\begin{aligned} \mathbf{V}^{1,1/2}_\rho(Q) = \big\{ \mathbf{v} \in \mathbf{V}^{1,1/2}(Q) \ : \ \operatorname{supp} \mathbf{v} \subset Q_{\rho+2}, \ \mathbf{v}\big|_{t=0} = \mathbf{0}, \\ \Delta\mathbf{v} \in \mathbf{L}^2(Q_{\rho+2}) + \mathbf{L}^2(0,T; \nabla H^{1/2}(\Omega_{\rho+2})) \big\} \end{aligned} \tag{2.19}$$

equipped with the norm

$$\|\mathbf{v}\|^2_{\mathbf{V}^{1,1/2}_\rho(Q)} = \|\mathbf{v}\|^2_{\mathbf{V}^{1,1/2}(Q)} + \|\Delta\mathbf{v}\|_{\mathbf{L}^2(Q_{\rho+2})+\mathbf{L}^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))},$$

where the space $\nabla H^{1/2}(\Omega_{\rho+2}) = \{\nabla q(\mathbf{x}) : q \in H^{1/2}(\Omega_{\rho+2})\}$ is equipped with the norm

$$\|\nabla q\|_{\nabla H^{1/2}(\Omega_{\rho+2})} \equiv \|\nabla q\|_{(H^{1/2}_{00}(\Omega_{\rho+2}))'}.$$

We recall from [14] that $H^{1/2}_{00}(\Omega_{\rho+2})$ and $[H^{1/2}_{00}(\Omega_{\rho+2})]'$ are defined as follows. Let $r(\mathbf{x}) \in C^\infty(\overline{\Omega_{\rho+2}})$, $r(\mathbf{x}) > 0$ for $\mathbf{x} \in \Omega_{\rho+2}$, and $r(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial\Omega_{\rho+2})$ in a sufficiently small neighborhood of $\partial\Omega_{\rho+2}$. Then,

$$H^{1/2}_{00}(\Omega_{\rho+2}) = \big\{ u \ : \ u \in H^{1/2}(\Omega_{\rho+2}), \ r^{-1/2}u \in L^2(\Omega_{\rho+2}) \big\}$$

with the norm

$$\|u\|^2_{H^{1/2}_{00}(\Omega_{\rho+2})} = \|u\|^2_{H^{1/2}(\Omega_{\rho+2})} + \|r^{-1/2}u\|^2_{L^2(\Omega_{\rho+2})}.$$

By $(H^{1/2}_{00}(\Omega_{\rho+2}))'$ we denote the dual space of $H^{1/2}_{00}(\Omega_{\rho+2})$ with the norm

$$\|f\|_{(H^{1/2}_{00}(\Omega_{\rho+2}))'} = \inf_{\phi \in H^{1/2}_{00}(\Omega_{\rho+2}),\, \phi \neq 0} \frac{\langle f, \phi\rangle}{\|\phi\|_{H^{1/2}_{00}(\Omega_{\rho+2})}},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality generated by the scalar product in $L^2(\Omega_{\rho+2})$.

Following the notation $\mathcal{V}^{(2)}(Q)$ introduced in [8], we define the spaces

$$\begin{aligned} \mathcal{V}^{(2)}_0(Q) = \big\{ \mathbf{v} \in L^2(0,T; \mathbf{V}^2(\Omega)) \ : \\ \partial_t \mathbf{v} \in L^2(0,T; \mathbf{V}^0(\Omega)), \ \mathbf{v}\big|_{t=0} = \mathbf{0}, \ \mathbf{v}\big|_\Sigma = \mathbf{0} \big\} \end{aligned} \tag{2.20}$$

The former is a subspace of the latter with the homogeneous boundary value. Analogous to (2.20), we introduce

$$\mathcal{V}_T^{(2)}(Q) = \big\{\mathbf{v} \in L^2(0,T;\mathbf{V}^2(\Omega)) \ : \ \partial_t \mathbf{v} \in L^2(0,T;\mathbf{V}^0(\Omega)), \mathbf{v}|_{t=T} = \mathbf{0}, \mathbf{v}|_\Sigma = \mathbf{0}\big\}. \tag{2.21}$$

Now, we can define the space in which the solution of the boundary value problem (2.7)–(2.10) is sought:

$$\mathbf{W} \equiv \mathbf{W}(Q) = \big\{\mathbf{v} \in \mathbf{V}_\rho^{1,1/2}(Q) + \mathcal{V}_0^{(2)}(Q) \ : \ \mathbf{v}\big|_\Sigma \in \mathbf{H}^1(\Sigma)\big\}. \tag{2.22}$$

The Sobolev space $\mathbf{H}^1(\Sigma)$ is defined with the help of partition of unity techniques; see [14]. The norm of the space $\mathbf{W}$ is defined by

$$\|\mathbf{v}\|^2_{\mathbf{W}} = \|\mathbf{v}\|^2_{\mathbf{V}_\rho^{1,1/2}(Q)+\mathcal{V}_0^{(2)}(Q)} + \|\mathbf{v}\|^2_{\mathbf{H}^1(\Sigma)}.$$

The boundary control will be sought in the following subspace of $\mathbf{H}^1(\Sigma)$ satisfying a compatibility condition at $t = 0$ (see (2.12)):

$$\widehat{\mathbf{H}}^1(\Sigma) = \Big\{\mathbf{v} \in \mathbf{H}^1(\Sigma) \ : \ \mathbf{v}|_{t=0} = \mathbf{0}, \ \int_{\partial\Omega} \mathbf{v}(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, ds = 0 \text{ a.e. } t \in [0,T]\Big\}. \tag{2.23}$$

We establish a compact embedding result that is used to prove, in Theorem 4.1, the existence of solutions of the optimization problems posed in section 2.5. Let ρ satisfying (2.16) be fixed, let $k > 0$ be arbitrary, and let $Q_{\rho+k}$ be defined as in (2.17). Let the function space $\mathbf{W}(Q_{\rho+k})$ be defined in exactly the same way as $\mathbf{W}(Q) \equiv \mathbf{W}$ (see (2.22)): we simply replace Q and Ω by $Q_{\rho+k}$ and $\Omega_{\rho+k}$, respectively.

LEMMA 2.2. *For each $k \in (0,\infty)$, the embedding $\mathbf{W}(Q_{\rho+k}) \hookrightarrow \mathbf{L}^2(Q_{\rho+k})$ is compact.*

Proof. From definitions (2.19), (2.20), and (2.22) for $\mathbf{W}(Q_{\rho+k})$ and definition (2.15) for $\mathbf{V}^{1,s}(Q_{\rho+k})$ (with Q replaced by $Q_{\rho+k}$ and Ω by $\Omega_{\rho+k}$ in all these relations), we easily conclude that the embedding $\mathbf{W}(Q_{\rho+k}) \hookrightarrow \mathbf{V}^{1,1/2}(Q_{\rho+k})$ is continuous. As is well known (see, e.g., [17, Chap. 4, sect. 3]), the embedding $\mathbf{V}^{1,1/2}(Q_{\rho+k}) \hookrightarrow \mathbf{L}^2(Q_{\rho+k})$ is compact. Combining these two results implies the desired assertion. ∎

2.5. Precise statement of the control problems. We now state precisely the optimal control problems to be studied.

PROBLEM I. *Let $\mathbf{v}_\infty \in \mathbb{R}^3$ and $M > 0$ be given, and suppose that $\mathbf{v}_0$ is constructed as in Proposition* 2.1. *Seek a $(\mathbf{w}, \nabla p) \in \mathbf{W} \times [L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2})) + \mathbf{L}^2(Q)]$ that minimizes the functional* (2.6) *subject to the constraints* (2.7)–(2.12).

As in [8], we may replace the constraint (2.11) by adding a corresponding penalty term in the functional and consider the following penalized variant of the above optimal control problem.

PROBLEM II. *Let $\mathbf{v}_\infty \in \mathbb{R}^3$ and $N > 0$ be given, and suppose that $\mathbf{v}_0$ is constructed as in Proposition* 2.1. *Seek a $(\mathbf{w}, \nabla p) \in \mathbf{W} \times [L^2(0,T;\nabla H^{1/2}(\Omega_{N+2})) + \mathbf{L}^2(Q)]$ that* *minimizes the functional*

$$
\begin{aligned}
\mathcal{J}_N(\mathbf{w}) = &\int_0^T \int_\Omega \mathcal{D}(\mathbf{w}+\mathbf{v}_0) : \mathcal{D}(\mathbf{w}+\mathbf{v}_0)\, d\mathbf{x}\, dt \\
&+\frac{1}{2}\int_\Omega (|\mathbf{w}(T,\mathbf{x})|^2 + 2[(\mathbf{w}(T,x)\cdot(\mathbf{v}_0(\mathbf{x})-\mathbf{v}_\infty)])\, d\mathbf{x} \\
&+\frac{1}{2}\int_0^T\int_{\partial\Omega} |\mathbf{w}-\mathbf{v}_\infty|^2 \mathbf{w}\cdot\mathbf{n}\, ds\, dt + \frac{N}{2}\int_\Sigma \Big(|\partial_t \mathbf{w}|^2 + |\nabla_\tau \mathbf{w}|^2 + |\mathbf{w}|^2\Big)\, ds\, dt
\end{aligned}
\tag{2.24}
$$

subject to the constraints (2.7)–(2.10) *and* (2.12).

DEFINITION 2.3. *An element* $\mathbf{w} \in W$ *is called admissible if it satisfies* (2.7)–(2.12) *in the case of Problem* I *and satisfies* (2.7)–(2.10) *and* (2.12) *in the case of Problem* II. *The set of admissible elements is denoted by* $\mathcal{U}_{\mathrm{ad}}$.

DEFINITION 2.4. *An element* $\widehat{\mathbf{w}} \in \mathcal{U}_{\mathrm{ad}}$ *is called a solution of Problem* I *if*

$$\mathcal{J}(\widehat{\mathbf{w}}) = \inf_{\mathbf{w}\in\mathcal{U}_{ad}} \mathcal{J}(\mathbf{w}),$$

where $\mathcal{J}$ *is defined by* (2.6). *An element* $\widehat{\mathbf{w}} \in \mathcal{U}_{\mathrm{ad}}$ *is called a solution of Problem* II *if*

$$\mathcal{J}_N(\widehat{\mathbf{w}}) = \inf_{\mathbf{w}\in\mathcal{U}_{\mathrm{ad}}} \mathcal{J}_N(\mathbf{w}),$$

where $\mathcal{J}_N$ *is defined by* (2.24).

3. Preliminary results.

In this section, we collect results that will be needed for the analysis of the optimal control problems defined in section 2.5.

3.1. Boundary value problems with inhomogeneous boundary conditions.

To prove the existence of solutions (see section 4) of the optimal control problems stated in section 2.5, we need results regarding the existence and uniqueness of a solution of the boundary value problem (2.7)–(2.10) with an inhomogeneous Dirichlet boundary condition on Σ satisfying the compatibility condition (2.12). We rewrite that boundary value problem in terms of $\mathbf{w}$ as follows:

$$
\begin{cases}
\partial_t \mathbf{w} - \Delta\mathbf{w} + [(\mathbf{w}+\mathbf{v}_0)\cdot\nabla]\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{v}_0 + \nabla p = \mathbf{0} \qquad \text{in } Q,\\
\operatorname{div}\mathbf{w} = 0 \qquad \text{in } Q,\\
\mathbf{w}|_{t=0} = \mathbf{0} \qquad \text{in } \Omega,\\
\mathbf{w}|_\Sigma = \mathbf{b},\\
\mathbf{w}\to\mathbf{0} \qquad \text{as } |\mathbf{x}|\to\infty.
\end{cases}
\tag{3.1}
$$

We assume that the Dirichlet boundary data $\mathbf{b} \in \widehat{\mathbf{H}}^1(\Sigma)$.

The solution $(\mathbf{w}, \nabla p)$ of (3.1) is sought in $\mathbf{W}\times[L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2})) + \mathbf{L}^2(Q)]$. This boundary value problem was analyzed in [10]. The key step was to prove the following result concerning the extension of the Dirichlet boundary condition $\mathbf{b}$ from Σ into Q.

PROPOSITION 3.1. *There exists a continuous extension operator*

$$\mathcal{E} : \widehat{\mathbf{H}}^1(\Sigma) \to \mathbf{V}_\rho^{1,1/2}(Q). \tag{3.2}$$

With the help of Proposition 3.1, the following result was also established in [10].

PROPOSITION 3.2. *Assume that* $\mathbf{b} \in \widehat{\mathbf{H}}^1(\Sigma)$ *and*

$$\|\mathbf{b}\|^2_{\mathbf{H}^1(\Sigma)} \le \epsilon, \tag{3.3}$$

where $\epsilon > 0$ *is sufficiently small. Assume further that* $\mathbf{v}_0 \in \mathbf{C}^\infty(\overline{\Omega})$ *and* $\mathbf{v}_0$ *satisfies* (2.4). *Then, there exists a unique solution* $(\mathbf{w}, \nabla p)$ *of problem* (3.1) *belonging to* $\mathbf{W} \times [L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2})) + \mathbf{L}^2(Q)]$ *and satisfying the estimate*

$$\|\mathbf{w}\|^2_{\mathbf{W}} + \|\nabla p\|^2_{L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))+\mathbf{L}^2(Q)} \le C(\epsilon), \tag{3.4}$$

where $C(\epsilon)$ *is a positive continuous function defined for all sufficiently small* ϵ.

3.2. The correctness of the functionals (2.6) and (2.24). Here we show that all integrals from (2.6) and (2.24) converge for each $\mathbf{w} \in \mathbf{W}$ satisfying (2.7)–(2.10). Indeed, the convergence of all terms except $\mathbf{w}(T,x)\cdot(\mathbf{v}_0(x) - \mathbf{v}_\infty)$ follow directly from the inclusion $\mathbf{w} \in \mathbf{W}$. Since by (2.4) we have $\mathbf{v}_0 - \mathbf{v}_\infty \in \mathbf{L}^6(\Omega)$, to prove the convergence of this aforementioned term we need to check that $\mathbf{w}(T,x) \in \mathbf{L}^{6/5}(\Omega)$.

LEMMA 3.3. *Let* $\mathbf{w} \in \mathbf{W}$ *satisfy* (2.7)–(2.10). *Then,* $\mathbf{w}(T,x) \in \mathbf{L}^{6/5}(\Omega)$.

Proof. Evidently, it is enough to prove the inclusion $\mathbf{w}(T,x) \in \mathbf{L}^{6/5}(\mathbb{R}^3 \setminus \Omega_{\rho+2})$ where ρ is the number from definitions (2.19) and (2.22). Using the techniques of [10] we extend a solution $\mathbf{w} \in \mathbf{W}$ of (2.7)–(2.10) from $[0,T]\times\mathbb{R}^3\setminus Q$ into a vector field $\widetilde{\mathbf{w}}$ on $[0,T]\times\mathbb{R}^3$ satisfying $\widetilde{\mathbf{w}} \in L^2(0,T;\mathbf{V}^2(\mathbb{R}^3))\cap H^1(0,T;\mathbf{V}^0(\mathbb{R}^3))$ and $\widetilde{\mathbf{w}}|_{t=0} = 0$. We also extend the gradient ∇p in (2.7)–(2.10) from $[0,T]\times\mathbb{R}^3\setminus Q$ into $\nabla\widetilde{p} \in L^2((0,T)\times\mathbb{R}^3)$, and extend $\mathbf{v}_0(x)$ into a C^∞-vector field on $\mathbb{R}^3$. Substituting $(\widetilde{\mathbf{w}}, \nabla\widetilde{p})$ into the left-hand side of (2.7), we obtain

$$\partial_t\widetilde{\mathbf{w}} - \Delta\widetilde{\mathbf{w}} = -[(\widetilde{\mathbf{w}} + \mathbf{v}_0)\cdot\nabla]\widetilde{\mathbf{w}} - (\widetilde{\mathbf{w}}\cdot\nabla)\mathbf{v}_0 + \nabla\widetilde{p} + \mathbf{g}(t,\mathbf{x}) \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{3.5}$$

$$\operatorname{div}\widetilde{\mathbf{w}} = 0 \quad \text{in } (0,T)\times\mathbb{R}^3, \qquad \widetilde{\mathbf{w}}|_{t=0} = \mathbf{0} \quad \text{in } \mathbb{R}^3, \tag{3.6}$$

where $\mathbf{g}(t,\mathbf{x}) \in L^2((0,T)\times\mathbb{R}^3)$ and $\operatorname{supp}\mathbf{g} \subset Q_{\rho+2}$. Evidently, $\mathbf{g} \in L^{6/5}((0,T)\times\mathbb{R}^3)$.

Recall that if $\Omega \subset \mathbb{R}^3$ is a domain and $Q = [0,T]\times\Omega$, then the Sobolev space $W^{1,2}_q(Q)$ with $1 \le q < \infty$ is defined as follows:

$$W^{1,2}_q(Q) = \Big\{u(t,x) \in L^q(Q) \ : \ \|u\|^q_{W^{1,2}_q(Q)} \equiv \int_Q |u|^q + |\partial_t u|^q + |\nabla u|^q + \sum_{i,j=1}^3 \left|\frac{\partial u}{\partial x_i \partial x_j}\right|^q \, dx < \infty\Big\}. \tag{3.7}$$

By virtue of Sobolev embedding theorem (see [3]), if $1 < p < q < \infty$ then the following embeddings are continuous:

$$W^{1,2}_p(Q) \subset L^q(Q) \quad \text{for } \frac{1}{p} - \frac{1}{q} \le \frac{2}{5} \tag{3.8}$$

and

$$\nabla W^{1,2}_p(Q) := \{\nabla f : f \in W^{1,2}_p(Q)\} \subset L^q(Q) \quad \text{for } \frac{1}{p} - \frac{1}{q} \le \frac{2}{5}. \tag{3.9}$$

Using Holder's inequality, (3.8), (3.9), and (2.4), we obtain

$$\|(\widetilde{\mathbf{w}}\cdot\nabla)\widetilde{\mathbf{w}}\|_{L^{6/5}(Q)} \le \|\widetilde{\mathbf{w}}\|_{\mathbf{L}^3(Q)}\|\nabla\widetilde{\mathbf{w}}\|_{\mathbf{L}^2(Q)} \le \|\widetilde{\mathbf{w}}\|^2_{\mathbf{W}_2^{1,2}(Q)},$$

$$\|(\widetilde{\mathbf{w}}\cdot\nabla)\mathbf{v}_0\|_{L^{6/5}(Q)} \le \|\widetilde{\mathbf{w}}\|_{\mathbf{L}^3(Q)}\|\nabla\mathbf{v}_0\|_{\mathbf{L}^2(Q)} \le c\|\widetilde{\mathbf{w}}\|_{\mathbf{W}_2^{1,2}(Q)},$$

and

$$\|(\mathbf{v}_0\cdot\nabla)\widetilde{\mathbf{w}}\|_{L^{3/2}(Q)} \le \|\mathbf{v}_0\|_{\mathbf{L}^6(Q)}\|\nabla\widetilde{\mathbf{w}}\|_{\mathbf{L}^2(Q)} \le C\|\widetilde{\mathbf{w}}\|_{\mathbf{W}_2^{1,2}(Q)}.$$

Thus, $(\widetilde{\mathbf{w}}\cdot\nabla)\widetilde{\mathbf{w}}+(\widetilde{\mathbf{w}}\cdot\nabla)\mathbf{v}_0 \in \mathbf{L}^{6/5}(Q)$ and $(\mathbf{v}_0\cdot\nabla)\widetilde{\mathbf{w}} \in \mathbf{L}^{3/2}(Q)$ so that the right-hand side of (3.5) belongs to $\mathbf{L}^{3/2}([0,T]\times\mathbb{R}^3)$. By virtue of the well-known estimates for solutions of the Cauchy problem for the Stokes equations (see [13, Chap. 4, sect. 6]) we obtain from (3.5) and (3.6) that $\widetilde{\mathbf{w}} \in \mathbf{W}^{1,2}_{3/2}([0,T]\times\mathbb{R}^3)$. Using this inclusion we deduce

$$\|(\mathbf{v}_0\cdot\nabla)\widetilde{\mathbf{w}}\|_{L^{6/5}(Q)} \le \|\mathbf{v}_0\|_{\mathbf{L}^6(Q)}\|\nabla\widetilde{\mathbf{w}}\|_{\mathbf{L}^{3/2}(Q)} \le C\|\widetilde{\mathbf{w}}\|_{\mathbf{W}^{1,2}_{3/2}(Q)}.$$

Therefore, the right-hand side of (3.5) belongs to $\mathbf{L}^{6/5}(Q)$, which implies that $\widetilde{\mathbf{w}} \in \mathbf{W}^{1,2}_{6/5}(Q)$ so that $\widetilde{\mathbf{w}}(T,\cdot) \in \mathbf{L}^{6/5}(\mathbb{R}^3)$. ☐

3.3. Linearized boundary value problems. To derive and analyze (see section 5) weak formulations of the optimality systems for the control problems, we will need the following theorem concerning the solvability of the (homogeneous) boundary value problem for the Oseen equations

$$\begin{cases} \partial_t\mathbf{h} - \Delta\mathbf{h} + [(\widehat{\mathbf{w}}+\mathbf{v}_0)\cdot\nabla]\mathbf{h} + [\mathbf{h}\cdot\nabla](\widehat{\mathbf{w}}+\mathbf{v}_0) + \nabla q = \mathbf{g} \quad \text{in } Q, \\ \operatorname{div}\mathbf{h} = 0 \quad \text{in } Q, \\ \mathbf{h}|_{t=0} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{h}|_\Sigma = \mathbf{0}, \\ \mathbf{h} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty \end{cases} \tag{3.10}$$

that are the linearization of (3.1) about a given vector field $\widehat{\mathbf{w}} \in \mathbf{W}$.

PROPOSITION 3.4. *Assume that* $\mathbf{v}_0 \in \mathbf{C}^\infty(\overline{\Omega})$ *satisfies* (2.4), $\widehat{\mathbf{w}} \in \mathbf{W}$, *and* $\mathbf{g} \in \mathbf{L}^2(Q)$. *Then, there exists a unique solution* $(\mathbf{h},\nabla q) \in \mathcal{V}_0^{(2)}(Q)\times\mathbf{L}^2(Q)$ *of the problem* (3.10). *Moreover,*

$$\|\mathbf{h}\|^2_{\mathcal{V}_0^{(2)}(Q)} + \|\nabla q\|^2_{\mathbf{L}^2(Q)} \le C\|\mathbf{g}\|^2_{\mathbf{L}^2(Q)}. \tag{3.11}$$

The proof is well known; see, e.g., [5, 10, 13, 16].

3.4. Adjoint boundary value problems. To derive strong forms (see sections 6 and 7) of the optimality systems, we will need results concerning the solvability and regularity of the adjoint boundary value problem for (3.10):

$$\begin{cases} \partial_t\mathbf{q} + \Delta\mathbf{q} + [(\widehat{\mathbf{w}}+\mathbf{v}_0)\cdot\nabla]\mathbf{q} - [\nabla(\widehat{\mathbf{w}}+\mathbf{v}_0)]^*\,\mathbf{q} + \nabla r = \mathbf{k} \quad \text{in } Q, \\ \operatorname{div}\mathbf{q} = 0 \quad \text{in } Q, \\ \mathbf{q}|_{t=T} = \mathbf{q}_0(\mathbf{x}) \qquad \text{in } \Omega, \\ \mathbf{q}|_\Sigma = \mathbf{0}, \\ \mathbf{q} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{3.12}$$

where $[\nabla\widehat{\mathbf{w}}]^*\,\mathbf{q} = (\sum_{i=1}^{3}\frac{\partial \widehat{w}_i}{\partial x_j}q_i,\ j=1,2,3)$ for $\widehat{\mathbf{w}} = (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$. The following assertion concerning the unique solvability of (3.12) in the case of $\mathbf{q}_0 = \mathbf{0}$ is completely analogous to Proposition 3.4; the proof is also identical to that of the proposition.

PROPOSITION 3.5. *Assume that* $\mathbf{v}_0 \in \mathbf{C}^\infty(\overline{\Omega})$ *satisfies* (2.4), $\widehat{\mathbf{w}} \in \mathbf{W}$, $\mathbf{k} \in \mathbf{L}^2(Q)$, *and* $\mathbf{q}_0 = \mathbf{0}$. *Then, there exists a unique solution* $(\mathbf{q}, \nabla r) \in \mathcal{V}_T^{(2)}(Q) \times \mathbf{L}^2(Q)$ *of the problem* (3.12) (*with* $\mathbf{q}_0 = \mathbf{0}$) *satisfying the estimate*

$$\|\mathbf{q}\|^2_{\mathcal{V}_T^{(2)}(Q)} + \|\nabla r\|^2_{\mathbf{L}^2(Q)} \le C\|\mathbf{k}\|^2_{\mathbf{L}^2(Q)}. \tag{3.13}$$

We will also need results concerning problem (3.12) with $\mathbf{k} = \mathbf{0}$ and $\mathbf{q}_0 \neq \mathbf{0}$. In this case, we reduce (3.12) to the following system through the change of variable $\tau = T - t$ and redenoting τ by t:

$$\begin{cases} \partial_t \mathbf{q} - \Delta\mathbf{q} - [(\widehat{\mathbf{w}} + \mathbf{v}_0)\cdot\nabla]\mathbf{q} + [\nabla(\widehat{\mathbf{w}} + \mathbf{v}_0)]^*\,\mathbf{q} - \nabla r = \mathbf{0} \quad \text{in } Q, \\ \operatorname{div}\mathbf{q} = 0 \quad \text{in } Q, \\ \mathbf{q}|_{t=0} = \mathbf{q}_0(\mathbf{x}) \qquad \text{in } \Omega, \\ \mathbf{q}|_\Sigma = \mathbf{0}, \\ \mathbf{q} \to \mathbf{0} \quad \text{as } |\mathbf{x}| \to \infty. \end{cases} \tag{3.14}$$

Using well-known energy methods (see [13, 16]), we can prove the following result.

LEMMA 3.6. *Assume that* $\mathbf{v}_0 \in \mathbf{C}^\infty(\overline{\Omega})$ *satisfies* (2.4), $\widehat{\mathbf{w}} \in \mathbf{W}$, *and* $\mathbf{q}_0 \in \mathbf{V}_0^0(\Omega)$. *Then there exists a solution* $(\mathbf{q}, \nabla r)$ *of the problem* (3.14) *satisfying the energy estimate*

$$\begin{aligned} &\|\mathbf{q}\|^2_{L^2(0,T;\mathbf{L}^2(\Omega))} + \|\nabla\mathbf{q}\|^2_{\mathbf{L}^2(Q)} \\ &\qquad \le C\big(\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}(\Omega)} + \|\nabla\mathbf{v}_0\|_{\mathbf{C}(\overline{\Omega})}\big)\|\mathbf{q}_0\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{3.15}$$

where $C(\gamma)$ *is a positive function increasing in* γ.

Improved estimates for $\mathbf{q}$ when $\mathbf{q}_0$ is smoother will also be needed in order to derive strong forms of the optimality systems; these estimates are obtained in Appendix A.3.

4. Existence of solutions for the optimal control problems.

4.1. The solvability of Problem I. We consider the solvability of Problem I formulated in section 2.5.

THEOREM 4.1. *Suppose that the constant* M *in* (2.11) *is sufficiently small. Then there exists a solution* $(\widehat{\mathbf{w}}, \nabla\widehat{p}) \in \mathbf{W} \times (\mathbf{L}^2(Q) + L^2(0,T;\nabla H^{1/2}(\Omega)))$ *for Problem* I.

Proof. Recall that a pair $(\mathbf{w}, p) \in \mathbf{W} \times \mathbf{L}^2(Q)$ is called admissible if it satisfies (2.7)–(2.12) and the functional (2.6) evaluated at $(\mathbf{w}, p)$ is finite. Evidently, the admissible set $\mathcal{U}_{\text{ad}} \neq \emptyset$, as $(\mathbf{w}, p) = (\mathbf{0}, 0) \in \mathcal{U}_{\text{ad}}$. Let $\{(\mathbf{w}_n, \nabla p_n)\} \in \mathcal{U}_{\text{ad}}$ be a minimizing sequence for the functional $\mathcal{J}(\mathbf{w})$:

$$\lim_{n\to\infty} \mathcal{J}(\mathbf{w}_n) = J_{\min} \equiv \inf_{(\mathbf{w},\nabla p)\in\mathcal{U}_{\text{ad}}} \mathcal{J}(\mathbf{w}).$$

By virtue of (2.11) we have $\|\mathbf{w}_n|_\Sigma\|^2_{\mathbf{H}^1(\Sigma)} \le M$. Let $\mathbf{b}_n \equiv \mathbf{w}_n|_\Sigma$; note that $\mathbf{b}_n \in \widehat{\mathbf{H}}^1(\Sigma)$. Consider the boundary value problem (3.1) with $\mathbf{b} = \mathbf{b}_n$. Let $\epsilon > 0$ be a sufficiently small number determined by Proposition 3.2 and suppose $M < \epsilon$. Proposition 3.2 then implies that

$$\|\mathbf{w}_n\|^2_{\mathbf{W}} + \|\nabla p_n\|_{L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))+\mathbf{L}^2(Q)} \leq C(M), \tag{4.1}$$

where $C(M)$ is a positive constant depending on M. The estimate (4.1) allows us to choose a subsequence of $\{\mathbf{w}_n\}$ (denoted by the same) such that

$$\mathbf{w}_n \rightharpoonup \widehat{\mathbf{w}} \quad \text{weakly in } \mathbf{W}.$$

The definition of $\mathbf{W}$ (see (2.22)) then implies that

$$\mathbf{w}_n|_\Sigma = \mathbf{b}_n \rightharpoonup \widehat{\mathbf{b}} \equiv \widehat{\mathbf{w}}|_\Sigma \quad \text{weakly in } \widehat{\mathbf{H}}^1(\Sigma). \tag{4.2}$$

Since $\mathbf{b}_n$ satisfies (2.11) and the set $\{\mathbf{w} \in \widehat{\mathbf{H}}^1(\Sigma) : \mathbf{w} \text{ satisfies (2.11)}\}$ is convex and closed (hence sequentially weakly closed), we see that $\widehat{\mathbf{b}} \in \widehat{\mathbf{H}}^1(\Sigma)$ and $\mathbf{b}$ satisfies (2.11). $\widehat{\mathbf{w}}$ obviously satisfies (2.8)–(2.10). To prove that $\widehat{\mathbf{w}}$ satisfies (2.7) with some $\nabla\widehat{p} \in L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2})) + \mathbf{L}^2(Q)$, we proceed by noting that

$$\begin{aligned}&\int_Q (\partial_t \mathbf{w}_n - \Delta \mathbf{w}_n)\cdot \boldsymbol{\phi}\,d\mathbf{x}\,dt \\ &\qquad \to \int_Q (\partial_t \widehat{\mathbf{w}} - \Delta\widehat{\mathbf{w}})\cdot\boldsymbol{\phi}\,d\mathbf{x}\,dt \qquad \forall\,\boldsymbol{\phi} \in L^2(0,T;\mathbf{V}^1_0(\Omega)),\end{aligned} \tag{4.3}$$

where $\mathbf{V}^1_0(\Omega)$ is defined by (A.2). Using Lemma 2.2 with $k = 1,2,3,\ldots$, we may choose subsequences $\{\mathbf{w}_{n,k}\}$ converging to $\widehat{\mathbf{w}}$ in $\mathbf{L}^2(Q_{\rho+k})$. Then, by choosing the diagonal subsequence $\{\mathbf{w}_j\}$, we infer that

$$\mathbf{w}_j \to \widehat{\mathbf{w}} \qquad \text{strongly in } \mathbf{L}^2(Q_{\rho+k}) \text{ for each } k = 1,2,3,\ldots. \tag{4.4}$$

We now take the $\mathbf{L}^2(Q)$ inner product between an arbitrary $\boldsymbol{\phi} \in L^2(0,T;\mathbf{V}^1_0(\Omega))$ and (2.7) for $(\mathbf{w}_j, \nabla p_j)$. The term involving ∇p_j obviously vanishes. Integrating by parts and passing to the limit with the help of (4.3) and (4.4), we obtain the following equation for $\widehat{\mathbf{w}}$:

$$\begin{aligned}\int_Q \Big(\boldsymbol{\phi}\cdot\partial_t\widehat{\mathbf{w}} + (\nabla\widehat{\mathbf{w}}) : (\nabla\boldsymbol{\phi}) + (\widehat{\mathbf{w}}\cdot\nabla)\mathbf{v}_0\cdot\boldsymbol{\phi} - [(\widehat{\mathbf{w}} + \mathbf{v}_0)\cdot\nabla]\boldsymbol{\phi}\cdot\widehat{\mathbf{w}}\Big)\,d\mathbf{x}\,dt& \\ = 0 \qquad \forall\,\boldsymbol{\phi} \in L^2(0,T;\mathbf{V}^1_0(\Omega)).&\end{aligned} \tag{4.5}$$

Since $\widehat{\mathbf{w}} \in \mathbf{W}$, we see from definition (2.22) that

$$\widehat{\mathbf{w}} = \mathcal{E}\widehat{\mathbf{b}} + \mathbf{w}, \tag{4.6}$$

where $\mathcal{E}$ is the extension operator of Proposition 3.1, $\mathcal{E}\widehat{\mathbf{b}} \in \mathbf{V}^{1,1/2}_\rho(Q)$, and $\mathbf{w} \in \mathcal{V}^{(2)}_0(Q)$. Substitution of (4.6) into (4.5) yields

$$\begin{aligned}&\int_Q \Big(\partial_t\mathbf{w} - \Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)(\mathbf{v}_0 + \mathcal{E}\widehat{\mathbf{b}}) \\ &\qquad\qquad + [(\mathcal{E}\widehat{\mathbf{b}} + \mathbf{v}_0 + \mathbf{w})\cdot\nabla]\mathbf{w} - \mathbf{f}_1 + \nabla p_1\Big)\cdot\boldsymbol{\phi}\,d\mathbf{x}\,dt \\ &\quad = \int_Q \Big(\partial_t\mathbf{w} - \Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)(\mathbf{v}_0 + \mathcal{E}\widehat{\mathbf{b}}) \\ &\qquad\qquad + [(\mathcal{E}\widehat{\mathbf{b}} + \mathbf{v}_0 + \mathbf{w})\cdot\nabla]\mathbf{w} - \mathbf{f}_1\Big)\cdot\boldsymbol{\phi}\,d\mathbf{x}\,dt \\ &\quad = 0 \qquad \forall\,\boldsymbol{\phi} \in L^2(0,T;\mathbf{V}^1_0(\Omega)),\end{aligned} \tag{4.7}$$

where $-\mathbf{f}_1 = \partial_t\mathcal{E}\widehat{\mathbf{b}} + \mathbf{g} + (\mathcal{E}\widehat{\mathbf{b}}\cdot\nabla)\mathbf{v}_0 + [(\mathcal{E}\widehat{\mathbf{b}}+\mathbf{v}_0)\cdot\nabla]\mathcal{E}\widehat{\mathbf{b}}$ and $-\Delta\mathcal{E}\widehat{\mathbf{b}} = \mathbf{g}+\nabla p_1$ with $\mathbf{g}\in\mathbf{L}^2(Q)$ and $\nabla p_1 \in L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))$. The inclusions $\mathbf{w}\in\mathcal{V}_0^{(2)}(Q)$ and $\mathcal{E}\widehat{\mathbf{b}}\in\mathbf{V}_\rho^{1,1/2}(Q)\subset\mathbf{V}^{1,1/2}(Q)$ and Sobolev embedding theorems imply that $\mathbf{f}_1\in\mathbf{L}^2(Q)$ and

$$\partial_t\mathbf{w} - \Delta\mathbf{w} + (\mathbf{w}\cdot\nabla)(\mathbf{v}_0+\mathcal{E}\widehat{\mathbf{b}}) + [(\mathcal{E}\widehat{\mathbf{b}}+\mathbf{v}_0+\mathbf{w})\cdot\nabla]\mathbf{w}\in\mathbf{L}^2(Q).$$

Recalling the Weyl decomposition

$$\mathbf{L}^2(Q) = \mathbf{V}_0^0(\Omega)\oplus\nabla H^1(\Omega),$$

where $\mathbf{V}_0^0(\Omega)$ is defined by (2.14) and

$$\nabla H^1(\Omega) = \{\mathbf{v}\in\mathbf{L}^2(\Omega)\ :\ \mathbf{v}=\nabla p,\ p\in H^1_{\mathrm{loc}}(\Omega)\},$$

we obtain from (4.7) that there exists a $\nabla p\in L^2(0,T;\nabla H^1(\Omega))$ such that

$$\begin{aligned}\partial_t\mathbf{w} - \Delta\mathbf{w} &+ (\mathbf{w}\cdot\nabla)(\mathbf{v}_0+\mathcal{E}\widehat{\mathbf{b}})\\ &+[(\mathcal{E}\widehat{\mathbf{b}}+\mathbf{v}_0+\mathbf{w})\cdot\nabla]\mathbf{w}+\nabla p-\mathbf{f}_1=\mathbf{0}\quad\text{in } Q,\end{aligned}\tag{4.8}$$

where (4.8) is understood as an equality in $\mathbf{L}^2(Q)$. Substituting (4.6) into (4.8) yields the equality

$$\partial_t\widehat{\mathbf{w}} - \Delta\widehat{\mathbf{w}} + (\widehat{\mathbf{w}}\cdot\nabla)\mathbf{v}_0 + [(\widehat{\mathbf{w}}+\mathbf{v}_0)\cdot\nabla]\widehat{\mathbf{w}} + \nabla\widehat{p} = 0 \qquad \text{in } Q,$$

where $\nabla\widehat{p} = \nabla p_1 + \nabla p \in L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2})) + L^2(0,T;\nabla H^1_{\mathrm{loc}}(\Omega))$. Thus, we have proved that $(\widehat{\mathbf{w}},\nabla\widehat{p})$ satisfies (2.7).

Since $\mathbf{w}_n \rightharpoonup \widehat{\mathbf{w}}$ in $\mathbf{W}$ and the functional

$$\begin{aligned}\mathcal{J}_1(\mathbf{w}) = &\int_0^T\int_\Omega \mathcal{D}(\mathbf{w}+\mathbf{v}_0):\mathcal{D}(\mathbf{w}+\mathbf{v}_0)\,d\mathbf{x}\,dt\\ &+\frac{1}{2}\int_\Omega(|\mathbf{w}(T,\mathbf{x})|^2 + 2[(\mathbf{w}(T,x)\cdot(\mathbf{v}_0(\mathbf{x})-\mathbf{v}_\infty))])\,d\mathbf{x}\end{aligned}$$

is convex (and therefore it is lower semicontinuous with respect to weak convergence), we deduce that

$$\mathcal{J}_1(\widehat{\mathbf{w}}) \le \liminf_{n\to\infty}\mathcal{J}_1(\mathbf{w}_n).\tag{4.9}$$

The facts that Σ is compact and $\dim\Sigma = 3$ allow us to use embedding theorems to deduce that the embedding $\widehat{\mathbf{H}}^1(\Sigma)\hookrightarrow\mathbf{L}^3(\Sigma)$ is compact. Then (4.2) implies $\mathbf{w}|_\Sigma = \mathbf{b}_n \to \widehat{\mathbf{b}} = \widehat{\mathbf{w}}|_\Sigma$ strongly in $\mathbf{L}^3(\Sigma)$. Thus, by defining the functional

$$\mathcal{J}_2(\mathbf{w}) = \frac{1}{2}\int_0^T\int_{\partial\Omega}|\mathbf{w}-\mathbf{v}_\infty|^2\mathbf{w}\cdot\mathbf{n}\,ds\,dt,$$

we have

$$\mathcal{J}_2(\widehat{\mathbf{w}}) \le \lim_{n\to\infty}\mathcal{J}_2(\mathbf{w}_n).\tag{4.10}$$

The relations (4.9) and (4.10) and the equality $\mathcal{J}(\mathbf{w}) = \mathcal{J}_1(\mathbf{w}) + \mathcal{J}_2(\mathbf{w})$ yield

$$\mathcal{J}(\widehat{\mathbf{w}}) \le \liminf_{n\to\infty}\mathcal{J}(\mathbf{w}_n) = J_{\mathrm{inf}}.$$

Therefore, the pair $(\widehat{\mathbf{w}},\widehat{p})$ is a solution of Problem I. $\square$

4.2. The solvability of Problem II. We next prove that there exists a solution to Problem II for sufficiently large N, where N is the parameter appearing in the definition of $\mathcal{J}_N(\mathbf{w})$; see (2.24).

THEOREM 4.2. *Suppose that the parameter N of the functional $\mathcal{J}_N(\mathbf{w})$ satisfies $N \geq N_0$ for a sufficiently large, fixed constant $N_0 > 0$. Then there exists a solution $(\widehat{\mathbf{w}}, \widehat{p}) \in \mathbf{W} \times [\mathbf{L}^2(Q) + L^2(0,T; \nabla H^{1/2}(\Omega_{\rho+2}))]$ for Problem* II.

Proof. Recall that a pair $(\mathbf{w}, \nabla p) \in \mathbf{W} \times \mathbf{L}^2(Q)$ is called admissible for Problem II if it satisfies (2.7)–(2.10) and (2.12). We denote by $\mathcal{U}_{\text{ad}}^N$ the set of all admissible pairs for Problem II.

Let $\epsilon > 0$ be a sufficiently small number such that for any boundary data $\mathbf{b} \in \widehat{\mathbf{H}}^1(\Sigma)$ satisfying (3.3), the assertions of Proposition 3.2 are true. We fix the constant M in (2.11) as

$$M \in (0, \epsilon). \tag{4.11}$$

Let $(\widehat{\mathbf{w}}, \widehat{p})$ be a solution of Problem I (the existence of such a pair is guaranteed by Theorem 4.1). We define $N_0 > 0$ by the relation

$$\frac{2}{N_0}\mathcal{J}_{N_0}(\widehat{\mathbf{w}}) \equiv \frac{2}{N_0}\mathcal{J}(\widehat{\mathbf{w}}) + R(\widehat{\mathbf{w}}) = \epsilon, \tag{4.12}$$

where the functionals $\mathcal{J}_N$, $\mathcal{J}$, and R are defined by (2.24), (2.6), and (2.11), respectively. The number N_0 satisfying (4.12) is well defined thanks to (4.11) and the estimate $R(\widehat{\mathbf{w}}) \leq M$ for every solution $\widehat{\mathbf{w}}$ of Problem I. Thus, for each $N > N_0$ we have

$$\frac{2}{N}\mathcal{J}_N(\widehat{\mathbf{w}}) \leq \frac{2}{N_0}\mathcal{J}_{N_0}(\widehat{\mathbf{w}}) = \epsilon. \tag{4.13}$$

Set

$$\mathcal{U}_{\text{ad}}^{N,\epsilon} = \left\{ (\mathbf{w}, \nabla p) \in \mathcal{U}_{\text{ad}}^N \ : \ \frac{1}{N}\mathcal{J}_N(\widehat{\mathbf{w}}) \leq \epsilon \right\}.$$

By virtue of (4.13), $\mathcal{U}_{\text{ad}}^{N,\epsilon}$ is not empty. We now choose a minimizing sequence $\{(\mathbf{w}_n, \nabla p_n)\} \subset \mathcal{U}_{\text{ad}}^{N,\epsilon}$ for Problem II:

$$\lim_{n\to\infty} \frac{2}{N}\mathcal{J}_N(\mathbf{w}_n) = J_{N,\inf} \equiv \inf_{(\mathbf{w},\nabla p) \in \mathcal{U}_{\text{ad}}^{N,\epsilon}} \frac{2}{N}\mathcal{J}_N(\mathbf{w}).$$

Since $\mathbf{w}_n \in \mathcal{U}_{\text{ad}}^{N,\epsilon}$, it follows from (4.13) and (2.6) that for every $N > N_0$,

$$R(\mathbf{w}_n|_\Sigma) \leq \epsilon. \tag{4.14}$$

Denoting $\mathbf{b}_n \equiv \mathbf{w}_n|_\Sigma$ and using (4.14) and (2.11) (the definition of functional R), we see that $\|\mathbf{b}_n\|_{\mathbf{H}^1(\Sigma)} < \epsilon$, i.e., the boundary condition $\mathbf{b}_n$ satisfies the assumptions of Proposition 3.1. Thus, $(\mathbf{w}_n, \nabla p_n)$, being the solution of (3.1) with $\mathbf{b}$ replaced by $\mathbf{b}_n$, satisfies the estimate (3.11) in which $\mathbf{w}$ and p are replaced by $\mathbf{w}_n$ and p_n, respectively. Then, by repeating the relevant segment of the proof of Theorem 4.1, we prove the existence of a solution $(\widehat{\widehat{\mathbf{w}}}, \widehat{\widehat{p}})$ for ProblemII. $\square$

5. A weak formulation of an optimality system for Problem II and regularity of the adjoint velocity.

5.1. Abstract Lagrange multiplier principles. We consider an abstract minimization problem. Let X_1 and X_2 be two Banach spaces. Let $f : X_1 \to \mathbb{R}$ and $g : X_1 \to \mathbb{R}$ be functionals and $F : X_1 \to X_2$ be a mapping. We seek a $z \in X_1$ such that

$$f(z) = \inf_{u \in \mathcal{U}_{\rm ad}} f(u), \tag{5.1}$$

where

$$\mathcal{U}_{\rm ad} = \big\{ u \in X_1 : F(u) = 0 \text{ and } g(u) \le 0 \big\}.$$

The Lagrange functional for the minimization problem (5.1) is defined by

$$\mathcal{L}(z, \lambda_0, \lambda, q) = \lambda_0 f(z) + \langle F(z), q\rangle + \lambda g(z) \tag{5.2}$$

for all $z \in X_1$, $\lambda_0 \in \mathbb{R}$, $\lambda \in \mathbb{R}$, and $q \in X_2^*$, where X_2^* is the dual space of X_2 and $\langle \cdot,\cdot\rangle$ denotes the duality pairing between X_2 and X_2^*. We quote a standard abstract Lagrange principle in the following particular form (see [2]).

THEOREM 5.1. *Let z be a solution of* (5.1)*. Assume that the mappings f, g, and F are continuously differentiable and that the image of the operator $F'(z) : X_1 \to X_2$ is closed. Then there exists a $q \in X_2^*$, $\lambda_0 \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ such that the triplet $(q, \lambda_0, \lambda) \ne (0,0,0)$ (i.e., the quantities in the triplet do not all vanish simultaneously),*

$$\langle \mathcal{L}_z(z, \lambda_0, \lambda, q), h\rangle = 0 \qquad \forall\, h \in X_1, \tag{5.3}$$

$$\lambda_0 \ge 0, \quad \lambda \ge 0, \quad \textit{and} \quad \lambda g(z) = 0, \tag{5.4}$$

where $\mathcal{L}_z(\cdot,\cdot,\cdot,\cdot)$ denotes the Fréchet derivative of $\mathcal{L}$ with respect to the first argument. Furthermore, if $F'(z) : X_1 \to X_2$ is an epimorphism and the constraint $g(z) \le 0$ is absent in problem (5.1)*, then $\lambda_0 \ne 0$ and λ_0 can be taken as* 1*.*

5.2. The weak formulation of an optimality system. In this subsection we apply Theorem 5.1 to derive a weak form of an optimality system of equations for Problem II by applying a trick employed in [6, Chap. 1, Thm. 1.8] that consists of using the space of variations which does not contain the solution of the considered extreme problem.

In order to apply Theorem 5.1, we first have to define the space $\mathbf{V}_{A^*}(Q)$ in which we search for the adjoint vector field for the optimality system. This space is determined in Appendix B; see (B.14).

THEOREM 5.2. *Assume that $(\widehat{\mathbf{w}}, \nabla\widehat{p}) \in \mathbf{W} \times [\mathbf{L}^2(Q) + L^2(0,T; \nabla H^{1/2}(\Omega_{\rho+2}))]$ is a solution for Problem* II*. Then there exists a $\widehat{\mathbf{q}} \in \mathbf{V}_{A^*}(Q)$ such that*

$$\begin{aligned}
&\int_Q \big\{\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}})\cdot\nabla]\mathbf{h} + (\mathbf{h}\cdot\nabla)(\mathbf{v}_0 + \widehat{\mathbf{w}})\big\} \cdot \widehat{\mathbf{q}}\, d\mathbf{x}\, dt \\
&+2\int_Q \mathcal{D}(\widehat{\mathbf{w}} + \mathbf{v}_0) : \mathcal{D}(\mathbf{h})\, d\mathbf{x}\, dt + \int_\Omega (\widehat{\mathbf{w}}(T,\mathbf{x}) + \mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty)\cdot \mathbf{h}\, d\mathbf{x} \\
&+\int_\Sigma \Big(\mathbf{h}\cdot(\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}}\cdot\mathbf{n} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2 \mathbf{h}\cdot\mathbf{n} \\
&\qquad + N[\partial_t\widehat{\mathbf{w}}\cdot\partial_t\mathbf{h} + \nabla_\tau\widehat{\mathbf{w}} : \nabla_\tau\mathbf{h} + \widehat{\mathbf{w}}\cdot\mathbf{h}]\Big)\, ds\, dt = 0 \qquad \forall\, \mathbf{h} \in \mathbf{V}_A^{1,2}(Q)
\end{aligned} \tag{5.5}$$

Proof. First we convert Problem II into an equivalent optimal control problem through the change of variables

$$\mathbf{w} = \widehat{\mathbf{w}} + \mathbf{z} \quad \text{and} \quad \nabla p = \nabla \widehat{p} + \nabla p_1. \tag{5.6}$$

The optimal control problem for $(\mathbf{z}, \nabla p_1)$ is

$$\mathcal{J}_N(\widehat{\mathbf{w}} + \mathbf{z}) \to \inf \tag{5.7}$$

subject to the constraints

$$\partial_t \mathbf{z} - \Delta \mathbf{z} + [(\mathbf{v}_0 + \widehat{\mathbf{w}} + \mathbf{z}) \cdot \nabla]\mathbf{z} + (\mathbf{z} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0) = -\nabla p_1, \qquad \operatorname{div} \mathbf{z} = 0, \qquad \text{in } Q, \tag{5.8}$$

$$\mathbf{z}|_{t=0} = \mathbf{0} \quad \text{and} \quad \mathbf{z} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty. \tag{5.9}$$

We consider the problem (5.7)–(5.9) for $\mathbf{z}$ running through the space $\mathbf{V}_A^{1,2}(Q)$ defined in Appendix B; see (B.21). Evidently, for each $\mathbf{z} \in \mathcal{W}_0^{(2)}(Q)$ (this space is defined in (B.21)), the left-hand side of the first equation in (5.8) belongs to $\mathbf{L}^2(Q)$. Let

$$P : \mathbf{L}^2(Q) \to L^2(0, T; \mathbf{V}_0^0(\Omega))$$

be the projection operator. Then Weyl's decomposition allows us to transform (5.8) into

$$P\Big(\partial_t \mathbf{z} - \Delta \mathbf{z} + [(\mathbf{v}_0 + \widehat{\mathbf{w}} + \mathbf{z}) \cdot \nabla]\mathbf{z} + (\mathbf{z} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0)\Big) = 0 \quad \text{and} \quad \operatorname{div} \mathbf{z} = 0. \tag{5.10}$$

The embedding $\mathcal{W}_0^{(2)}(Q) \hookrightarrow \mathbf{W}$ and the assumption

$$(\widehat{\mathbf{w}}, \widehat{p}) \in \mathbf{W} \times [\mathbf{L}^2(Q) + L^2(0, T; \nabla H^{1/2}(\Omega_{\rho+2}))]$$

being a solution for Problem II imply that $\widehat{\mathbf{z}} \equiv \mathbf{0}$ is a solution of optimal control problem (5.7) and (5.9)–(5.10). We now apply Theorem 5.1 to this control problem. We set $X_1 = \mathbf{V}_A^{1,2}(Q)$ and $X_2 = \mathbf{V}_A(Q)$ (see (B.21) and (B.12)). We define the mappings $f : X_1 \to \mathbb{R}$ and $F : X_1 \to X_2$ as follows:

$$\begin{aligned} &f(\mathbf{z}) = \mathcal{J}_N(\mathbf{z} + \widehat{\mathbf{w}}), \\ &F(\mathbf{z}) = P\Big(\partial_t \mathbf{z} - \Delta \mathbf{z} + [(\mathbf{v}_0 + \widehat{\mathbf{w}} + \mathbf{z}) \cdot \nabla]\mathbf{z} + (\mathbf{z} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0)\Big). \end{aligned} \tag{5.11}$$

Note that the constraint (5.9) is built into the space X_1 and the inequality constraint $g \le 0$ is absent in Problem II. We have to show that f and F defined in (5.11) are continuously differentiable. Here, we will only prove the continuous differentiability of F since this is more difficult than the corresponding property of f. The proof that the operator defined by the left-hand side of (5.8) acts continuously from X_1 to $\mathbf{L}^2(Q)$ is evident. To prove its continuity from X_1 to $\mathbf{L}^{6/5}(Q)$ we have to repeat the proof of Lemma 3.3 for all terms except for $(\mathbf{v}_0 \cdot \nabla)\mathbf{z}$. Using Holder's inequality and interpolation bound, we have

$$\begin{aligned} \|(\mathbf{v}_0 \cdot \nabla)\mathbf{z}\|_{L^{6/5}(Q)} &\le \|\mathbf{v}_0\|_{\mathbf{L}^6(Q)} \|\nabla \mathbf{z}\|_{\mathbf{L}^{3/2}(Q)} \\ &\le \|\mathbf{v}_0\|_{\mathbf{L}^6(Q)} \|\nabla \mathbf{z}\|^{1/2}_{\mathbf{L}^{6/5}(Q)} \|\nabla \mathbf{z}\|^{1/2}_{\mathbf{L}^2(Q)} \le C \|\nabla \mathbf{z}\|_{\mathbf{V}_A^{1/2}(Q)}. \end{aligned}$$

Therefore, the continuity of $F(z)$ from (5.11) is reduced to the proof of the continuity of the projector $P : \mathbf{L}_A(Q) \to \mathbf{V}_A(Q)$ that is actually contained in the proof of decomposition (B.16). Hence we have proved the continuity of $P : \mathbf{L}_A \to \mathbf{V}_A$. Consequently, we have proved the continuity of $F : \mathbf{V}_A^{1,2}(Q) \to \mathbf{V}_A(Q)$.

The derivative of F at the solution $\mathbf{0}$ for the problem (5.7) and (5.9)–(5.10) is given by

$$F'(\mathbf{0})\mathbf{h} = P\Big(\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{h} + (\mathbf{h} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0)\Big)$$

and the operator $F'(0) : X_1 \to X_2$ is continuous, which can be proved in a way analogous to the proof of the continuity of operator F. To show $F'(0)$ is surjective, it suffices to prove that for each $\mathbf{f} \in \mathbf{V}_A(Q)$ there exists a solution $\mathbf{h} \in \mathbf{V}_A^{1,2}(Q)$ for the problem

$$\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{h} + (\mathbf{h} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0) + \nabla p = \mathbf{f} \tag{5.12}$$

$$\operatorname{div} \mathbf{h} = 0, \quad \mathbf{h}|_{t=0} = \mathbf{0}, \quad \mathbf{h}|_\Sigma = \mathbf{0}, \quad \text{and} \quad \mathbf{h} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \tag{5.13}$$

with some $\nabla p \in \mathbf{L}^2(Q) \cap \mathbf{L}^{6/5}(Q)$.

Since $\mathbf{V}_A(Q) \subset \mathbf{L}^2(Q)$, by virtue of Proposition 3.4 there exists a unique solution $\mathbf{h} \in \mathcal{W}_0^{(2)}(Q)$ of (5.12) and (5.13). Moving the last three terms in the left-hand side of (5.12) to the right-hand side and using arguments of the proof of Lemma 3.3, we see that this new right-hand side belongs to $\mathbf{L}^{6/5}(Q)$. Extending $\mathbf{h}$ in (5.12) from $(0,T) \times \Omega$ into $\widetilde{\mathbf{h}} \in \mathcal{W}_0^{(2)}((0,T) \times \mathbb{R}^3)$ and using estimates of solutions of the Cauchy problem for the Stokes equations, we obtain, as in the proof of Lemma 3.3, that $\mathbf{h} \in \mathcal{W}_0^{(2)}((0,T) \times \mathbb{R}^3) \cap \mathbf{W}_{6/5}^{1,2}$. Hence $\widetilde{\mathbf{h}} = \widetilde{\mathbf{h}}|_Q \in \mathbf{V}_A^{1,2}(Q)$.

Hence, we have verified all assumptions of Theorem 5.1 and that theorem implies that there exists a $\hat{\mathbf{q}} \in \mathbf{V}_{A^*}(Q)$ such that (5.3) holds with $\lambda_0 = 1$, λ absent, and

$$\begin{aligned} \mathcal{L}(\mathbf{z}, \lambda_0, \mathbf{q}) &= \mathcal{J}_N(\mathbf{z} + \widehat{\mathbf{w}}) \\ &\quad + \int_Q \Big(\partial_t \mathbf{z} - \Delta \mathbf{z} + [(\mathbf{v}_0 + \widehat{\mathbf{w}} + \mathbf{z}) \cdot \nabla]\mathbf{z} + (\mathbf{z} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0)\Big) \cdot \hat{\mathbf{q}}\, d\mathbf{x}\, dt. \end{aligned} \tag{5.14}$$

Equation (5.3) with $\mathcal{L}$ defined by (5.14) takes on the form of (5.5). $\square$

We express $\widehat{\mathbf{q}}(t, \mathbf{x})$ in the form

$$\widehat{\mathbf{q}}(t, \mathbf{x}) = \mathbf{q}(t, \mathbf{x}) - (\mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty), \tag{5.15}$$

where $\mathbf{v}_0(\mathbf{x})$ is the steady state solution from Proposition 2.1 and $\mathbf{v}_\infty \in \mathbb{R}^3$ is the vector from (2.4). By virtue of definitions (B.3), (B.4), and (B.14), the inclusion $\mathbf{v}_0(x) - \mathbf{v}_\infty \in \mathbf{L}^6(Q) \cap \mathbf{C}^\infty(Q)$ implies $\mathbf{v}_0(x) - \mathbf{v}_\infty \in \mathbf{V}_{A^*}(Q)$. Since $\hat{\mathbf{q}} \in \mathbf{V}_{A^*}$, we have $\mathbf{q} \in \mathbf{V}_{A^*}$.

5.3. Regularity of the adjoint velocity in the optimality system. In this subsection we will derive some regularity estimates for the adjoint variable $\mathbf{q}$ defined by Theorem 5.2 and (5.15).

We substitute (5.15) in (5.5) and restrict, in (5.5), $\mathbf{h}$ to $\mathbf{V}_A^{1,2}(Q) \cap \mathcal{V}_0^{(2)}(Q)$ (see (2.20)). Then we obtain

$$\begin{aligned} &\int_Q \Big[\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{h} + (\mathbf{h} \cdot \nabla)(\mathbf{v}_0 + \widehat{\mathbf{w}})\Big] \cdot \Big[\mathbf{q} - (\mathbf{v}_0 - \mathbf{v}_\infty)\Big]\, d\mathbf{x}\, dt \\ &+ \int_Q 2\mathcal{D}(\mathbf{h}) : \mathcal{D}(\mathbf{v}_0 + \widehat{\mathbf{w}})\, d\mathbf{x}\, dt \\ &+ \int_\Omega \mathbf{h}(T, x) \cdot \big[\widehat{\mathbf{w}}(T, x) + (\mathbf{v}_0(x) - \mathbf{v}_\infty)\big]\, d\mathbf{x} = 0 \quad \forall\, \mathbf{h} \in \mathbf{V}_A^{1,2} \cap \mathcal{V}_0^{(2)}(Q). \end{aligned} \tag{5.16}$$

The relation (5.16) implies that $\mathbf{q} \in L^2(0, T; \mathbf{V}_0^0(\Omega))$ is the generalized solution of the boundary value problem

$$P\Big(\partial_t \mathbf{q} + \Delta \mathbf{q} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{q} - [\nabla(\mathbf{v}_0 + \widehat{\mathbf{w}})]^* \mathbf{q}\Big) = P\Phi, \tag{5.17}$$

$$\operatorname{div} \mathbf{q} = 0, \qquad \mathbf{q}\big|_\Sigma = \mathbf{0}, \qquad \mathbf{q} \to \mathbf{0} \text{ as } |\mathbf{x}| \to \infty \tag{5.18}$$

and

$$\mathbf{q}\big|_{t=T} = -P\widehat{\mathbf{w}}(T, \cdot). \tag{5.19}$$

Here

$$\Phi(t, x) = \Delta \mathbf{v}_0 + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{v}_0 - [\nabla(\mathbf{v}_0 + \widehat{\mathbf{w}})]^*(\mathbf{v}_0 - \mathbf{v}_\infty) - \Delta(\mathbf{v}_0 + \widehat{\mathbf{w}}). \tag{5.20}$$

We emphasize that the "initial" condition for $\mathbf{q}$, i.e., the right-hand side of (5.19), does not contain $\mathbf{v}_0 - \mathbf{v}_\infty$ although this term is present in integral over Ω in (5.16). We have the following regularity result for $\mathbf{q}(T, \cdot)$.

LEMMA 5.3. *Let $\widehat{\mathbf{w}} \in \mathbf{W}$ be a solution of Problem* I *or* II. *Then $P\widehat{\mathbf{w}}(T, \cdot) \in \mathbf{H}^{3/4}(\Omega) \cap \mathbf{V}_0^0(\Omega)$, where P is the projection operator defined by* (A.16).

Proof. We consider the operator $P = P_0 : \mathbf{L}^2(\Omega) \to \mathbf{V}_0^0(\Omega)$, which was defined as the Weyl orthogonal projection operator. We recall that for $s \in [0, 2]$, $P\mathbf{H}^s(\Omega) \subset \mathbf{H}^s(\Omega)$ and the operator $P : \mathbf{H}^s(\Omega) \to \mathbf{H}^s(\Omega)$ is bounded. Indeed, for each $\mathbf{u} \in \mathbf{L}^2(\Omega)$, $P\mathbf{u} = \mathbf{u} - \nabla p$, where $\nabla p \in G_0 \equiv \{\nabla\phi \in \mathbf{L}^2(\Omega) : \phi \in H^1_{\mathrm{loc}}(\Omega)\}$ is the solution of the variational problem

$$\int_\Omega \nabla p(\mathbf{x}) \cdot \nabla\phi(\mathbf{x})\, d\mathbf{x} = \int_\Omega \mathbf{u} \cdot \nabla\phi(\mathbf{x})\, d\mathbf{x} \quad \forall\, \nabla\phi \in G_0. \tag{5.21}$$

The existence and uniqueness of a solution for this problem is well known (see [13]). Let $\mathbf{u} \in \mathbf{H}^2(\Omega)$. Integration by parts in (5.21) yields that ∇p is the solution of the boundary value problem: $\nabla p \in \mathbf{L}^2(\Omega)$,

$$-\Delta p = \operatorname{div} \mathbf{u} \quad \text{in } \Omega \qquad \text{and} \qquad \frac{\partial p}{\partial n}\Big|_{\partial\Omega} = (\mathbf{u} \cdot \mathbf{n})\big|_{\partial\Omega}.$$

By elliptic regularity and the regularity for div-curl problems (see [16]), we obtain that $p \in H^2_{\mathrm{loc}}(\Omega)$ and

$$\|\nabla p\|_{\mathbf{H}^2(\Omega)} \le C\Big(\|\operatorname{div} \mathbf{u}\|_{H^1(\Omega)} + \|(\mathbf{u} \cdot \mathbf{n})\big|_{\partial\Omega}\|_{H^{3/2}(\partial\Omega)}\Big) \le C\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}.$$

Thus, we have that the operators $I - P : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ and $I - P : \mathbf{H}^2(\Omega) \to \mathbf{H}^2(\Omega)$ are bounded. By interpolation theorems, the operator $P : \mathbf{H}^s(\Omega) \to \mathbf{H}^s(\Omega)$ and $I - P : \mathbf{H}^s(\Omega) \to \mathbf{H}^s(\Omega)$ are bounded for each $s \in [0, 2]$.

Since $\widehat{\mathbf{w}} \in \mathbf{W} \subset L^2(0,T;\mathbf{H}^{3/2}(\Omega))$, we deduce that

$$P\widehat{\mathbf{w}} \in L^2(0,T;\mathbf{H}^{3/2}(\Omega)). \tag{5.22}$$

Because $\widehat{\mathbf{w}}$ is a solution of Problem I or II, it satisfies (2.7)–(2.10). Integrating (2.7) over $t \in [0,\tau]$ and then applying the operator P, we obtain

$$P\widehat{\mathbf{w}}(\tau,\cdot) = \int_0^\tau \Big(P\Delta\widehat{\mathbf{w}}(t,\cdot) - P[(\widehat{\mathbf{w}}+\mathbf{v}_0)\cdot\nabla]\widehat{\mathbf{w}} - P(\widehat{\mathbf{w}}\cdot\nabla)\mathbf{v}_0\Big)\,dt. \tag{5.23}$$

Since $\widehat{\mathbf{w}} \in \mathbf{W}$ and therefore $\Delta\widehat{\mathbf{w}} = g + \nabla p$ with $g \in \mathbf{L}^2(Q)$, $\nabla p \in L^2(0,T;\nabla H^{1/2}(\Omega))$, we easily see that

$$P\widehat{\mathbf{w}}(t,\cdot) \in L^2(0,T;\mathbf{V}_0^0(\Omega)) \text{ and } \widehat{\mathbf{w}} \in L^2(0,T;\mathbf{H}^{3/2}(\Omega)) \cap L^\infty(0,T;\mathbf{H}^1(\Omega)).$$

From the last inclusion, the inclusion $\mathbf{v}_0 \in \mathbf{C}^2(\overline{\Omega})$, and Sobolev embedding theorems, we conclude that the integrand from the right-hand side of (5.23) belongs to $L^2(0,T;\mathbf{V}_0^0(\Omega))$. Hence, by differentiating (5.23) with respect to τ we obtain

$$\partial_t P\widehat{\mathbf{w}}(t,\cdot) \in L^2(0,T;\mathbf{V}_0^0(\Omega)) \subset L^2(0,T;\mathbf{L}^2(\Omega)). \tag{5.24}$$

Then (5.22), (5.24), and the trace theorems of [14] imply that

$$P\widehat{\mathbf{w}}(t,\cdot) \in \mathbf{C}(0,T;\mathbf{H}^{3/4}(\Omega)). \qquad \square$$

The spaces $\mathbf{V}_\sigma^s$ used below are defined and studied in Appendix A.1.

THEOREM 5.4. *Assume that* $\mathbf{q} \in \mathbf{V}_{A^*}(Q)$ *satisfies* (5.16). *Then for each* $\delta > 0$, $\mathbf{q} \in L^2(0,T-\delta;\mathbf{V}_\sigma^2(\Omega)) \cap H^1(0,T-\delta;\mathbf{V}_\sigma^0(\Omega))$. *Furthermore, there exists a constant* $C > 0$ *such that for each* $\delta \in [0,T]$ *and each* $\epsilon \in (0,1/2)$, $\mathbf{q}$ *satisfies the estimate*

$$\begin{aligned}&\int_0^{T-\delta} \Big(\|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}_\sigma^2(\Omega)} + \|\partial_t\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}_\sigma^0(\Omega)}\Big)\,dt \\ &\le C\Big(\delta^{-\epsilon-1/2}\|P\widehat{\mathbf{w}}(T,\cdot)\|^2_{\mathbf{V}_\sigma^{-\epsilon+1/2}(\Omega)} + \|P\Phi\|^2_{L^2(0,T;\mathbf{V}_\sigma^0(\Omega))}\Big).\end{aligned} \tag{5.25}$$

In particular, $\mathbf{q}$ *satisfies* (5.17) *in* $L^2(0,T-\delta;\mathbf{V}_\sigma^0(\Omega))$ *and* (5.18) *for every* $\delta \in (0,T)$ *and satisfies* (5.19) *in the space* $\mathbf{V}_\sigma^{-\epsilon+1/2}(\Omega)$ *for every* $\epsilon \in (0,1/2)$.

Proof. Since $\widehat{\mathbf{w}} \in \mathbf{W}$ and $\mathbf{v}_0 \in C^\infty(\overline{\Omega})$ satisfies (2.4), the vector field Φ defined by (5.20) belongs to $L^2(0,T;\mathbf{V}_0^0(\Omega))$. Therefore problem (5.17)–(5.19) has a solution $\widetilde{\mathbf{q}} \in L^2(0,T;\mathbf{V}_\sigma^1(\Omega))$. This follows directly from Proposition 3.5 and Lemma 3.6. Moreover, since $\Phi \in \mathbf{L}^2(Q)$, Proposition 3.5 and Theorem A.8 imply that $\widetilde{\mathbf{q}} \in L^2(0,T-\delta;\mathbf{V}_\sigma^2(\Omega)) \cap H^1(0,T-\delta;\mathbf{V}_\sigma^0(\Omega))$ for all $\delta \in (0,T)$ and $\widetilde{\mathbf{q}}$ satisfies (5.25).

Multiplying (5.17) by $\mathbf{h} \in \mathcal{V}_0^{(2)}(Q)$, integrating over Q, and performing integration by parts, we see that $\widetilde{\mathbf{q}}$ is a generalized solution of (5.17)–(5.19), i.e., $\widetilde{\mathbf{q}}$ satisfies (5.16).

Now we prove the uniqueness of the generalized solution in the space $\mathbf{V}_{A^*}(Q)$ for (5.17)–(5.19) (recall that the space $\mathbf{V}_{A^*}(\Omega)$ contains $L^2(0,T;\mathbf{V}_0^0(\Omega))$.) Let $\mathbf{q}$ and $\widetilde{\mathbf{q}}$ both belong to $\mathbf{V}_{A^*}(Q)$ and satisfy (5.16). Denote $\mathbf{g} = \mathbf{q} - \widetilde{\mathbf{q}}$. Substituting (5.16) for $\widetilde{\mathbf{q}}$ from (5.16) for $\mathbf{q}$ we obtain

$$\begin{aligned}&\int_Q \Big([\partial_t\mathbf{h} - \Delta\mathbf{h} + [(\mathbf{v}_0+\widehat{\mathbf{w}})\cdot\nabla]\mathbf{h} + (\mathbf{h}\cdot\nabla)(\widehat{\mathbf{w}}+\mathbf{v}_0)\Big)\cdot\mathbf{g}\,d\mathbf{x}\,dt = 0 \\ &\qquad\qquad\qquad \forall\,\mathbf{h} \in \mathbf{V}_A^{1,2}(Q) \cap \mathcal{V}_0^{(2)}(Q).\end{aligned} \tag{5.26}$$

Let us consider the boundary value problem

$$P\big(\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{h} + (\mathbf{h} \cdot \nabla)(\widehat{\mathbf{w}} + \mathbf{v}_0)\big) = \mathbf{g}_1, \tag{5.27}$$

$$\operatorname{div} \mathbf{h} = 0, \qquad \mathbf{h}\big|_\Sigma = \mathbf{0}, \qquad \mathbf{h} \to 0 \text{ as } |\mathbf{x}| \to \infty, \qquad \text{and} \qquad \mathbf{h}\big|_{t=0} = \mathbf{0}, \tag{5.28}$$

where $\mathbf{g}_1 \in \mathbf{V}_A(Q)$. The problem (5.27)–(5.28) is equivalent to the problem (5.12)–(5.13), whose solvability has been established in the proof of Theorem 5.2. Thus, for each $\mathbf{g}_1 \in \mathbf{V}_A(Q)$ there exists the unique solution $\mathbf{h} \in \mathbf{V}_A^{1,2}(Q)$ of (5.27)–(5.28). The spaces $\mathbf{V}_A(Q)$ and $\mathbf{V}_{A^*}(Q)$ are dual, and therefore by a well-known corollary of the Hanh–Banach theorem, for a given $\mathbf{g} \in \mathbf{V}_{A^*}$ there exists a $\mathbf{g}_1 \in \mathbf{V}_A(Q)$ (which we consider here as a functional on $\mathbf{V}_{A^*}(Q)$) such that $\|\mathbf{g}_1\|_{\mathbf{V}_A(Q)} = 1$ and

$$\int_Q \mathbf{g}_1 \cdot \mathbf{g}\, d\mathbf{x}\, dt = \|\mathbf{g}\|_{\mathbf{L}_{A^*}(Q)}. \tag{5.29}$$

If we substitute into (5.26) the solution $\mathbf{h}$ of (5.27)–(5.28), we obtain that the left-hand side of (5.26) is equal to (5.29). Hence, $\mathbf{g} \equiv \mathbf{0}$ and uniqueness is proved. Equality (5.19) is true in the space $\mathbf{V}_\sigma^{-\epsilon+1/2}(\Omega)$ by virtue of Lemmas 5.3, A.4, A.6, and A.7. To summarize, we have proved all the assertions of Theorem 5.4. ◻

6. The strong form of the optimality system for Problem II. Using the regularity results for the adjoint velocity field established in Theorem 5.4, we now proceed to derive (see Theorem 6.4 below) the optimality system of partial differential equations and boundary, initial, and terminal conditions for Problem II.

6.1. The adjoint pressure. We first establish the existence of an adjoint pressure variable.

LEMMA 6.1. *Let $\mathbf{q} \in L^2(0.T; \mathbf{V}_0^0(\Omega))$ be the adjoint variable defined in* (5.15) *by $\widehat{\mathbf{q}}$ found in Theorem* 5.2. *Then there exists a distribution $\widetilde{r}(t, \mathbf{x})$ on Q such that the pair $(\mathbf{q}, \nabla\widetilde{r})$ is a solution of the problem* (5.18) *and*

$$\partial_t \mathbf{q} + \Delta \mathbf{q} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{q} - [\nabla(\mathbf{v}_0 + \widehat{\mathbf{w}})]^* \mathbf{q} + \nabla\widetilde{r} = \Phi, \tag{6.1}$$

where Φ is defined in (5.20). *Furthermore, $\nabla\widetilde{r}$ has the decomposition*

$$\nabla\widetilde{r} = \nabla r_1 + \nabla r_2 + \nabla r_3, \tag{6.2}$$

with ∇r_i, $i = 1, 2, 3$, satisfying the estimates

$$\begin{aligned}\int_0^{T-\delta} \|\nabla r_1(t, \cdot)\|^2_{\mathbf{L}^2(\Omega)}\, dt \le C\Big(&\delta^{-\epsilon-1/2} \|P\widehat{\mathbf{w}}(T, \cdot)\|^2_{\mathbf{V}^{-\epsilon+1/2}(\Omega)} \\ &+ \|P\Phi\|^2_{L^2(0,T;\mathbf{V}^0_\sigma(\Omega))}\Big),\end{aligned} \tag{6.3}$$

$$\operatorname{supp}(\nabla r_2) \in Q_{\rho+2}, \qquad \|\nabla r_2\|_{L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))} \le C\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}_\rho(Q)}, \tag{6.4}$$

and

$$\|\nabla r_3\|_{\mathbf{L}^2(Q)} \le C\|(\Phi + \Delta\widehat{\mathbf{w}}) - \mathbf{w}_2\|_{\mathbf{L}^2(Q)} \le C_1\Big(\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}_\rho(Q)} + \|\Delta \mathbf{v}_0\|_{\mathbf{L}^2(Q)}\Big), \tag{6.5}$$

where C in (6.3) *depends on* $\|\mathbf{v}_0\|_{\mathbf{C}^2(\overline{\Omega})}+\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}(Q)}$, C_1 *in* (6.5) *depends on* $\|\mathbf{v}_0\|_{\mathbf{C}^2(\Omega)}+\|\nabla\mathbf{v}_0\|_{\mathbf{L}^2(\Omega)}+|\mathbf{v}_\infty|$, *and* $\mathbf{w}$ *is defined below in* (6.7).

Proof. First, we claim that (5.17) can be rewritten as (6.1). Indeed, since $\partial_t\mathbf{q}+\Delta\mathbf{q}+[(\mathbf{v}_0+\widehat{\mathbf{w}})\cdot\nabla]\mathbf{q}-[\nabla(\mathbf{v}_0+\widehat{\mathbf{w}})]^*\mathbf{q}\in\mathbf{L}^2((0,T-\delta)\times\Omega)$ for each $\delta\in(0,T)$, we obtain from Weyl's decomposition that there exists a $\nabla r_1\in\mathbf{L}^2((0,T-\delta)\times\Omega)$ for each $\delta\in(0,T)$ such that

$$\partial_t\mathbf{q}+\Delta\mathbf{q}+[(\mathbf{v}_0+\widehat{\mathbf{w}})\cdot\nabla]\mathbf{q}-[\nabla(\mathbf{v}_0+\widehat{\mathbf{w}})]^*\mathbf{q}+\nabla r_1=P\Phi. \tag{6.6}$$

Expressing ∇r_1 through (6.6) and using (5.25) and Sobolev embedding theorems, we deduce estimate (6.3). On the other hand, it follows from the definition of $\mathbf{W}$ and $\mathbf{V}_\rho^{1,1/2}(Q)$ (see (2.22) and (2.19)) and the inclusion $\widehat{\mathbf{w}}\in\mathbf{W}$, that

$$\widehat{\mathbf{w}}=\mathbf{v}_1+\mathbf{v}_2 \quad \text{where} \quad \Delta\mathbf{v}_2=\mathbf{w}_2+\nabla r_2 \tag{6.7}$$

and where $\mathbf{v}_1\in\mathcal{V}_0^{(2)}(Q), \mathbf{v}_2\in\mathbf{V}_\rho^{1,1/2}(Q), \mathbf{w}_2\in\mathbf{L}^2(Q_{\rho+2})$, and $\nabla r_2\in L^2(0,T;\nabla H^{1/2}(\Omega_{\rho+2}))$. Evidently, the estimate (6.4) holds. Moreover, $P(\Phi+\Delta\widehat{\mathbf{w}}-\mathbf{w}_2)=\Phi+\Delta\widehat{\mathbf{w}}-\mathbf{w}_2-\nabla r_3$ so that (6.1) and the estimate (6.5) hold. □

We next show that the traces of $\partial\widehat{\mathbf{w}}/\partial n$ and r_2 with respect to Σ are well defined.

LEMMA 6.2. *Let* r_2 *be defined in* (6.7). *The restrictions of* r_2 *and* $\partial\widehat{\mathbf{w}}/\partial n$ *on* Σ *are well defined and*

$$\frac{\partial\widehat{\mathbf{w}}}{\partial n}\Big|_\Sigma\in\mathbf{L}^2(\Sigma) \qquad \text{and} \qquad r_2|_\Sigma\in L^2(\Sigma). \tag{6.8}$$

Proof. To define r_2 from (6.7) uniquely, we assume that $\int_{\Omega_{\rho+2}}r_2(t,\mathbf{x})\,d\mathbf{x}=0$ a.e. $t\in[0,T]$. Since the inclusion $\mathbf{v}_1\in\mathcal{V}_0^{(2)}(Q)$ implies that $\frac{\partial\mathbf{v}_1}{\partial n}|_\Sigma\in L^2(\Sigma)$, the inclusion $\frac{\partial\widehat{\mathbf{w}}}{\partial n}\in L^2(\Sigma)$ follows from (6.7) and $\frac{\partial\mathbf{v}_2}{\partial n}|_\Sigma\in L^2(\Sigma)$.

Using Lemma 4.4 from [9], we introduce coordinates (y_1,y_2,y_3) in $\Omega(\varepsilon)$ (an ε-neighborhood of $\partial\Omega$), where $y_3(\mathbf{x})=\operatorname{dist}(\mathbf{x},\partial\Omega)$ for $\mathbf{x}\in\Omega(\varepsilon)$ and $(y_1,y_2)=(y_1^k(\mathbf{x}),y_2^k(\mathbf{x}))$ are local coordinates in $\mathcal{U}^k$ and $\cup_k\mathcal{U}^k$ is a finite covering of $\partial\Omega$. For each k, the coordinates $(y_1,y_2,y_3)=(y_1^k,y_2^k,y_3)$ are orthogonal, i.e.,

$$\nabla y_i(\mathbf{x})\cdot\nabla y_j(\mathbf{x})=\delta_{ij}, \quad i,j=1,2,3.$$

This property implies that if $\mathbf{v}(\mathbf{y})$ is a rewriting of a vector field $\mathbf{w}(\mathbf{x})$ in terms of the coordinates (y_1,y_2,y_3) and $\operatorname{div}\mathbf{w}(\mathbf{x})=\sum_{i=1}^3\partial_{x_i}w_i=0$, then $\operatorname{div}\mathbf{v}(\mathbf{y})=\sum_{i=1}^3\partial_{y_i}v_i=0$; see [4].

Let $\widehat{\mathbf{v}}(t,\mathbf{y}),\mathbf{z}(t,\mathbf{y})$, and $\nabla_\mathbf{y}s(t,\mathbf{y})$ be rewritings in terms of the coordinates (y_1,y_2,y_3) of the corresponding vector fields $\mathbf{v}_2(t,\mathbf{x})$, $\mathbf{w}_2(t,\mathbf{x})$, and $\nabla_\mathbf{x}r_2(t,\mathbf{x})$. Then, the decomposition $\Delta\mathbf{v}_2=\mathbf{w}_2+\nabla r_2$ can be rewritten as follows:

$$\Delta_y\widehat{\mathbf{v}}=\mathbf{z}+\sum_{i=1}^3C_i(\mathbf{y})\frac{\partial\widehat{\mathbf{v}}}{\partial y_i}+\nabla_\mathbf{y}s, \tag{6.9}$$

where $C_i(\mathbf{y})$ are certain infinitely smooth vector fields. We denote $Q(\varepsilon)=(0,T)\times\Omega(\varepsilon)$. Taking into account the fact that $\widehat{\mathbf{v}}\in\mathbf{H}^{1,1/2}(Q(\varepsilon))$, which in turn implies

$$\partial\widehat{\mathbf{v}}/\partial y_3\in L^2(0,T;\mathbf{H}^{1/2}(\Omega(\varepsilon)))\cap H^1(0,T;\mathbf{H}^{-1/2}(\Omega(\varepsilon))),$$

we obtain that, for $j=1,2$,

$$\frac{\partial^2\widehat{\mathbf{v}}}{\partial y_3\partial y_j}\in L^2(0,T;\mathbf{L}^2(0,\varepsilon;\mathbf{H}^{-1/2}(\partial\Omega_{(\cdot)}))), \tag{6.10}$$

where $(0,\varepsilon)$ is the interval for the local variables y_3 and $\partial\Omega_{(z)} = \{\mathbf{x} \in \Omega(\varepsilon) : y_3(\mathbf{x}) = z\}$, $z \in (0,\varepsilon)$. Relation (6.10) means that $\partial^2_{y_3 y_j}\widehat{\mathbf{v}}(\cdot, y_3) \in L^2(0,T;\mathbf{H}^{-1/2}(\partial\Omega_{(y_3)}))$ for almost all y_3 and the function $y_3 \mapsto \|\partial^2\widehat{\mathbf{v}}/\partial y_3\partial y_j(\cdot,y_3)\|_{L^2(0,T;\mathbf{H}^{-1/2}(\partial\Omega_{(y_3)}))} \in L^2(0,\varepsilon)$. Differentiating the equality $\operatorname{div}\widehat{\mathbf{v}} = 0$ with respect to y_3 and taking into account (6.10), we obtain

$$\frac{\partial^2\widehat{v}_3}{\partial y_3^2} = -\frac{\partial^2\widehat{v}_1}{\partial y_3\partial y_1} - \frac{\partial^2\widehat{v}_2}{\partial y_3\partial y_2} \in L^2(0,\varepsilon;L^2(0,T;\mathbf{H}^{-1/2}(\partial\Omega_{(\cdot)}))). \tag{6.11}$$

Since $\widehat{\mathbf{v}} \in \mathbf{H}^{1,1/2}(Q(\varepsilon)) \subset L^2(0,T;\mathbf{H}^{3/2}(\Omega(\varepsilon)))$, we obtain

$$\widehat{v}_j \in L^2(0,\varepsilon;L^2(0,T;H^{3/2}(\partial\Omega_{(\cdot)}))), \quad j=1,2,3. \tag{6.12}$$

Define the diffeomorphism $\kappa : \Omega(\varepsilon) \to (0,\varepsilon)\times\partial\Omega$ which transforms each point $\mathbf{x} \in \partial\Omega(\varepsilon)$ possessing coordinates $(y_1^k(\mathbf{x}), y_2^k(\mathbf{x}), y_3(\mathbf{x}))$ into the pair $(y_3(\mathbf{x}), \mathbf{z})$, where $y_3 \in (0,\epsilon)$ and $\mathbf{z} \in \partial\Omega$ is a point possessing coordinates $(y_1(\mathbf{x}), y_2(\mathbf{x}))$. Evidently, under this diffeomorphism the inclusions (6.11) and (6.12) transform into the inclusions

$$\frac{\partial^2\widehat{v}_3}{\partial y_3^2} \in L^2(0,\varepsilon;L^2(0,T;H^{-1/2}(\partial\Omega))), \quad \widehat{v}_3 \in L^2(0,\varepsilon;L^2(0,T;H^{3/2}(\partial\Omega))). \tag{6.13}$$

The inclusions in (6.13) and trace theorems (see [14]) yield that

$$\frac{\partial\widehat{v}_3}{\partial y_3} \in C(0,\varepsilon;L^2(0,T;L^2(\partial\Omega))) \qquad \text{and} \qquad \frac{\partial\widehat{v}_3}{\partial y_3}\Big|_\Sigma \in L^2(\Sigma). \tag{6.14}$$

Since $\widehat{v}_3 \in H^{1,1/2}(Q(\varepsilon))$ and $z \in \mathbf{L}^2(Q(\varepsilon))$, (6.9) implies that

$$\begin{aligned}\partial_{y_3}\big(\partial_{y_3}\widehat{v}_3 - s\big) = -\big(\partial^2_{y_1y_1} + \partial^2_{y_2y_2}\big)\widehat{v}_3 - z_3 + \sum_{j=1}^3 C_j^3(\mathbf{y})\frac{\partial\widehat{v}^3}{\partial y_j}& \\ \in L^2(0,\varepsilon;L^2(0,T;H^{-1/2}(\partial\Omega))).&\end{aligned} \tag{6.15}$$

Moreover, since $\partial_{y_3}\widehat{v}_3 \in L^2(0,T;H^{1/2}(\Omega(\varepsilon)))$ and $s \in L^2(0,T;H^{1/2}(\Omega(\varepsilon)))$, we have that

$$\partial_{y_3}\widehat{v}_3 - s \in L^2(0,\varepsilon;L^2(0,T;H^{1/2}(\partial\Omega_{(\cdot)}))). \tag{6.16}$$

Applying the diffeomorphysm κ and after using trace theorems, we obtain from (6.15) and (6.16) that

$$\partial_{y_3}\widehat{v}_3 - s \in C(0,\varepsilon;L^2(0,T;L^2(\partial\Omega_{(\cdot)}))) \quad \text{and} \quad (\partial_{y_3}\widehat{v}_3 - s)\big|_\Sigma \in L^2(\Sigma). \tag{6.17}$$

Then (6.14) and (6.17) imply that

$$s|_\Sigma \in L^2(\Sigma). \tag{6.18}$$

Finally, (6.9) implies, for $k=1,2$,

$$\frac{\partial^2\widehat{v}_k}{\partial y_3^2} = -\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)\widehat{v}_k - z_3 + \sum_{j=1}^3 C_j^k\frac{\partial\widehat{v}}{\partial y_j} - \frac{\partial s}{\partial y_k} \in L^2(0,\varepsilon;L^2(0,T;H^{-1/2}(\partial\Omega_{(\cdot)}))).$$

This relation together with (6.12) implies, by virtue of trace theorems, that

$$\frac{\partial \widehat{v}_k}{\partial y_3} \in C(0,\varepsilon; L^2(0,T; L^2(\partial\Omega_{(\cdot)}))) \quad \text{and} \quad \frac{\partial \widehat{v}_k}{\partial y_3}\Big|_\Sigma \in L^2(\Sigma), \quad k = 1,2. \tag{6.19}$$

The relations (6.7), (6.14), (6.18), and (6.19) imply (6.8). □

Note that (6.3), (6.4), and (6.5) imply that $\widetilde{r}(t,\mathbf{x})$ determined in (6.2) has a well-defined trace on Σ and

$$\widetilde{r}|_\Sigma \in L^2(\Sigma). \tag{6.20}$$

6.2. An additional condition on the boundary. We now derive an additional condition that holds on the boundary. This is done, roughly speaking, by integration by parts in (5.5).

Let $\widehat{p}$ be the pressure in (2.7), where $\mathbf{w} = \widehat{\mathbf{w}}$. Using (6.7) and Lemma 6.2 and repeating relevant arguments from (6.1) to (6.20), we can derive from (2.7) that $\nabla\widehat{p}$ has a well-defined trace on Σ and

$$\widehat{p}|_\Sigma \in L^2(\Sigma). \tag{6.21}$$

Note that through the change of variable

$$\nabla\widetilde{r} = -\nabla r - \nabla\widehat{p}, \tag{6.22}$$

where $\widehat{p}$ is the pressure in the first equation in (2.7), we may convert (6.1) into

$$\partial_t\mathbf{q} + \Delta\mathbf{q} + [(\mathbf{v}_0 + \widehat{\mathbf{w}})\cdot\nabla]\mathbf{q} - [\nabla(\mathbf{v}_0 + \widehat{\mathbf{w}})]^*\mathbf{q} - \nabla r = \Phi + \nabla\widehat{p}. \tag{6.23}$$

By (6.20), (6.21), and (6.22), $r|_\Sigma$ is well defined and

$$r|_\Sigma \in L^2(\Sigma). \tag{6.24}$$

In order to define r and $\widehat{p}$ uniquely we assume that

$$\int_{\partial\Omega} \widehat{p}(t,\mathbf{x})\,ds = 0 \qquad \text{and} \qquad \int_{\partial\Omega} r(t,\mathbf{x})\,ds = 0 \qquad \text{a.e. } t \in [0,T]. \tag{6.25}$$

LEMMA 6.3. *Let* $\mathbf{q} \in L^2(0.T; \mathbf{V}_0^0(\Omega))$ *be the adjoint variable defined in* (5.15) *by* $\widehat{\mathbf{q}}$ *found in Theorem* 5.2. *Then*

$$\begin{aligned} &\int_\Sigma \mathbf{h}\cdot\Big(\mathcal{T}(\widehat{\mathbf{w}},\widehat{p})\mathbf{n} + \mathcal{T}(\mathbf{q},r)\mathbf{n} + 2\mathcal{D}(\mathbf{v}_0)\mathbf{n} + (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}}\cdot\mathbf{n} \\ &+ \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2\mathbf{n} + N(-\partial_{tt}\widehat{\mathbf{w}} + \widehat{\mathbf{w}})\Big)\,ds\,dt \\ &+ N\int_\Sigma \nabla_\tau\widehat{\mathbf{w}} : \nabla_\tau\mathbf{h}\,ds\,dt = 0 \quad \forall\,\mathbf{h} \in \widetilde{\mathcal{V}}_0^{(2)}(Q) \quad \textit{with} \quad \mathbf{h}(T,\cdot) = 0, \end{aligned} \tag{6.26}$$

where

$$\mathcal{T}(\widehat{\mathbf{w}},\widehat{p}) = -\widehat{p}I + 2\mathcal{D}(\widehat{\mathbf{w}}) \qquad \textit{and} \qquad \mathcal{T}(\mathbf{q},r) = -rI + 2\mathcal{D}(\mathbf{q}). \tag{6.27}$$

Proof. We substitute (5.15) into (5.5) and manipulate the resulting expression with $\mathbf{q}$ as follows. We subdivide Q into two disjoint cylinders $Q_{T-\delta} = (0, T-\delta)\times\Omega$ and $Q_\delta = (T-\delta, T)\times\Omega$. Denote $\Sigma_{T-\delta} = (0, T-\delta)\times\partial\Omega$. We express the integral over Q in (5.5) as the sum of integrals over $Q_{T-\delta}$ and Q_δ and perform integration by parts over $Q_{T-\delta}$. This leads us to

$$\begin{aligned}
&\int_{Q_{T-\delta}} \Big(- \big[\partial_t \mathbf{q} + \Delta \mathbf{q} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{q} \\
&\qquad - [\nabla(\mathbf{v}_0 + \widehat{\mathbf{w}})]^* \mathbf{q}\big] \cdot \mathbf{h}\Big)\, d\mathbf{x}\, dt + \int_{Q_{T-\delta}} \mathbf{h} \cdot \Phi \, dx\, dt \\
&\quad + \int_{Q_\delta} \Big(2\mathcal{D}(\widehat{\mathbf{w}} + \mathbf{v}_0) : \mathcal{D}(\mathbf{h}) + \big[\partial_t \mathbf{h} - \Delta \mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla]\mathbf{h} \\
&\qquad + (\mathbf{h} \cdot \nabla)(\mathbf{v}_0 + \widehat{\mathbf{w}})\big] \cdot (\mathbf{q} - \mathbf{v}_0 + \mathbf{v}_\infty)\Big)\, d\mathbf{x}\, dt \\
&\quad + \int_{\Sigma_{T-\delta}} \Big(\mathbf{h} \cdot 2\mathcal{D}(\mathbf{v}_0 + \widehat{\mathbf{w}})\mathbf{n} + \mathbf{h} \cdot \partial_n \mathbf{q}\Big)\, ds\, dt \\
&\quad + \int_\Omega (\mathbf{q}(T-\delta, \mathbf{x}) - \mathbf{v}_0 + \mathbf{v}_\infty) \cdot \mathbf{h}(T-\delta, \mathbf{x})\, d\mathbf{x} \\
&\quad + \int_\Omega (\widehat{\mathbf{w}}(T, \mathbf{x}) + \mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty) \cdot \mathbf{h}(T, \mathbf{x})\, d\mathbf{x} \\
&\quad + \int_\Sigma \Big(\mathbf{h} \cdot (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}} \cdot \mathbf{n} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2 \mathbf{h} \cdot \mathbf{n} \\
&\qquad + N[\partial_t \widehat{\mathbf{w}} \cdot \partial_t \mathbf{h} + \nabla_\tau \widehat{\mathbf{w}} : \nabla_\tau \mathbf{h} + \widehat{\mathbf{w}} \cdot \mathbf{h}]\Big)\, ds\, dt = 0 \quad \forall\, \mathbf{h} \in \mathbf{V}_A^{1,2}(Q).
\end{aligned} \tag{6.28}$$

Note that (6.22) and (6.25) imply

$$\int_{\partial\Omega} \widetilde{r}(t, \cdot)\, ds = 0 \quad \text{a.e. } t \in (0, T). \tag{6.29}$$

Expressing $\nabla \widetilde{r}$ by (6.1), we see that the integrals over $Q_{T-\delta}$ in (6.28) are equal to

$$\int_{Q_{T-\delta}} \mathbf{h} \cdot \nabla \widetilde{r}\, d\mathbf{x}\, dt = \int_{\Sigma_{T-\delta}} \mathbf{h} \cdot \widetilde{r}\, \mathbf{n}\, ds,$$

where the integration by parts is valid because of (6.8). Thus, (6.28) and the relation $\mathbf{v}_0|_{\partial\Omega} = 0$ reduce to

$$\begin{aligned}
&\int_{Q_\delta} \Big\{ 2\mathcal{D}(\mathbf{h}) : \mathcal{D}(\mathbf{v}_0 + \widehat{\mathbf{w}}) + [\partial_t \mathbf{h} - \Delta \mathbf{h} + ((\mathbf{v}_0 + \widehat{\mathbf{w}}) \cdot \nabla)\mathbf{h} \\
&\qquad\qquad + (\mathbf{h} \cdot \nabla)(\mathbf{v}_0 + \widehat{\mathbf{w}})] \cdot (\mathbf{q} - \mathbf{v}_0 + \mathbf{v}_\infty)\Big\}\, d\mathbf{x}\, dt \\
&\quad + \int_{\Sigma_{T-\delta}} \mathbf{h} \cdot 2\mathcal{D}(\widehat{\mathbf{w}})\mathbf{n}\, ds\, dt + \int_{\Sigma_{T-\delta}} \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n})\, ds\, dt \\
&\quad + \int_\Omega \Big(\mathbf{h}(T, \mathbf{x}) \cdot (\widehat{\mathbf{w}}(T, \mathbf{x}) + \mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty) \\
&\qquad + \mathbf{h}(T-\delta, \mathbf{x}) \cdot (\mathbf{q}(T-\delta, \mathbf{x}) - \mathbf{v}_0 + \mathbf{v}_\infty)\Big)\, d\mathbf{x} \\
&\quad + \int_\Sigma \Big(\mathbf{h} \cdot (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}} \cdot \mathbf{n} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2 \mathbf{h} \cdot \mathbf{n}\Big)\, ds\, dt \\
&\quad + \int_\Sigma \Big(N(\partial_t \mathbf{w} \cdot \partial_t \mathbf{h} - \nabla_\tau \mathbf{w} : \nabla_\tau \mathbf{h} + \widehat{\mathbf{w}} \cdot \mathbf{h})\Big)\, ds\, dt = 0 \quad \forall\, \mathbf{h} \in \mathbf{V}_A^{1,2}(Q).
\end{aligned} \tag{6.30}$$

Now we examine passage to the limit in (6.30) as $\delta \to 0$. The integral over Q_δ tends to zero as $\delta \to 0$ since $\mathbf{h} \in \mathbf{V}_A^{1,2}(Q)$, $(q - \mathbf{v}_0 + \mathbf{v}_\infty) \in L_{A^*}(Q)$, and $\operatorname{meas}(Q_\delta) \to 0$. By virtue of Lemma 6.2,

$$\int_{\Sigma_{T-\delta}} \mathbf{h} \cdot 2\mathcal{D}(\widehat{\mathbf{w}})\mathbf{n}\, ds\, dt \to \int_{\Sigma_T} \mathbf{h} \cdot 2\mathcal{D}(\widehat{\mathbf{w}})\mathbf{n}\, ds\, dt \quad \text{as } \delta \to 0.$$

Since $\mathbf{q}$ is a solution of (3.12), by Proposition 3.5 and Lemma 3.6 it satisfies the energy estimate so that $\mathbf{q} \in C([0,T]; \mathbf{L}^2_w(\Omega))$, where $\mathbf{L}^2_w(\Omega)$ is $\mathbf{L}^2(\Omega)$ endowed with the weak topology. Thus, the integral over Ω in (6.30) tends to zero as $\delta \to 0$ thanks to Theorem 5.4 and the fact that $\mathbf{h} \in \mathbf{V}_A^{1,2}(Q) \subset C([0,T]; \mathbf{L}_A(\Omega))$. We also have

$$\int_{\Sigma_{T-\delta}} \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n})\, ds\, dt \to \int_{\Sigma_T} \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n})\, ds\, dt \quad \text{as } \delta \to 0. \tag{6.31}$$

Indeed, (6.30) is valid for all $\delta > 0$ and each term in equality (6.30), except for the term $\int_{\Sigma_{T-\delta}} \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n})\, ds\, dt$, has a limit as $\delta \to 0$. These facts imply that the integral $\int_{\Sigma_{T-\delta}} \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n})\, ds\, dt$ has a limit which, by the definition of improper integrals, equals the right-hand side of (6.31). Hence, passing to limit in (6.30) as $\delta \to 0$ yields

(6.32)
$$\begin{aligned}
&\int_\Sigma \Big(\mathbf{h} \cdot 2\mathcal{D}(\widehat{\mathbf{w}} + \mathbf{v}_0)\mathbf{n} + \mathbf{h} \cdot (\partial_n \mathbf{q} + \widetilde{r}\mathbf{n}) + \mathbf{h} \cdot (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}} \cdot \mathbf{n} \\
&\qquad + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2 \mathbf{h} \cdot \mathbf{n} + N(\partial_t \widehat{\mathbf{w}} \cdot \partial_t \mathbf{h} + \nabla_\tau \widehat{\mathbf{w}} : \nabla_\tau \mathbf{h} + \widehat{\mathbf{w}} \cdot \mathbf{n}) \Big)\, ds\, dt = 0 \\
&\forall\, \mathbf{h} \in \mathbf{V}_A^{1,2}(Q).
\end{aligned}$$

We now show that

$$\int_{\partial\Omega} \partial_n \mathbf{q} \cdot \mathbf{h}\, ds = 2 \int_{\partial\Omega} \mathbf{h} \cdot \mathcal{D}(\mathbf{q})\mathbf{n}\, ds. \tag{6.33}$$

For almost every $t \in [0,T]$, we have Green's identity

$$\int_\Omega \nabla \mathbf{q} : \nabla \mathbf{h}\, d\mathbf{x} = \int_{\partial\Omega} \partial_n \mathbf{q} \cdot \mathbf{h}\, ds - \int_\Omega \mathbf{h} \cdot \Delta \mathbf{q}\, d\mathbf{x}. \tag{6.34}$$

Since $\mathbf{q}\big|_{\partial\Omega} = \mathbf{0}$ and $\operatorname{div} \mathbf{h} = 0$, we obtain

$$\int_\Omega \nabla \mathbf{q} : \nabla \mathbf{h}\, d\mathbf{x} = \int_\Omega [\nabla \mathbf{q} + (\nabla \mathbf{q})^T] : \nabla \mathbf{h}\, d\mathbf{x} = 2 \int_\Omega \nabla \mathbf{h} \cdot \mathcal{D}(\mathbf{q})\, d\mathbf{x}$$

so that by applying Green's identity to the last term we have

$$\int_\Omega \nabla \mathbf{q} : \nabla \mathbf{h}\, d\mathbf{x} = 2 \int_{\partial\Omega} \mathbf{h} \cdot \mathcal{D}(\mathbf{q})\mathbf{n}\, ds - \int_\Omega \mathbf{h} \cdot \Delta \mathbf{q}\, d\mathbf{x}. \tag{6.35}$$

Equalities (6.34) and (6.35) imply (6.33).

Taking into account (6.21) and (6.24) and using (6.22) and (6.33), we can rewrite (6.32) as (6.26). $\square$

6.3. The optimality system. We are now in a position to derive an optimality system for Problem II in the form of a boundary value problem. The surface Laplacian $\Delta_\tau \mathbf{u}$ defined on $\partial\Omega$ (and on Σ) can be determined by

$$\int_{\partial\Omega} \nabla_\tau \mathbf{u} : \nabla_\tau \mathbf{v}\, ds = -\int_{\partial\Omega} \mathbf{v}\cdot \Delta_\tau \mathbf{u}\, ds, \tag{6.36}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the restrictions of $\mathbf{w}_i \in \mathbf{V}^2(\Omega)$, $i=1,2$, onto $\partial\Omega$, i.e., $\mathbf{u} = \mathbf{w}_1\big|_{\partial\Omega}$ and $\mathbf{v} = \mathbf{w}_2\big|_{\partial\Omega}$.

THEOREM 6.4. *Assume* $(\widehat{\mathbf{w}}, \widehat{p}) \in \mathbf{W}\times \mathbf{L}^2(Q)$ *is a solution for Problem* II *and* $\mathbf{q}\in L^2(0,T;\mathbf{V}^1_\sigma(\Omega))$ *is the Lagrange multiplier defined in* (5.15) *by* $\widehat{\mathbf{q}}$ *introduced in Theorem* 5.2. *Then there exists an* r *defined in the form of* (6.22), *with* $\widetilde{r}$ *defined in Lemma* 6.1 *such that the quadruple* $(\widehat{\mathbf{w}}, \widehat{p}, \mathbf{q}, r)$ *satisfies the partial differential equations*

$$\partial_t \widehat{\mathbf{w}} - \Delta\widehat{\mathbf{w}} + [(\mathbf{v}_0 + \widehat{\mathbf{w}})\cdot\nabla]\widehat{\mathbf{w}} + (\widehat{\mathbf{w}}\cdot\nabla)\mathbf{v}_0 + \nabla\widehat{p} = \mathbf{0} \qquad \text{in } Q, \tag{6.37}$$

$$\operatorname{div}\widehat{\mathbf{w}} = 0 \qquad \text{in } Q, \tag{6.38}$$

$$\begin{aligned} -\partial_t\mathbf{q} - \Delta\mathbf{q} - [(\mathbf{v}_0+\widehat{\mathbf{w}})\cdot\nabla]\mathbf{q} + [\nabla(\mathbf{v}_0+\widehat{\mathbf{w}})]^T\mathbf{q} + \nabla r& \\ = \Phi - \nabla\widehat{p} \qquad \text{in } Q&, \end{aligned} \tag{6.39}$$

where Φ *is defined by* (5.20) *and*

$$\operatorname{div}\mathbf{q} = 0 \qquad \text{in } Q, \tag{6.40}$$

where $\widehat{p}$ *and* r *satisfy* (6.25). *Furthermore,* $(\widehat{\mathbf{w}}, \mathbf{q})$ *satisfy the initial and terminal conditions*

$$\widehat{\mathbf{w}}(0,\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega \tag{6.41}$$

and

$$\mathbf{q}(T,\mathbf{x}) + \widehat{\mathbf{w}}_\sigma(T,\mathbf{x}) = 0 \quad \text{in } \Omega, \tag{6.42}$$

where, by definition, $\mathbf{v}_\sigma = P\mathbf{v}$ *is the* $\mathbf{V}^0_0(\Omega)$*-projection of* $\mathbf{v}$*, and the lateral boundary conditions*

$$\mathbf{q}\big|_\Sigma = \mathbf{0} \tag{6.43}$$

and

$$\begin{aligned} \big[N(-\partial_{tt}\widehat{\mathbf{w}} - \Delta_\tau\widehat{\mathbf{w}}) + \mathcal{A}(\widehat{\mathbf{w}}) + \mathcal{T}(\mathbf{q},\tau)\mathbf{n}& \\ +\mathcal{T}(\widehat{\mathbf{w}},\widehat{p})\mathbf{n} + 2\mathcal{D}(\mathbf{v}_0)\mathbf{n}\big]\big|_\Sigma = -\boldsymbol{\eta}(t)\mathbf{n}\big|_\Sigma&, \end{aligned} \tag{6.44}$$

where $\mathcal{T}(\widehat{\mathbf{w}},\widehat{p})$ *and* $\mathcal{T}(\mathbf{q},\tau)$ *are defined by* (6.27),

$$\mathcal{A}(\widehat{\mathbf{w}}) = N\widehat{\mathbf{w}} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2\mathbf{n} + (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}}\cdot\mathbf{n}, \tag{6.45}$$

and

$$\boldsymbol{\eta}(t)=\frac{\displaystyle\int_{\partial\Omega}\big[N\Delta_\tau\widehat{\mathbf{w}}-\mathcal{A}(\widehat{\mathbf{w}})\big]\cdot\mathbf{n}\,ds}{\displaystyle\int_{\partial\Omega}ds}. \tag{6.46}$$

Furthermore, the following compatibility conditions hold:

$$\begin{aligned}&\Big(\widehat{\mathbf{w}}\big|_\Sigma\Big)\Big|_{t=0}=\mathbf{0},\quad \partial_t\big(\widehat{\mathbf{w}}\big|_\Sigma\big)_\tau\Big|_{t=T}=\mathbf{0},\\&\Big(\big|(\widehat{\mathbf{w}}-\mathbf{v}_\infty)_\pi\big|_\Sigma+2N\partial_t\big(\widehat{\mathbf{w}}\big|_\Sigma\big)\cdot\mathbf{n}\Big)\Big|_{t=T}=0,\end{aligned} \tag{6.47}$$

where $(\widehat{\mathbf{w}}|_\Sigma)_\tau$ *is the tangential projection of* $\widehat{\mathbf{w}}|_\Sigma$ *onto* Σ *and* $\mathbf{w}_\pi$ *is the projection of* $\mathbf{w}$ *in the following Weyl decomposition of* $\mathbf{V}^0(\Omega)$*:*

$$\mathbf{w}=\mathbf{w}_\sigma+\nabla w_\pi, \tag{6.48}$$

where $\mathbf{w}_\sigma\in\mathbf{V}_0^0(\Omega)$ *and* $\nabla w_\pi\in\nabla H_\pi(\Omega)\equiv\{\nabla\tau\in\mathbf{L}^2(\Omega):\tau\in H^1_{\mathrm{loc}}(\Omega),\Delta\tau=0\}$*. The primitive* w_π *of* ∇w_π *is determined by the equality*

$$\int_{\partial\Omega}w_\pi\,ds=0. \tag{6.49}$$

Proof. The equations in (6.25) are simply conditions for pinning down $\widehat{p}$ and r uniquely and can always be satisfied by redefining $\widehat{p}$ and r if necessary.

The relations (6.37)–(6.38) and (6.41) were built into the formulation of Problem II and are automatically satisfied. Equalities (6.39)–(6.40) and (6.43) were proved in Theorem 5.4.

The equalities (6.26) and (6.36) yield

$$\begin{aligned}\int_\Sigma\mathbf{h}\cdot\Big(&\mathcal{T}(\widehat{\mathbf{w}},\widehat{p})\mathbf{n}+\mathcal{T}(\mathbf{q},r)\mathbf{n}+2\mathcal{D}(\mathbf{v}_0)\mathbf{n}+\mathcal{A}(\widehat{\mathbf{w}})\\&+\big[N(-\partial_{tt}\widehat{\mathbf{w}}-\Delta_\tau\widehat{\mathbf{w}})\big]\Big)\,ds\,dt=0\end{aligned} \tag{6.50}$$

for all $\mathbf{h}\in\mathbf{V}_A^{1,2}(Q)$, $\mathbf{h}(T,\cdot)=0$, where $\mathcal{T}(\mathbf{w},\widehat{p})$, $\mathcal{T}(\mathbf{q},r)$, and $\mathcal{A}(\widehat{\mathbf{w}})$ are defined in (6.27) and (6.45). In [8], it was proved that

$$\int_\Sigma\mathbf{n}\cdot\Big(\mathcal{T}(\widehat{\mathbf{w}},\widehat{p})\mathbf{n}+\mathcal{T}(\mathbf{q},r)\mathbf{n}+2\mathcal{D}(\mathbf{v}_0)\mathbf{n}-N\partial_{tt}\widehat{\mathbf{w}}\Big)\,ds\,dt=0. \tag{6.51}$$

Recall that, by assumption, $\partial\Omega$ is connected set. (In the case of unconnected sets, the proof would be the same, but in the formulas (6.44) and (6.46) we would have to change $\eta(t)$ to $\eta_j(t)$ and $\partial\Omega$, Σ to $\partial\Omega_j$, $\Sigma_j=(0,T)\times\partial\Omega_j$, where $\partial\Omega_j$ is a connected component of $\partial\Omega$.) As is well known (see [9]), a vector field $\mathbf{g}(x'),x'\in\partial\Omega$, is a restriction on $\partial\Omega$ of a solenoidal vector field defined on Ω if and only if $\int_{\partial\Omega}\mathbf{g}\cdot\mathbf{n}\,ds=0$. Hence, (6.50) and (6.51) imply (6.44) with $\eta(t)$ defined by (6.46).

The relations (6.47) can be proved in the same way as in [8]. $\square$

7. The optimality system for Problem I. We now derive the optimality system for Problem I.

THEOREM 7.1. *Assume* $(\widehat{\mathbf{w}}, \widehat{p}) \in \mathbf{W} \times \mathbf{L}^2(Q)$ *is a solution of Problem* I. *Then there exists a triple* $(\mathbf{q}, \nabla\tau, \lambda) \in L^2(0,T; \mathbf{V}^1_\sigma)(\Omega) \times \mathbf{L}^2(Q) \times \mathbb{R}_+$ *such that* $(\mathbf{q}, \nabla\tau, \lambda) \neq (\mathbf{0}, \mathbf{0}, 0)$ *and* $(\widehat{\mathbf{w}}, \widehat{p}, \mathbf{q}, \tau, \lambda)$ *satisfies* (6.37)–(6.43), (6.25), *and the boundary conditions*

$$\lambda(-\partial_{tt}\widehat{\mathbf{w}} - \Delta_\tau \widehat{\mathbf{w}}) + \widetilde{\mathcal{A}}(\widehat{\mathbf{w}}) + \mathcal{T}(\mathbf{q}, \tau)\mathbf{n} + \mathcal{T}(\widehat{\mathbf{w}}, \widehat{p})\mathbf{n} + \mathcal{D}(\mathbf{v}_0)\mathbf{n} = -\widetilde{\boldsymbol{\eta}}(t)\mathbf{n}, \tag{7.1}$$

where $\mathcal{T}(\widehat{\mathbf{w}}, \widehat{p})$ *and* $\mathcal{T}(\mathbf{q}, \tau)$ *are defined by* (6.30),

$$\widetilde{\mathcal{A}}(\widehat{\mathbf{w}}) = \lambda\widehat{\mathbf{w}} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2\mathbf{n} + (\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}} \cdot \mathbf{n}, \tag{7.2}$$

and

$$\widetilde{\boldsymbol{\eta}}(t) = \frac{\displaystyle\int_{\partial\Omega} \Big(\lambda\Delta_\tau\widehat{\mathbf{w}} - \widetilde{\mathcal{A}}(\widehat{\mathbf{w}})\mathbf{n}\Big)\, ds}{\displaystyle\int_{\partial\Omega} ds}. \tag{7.3}$$

In addition, the following compatibility conditions hold:

$$\lambda\partial_t\big(\widehat{\mathbf{w}}\big|_\Sigma\big)_\tau\Big|_{t=T} = \mathbf{0} \quad \textit{and} \quad \Big(\widehat{w}_\pi\big|_\Sigma + 2\lambda\partial_t\big(\widehat{\mathbf{w}}\big|_\Sigma\big)\cdot\mathbf{n}\Big)\Big|_{t=T} = 0, \tag{7.4}$$

where $\widehat{\mathbf{w}}_\tau$ *is the tangential projection of* $\widehat{\mathbf{w}}$ *onto* Σ *and* $\widehat{w}_\pi$ *is defined by* (6.48) *and* (6.49). *Moreover, the nonnegativity and complementary slackness conditions hold, i.e.,*

$$\lambda \geq 0 \tag{7.5}$$

and

$$\lambda\Big[\int_\Sigma \Big(|\partial_t\widehat{\mathbf{w}}|^2 + |\nabla_\tau\widehat{\mathbf{w}}|^2 + |\widehat{\mathbf{w}}|^2\Big)\, ds\, dt - M\Big] = 0. \tag{7.6}$$

Proof. First, we derive a weak form of optimality system for Problem I in exactly the same manner as was done in the appropriate part of Theorem 6.9 in [8] for the two-dimensional case:

$$\begin{aligned}
&\lambda_0 \int_Q \mathcal{D}(\widehat{\mathbf{w}} + \mathbf{v}_0) : \mathcal{D}(\mathbf{h})\, d\mathbf{x}\, dt \\
&\quad + \int_Q \big\{\partial_t\mathbf{h} - \Delta\mathbf{h} + [(\mathbf{v}_0 + \widehat{\mathbf{w}})\cdot\nabla]\mathbf{h} + (\mathbf{h}\cdot\nabla)(\mathbf{v}_0 + \widehat{\mathbf{w}})\big\} \cdot \widehat{\mathbf{q}}\, d\mathbf{x}\, dt \\
&\quad + \int_\Omega (\widehat{\mathbf{w}}(T, \mathbf{x}) + \mathbf{v}_0(\mathbf{x}) - \mathbf{v}_\infty) \cdot \mathbf{h}\, d\mathbf{x} \\
&\quad + \int_\Sigma \Big(\mathbf{h}\cdot(\widehat{\mathbf{w}} - \mathbf{v}_\infty)\widehat{\mathbf{w}}\cdot\mathbf{n} + \frac{1}{2}|\widehat{\mathbf{w}} - \mathbf{v}_\infty|^2\mathbf{h}\cdot\mathbf{n} \\
&\qquad + \lambda[\partial_t\widehat{\mathbf{w}}\cdot\partial_t\mathbf{h} + \nabla_\tau\widehat{\mathbf{w}} : \nabla_\tau\mathbf{h} + \widehat{\mathbf{w}}\cdot\mathbf{h}]\Big)\, ds\, dt = 0 \quad \forall\, \mathbf{h} \in \mathbf{V}^{1,2}_A(Q).
\end{aligned} \tag{7.7}$$

The difference between this integral equality and (5.5) is that, in (7.7), the parameter N is renamed to λ and λ_0 multiplies the first term. Therefore, if we repeat all the arguments that lead to the assertions of Theorem 6.4, we obtain the optimality system (6.37)–(6.49), where N is renamed as λ and all terms generated by the first term in (5.5) are multiplied by λ_0. Hence, to prove Theorem 7.1, we have to show that $\lambda_0 \neq 0$. We do this analogously to the proof of the appropriate part of Theorem 6.9 in [8]. As in that theorem, the assumption that $\lambda_0 = 0$ leads to the inequality $\lambda > 0$. As a result, the proof is reduced to establishing the following assertion:

$$\begin{aligned} &\text{there exists a } \mathbf{y} \in \gamma_\Sigma \mathbf{V}_A^{1,2}(Q) \text{ such that} \\ &\int_\Sigma \Big(\partial_t \widehat{\mathbf{w}} \cdot \partial_t \mathbf{y} + \nabla_\tau \widehat{\mathbf{w}} : \nabla_\tau \mathbf{y} + \widehat{\mathbf{w}} \cdot \mathbf{y} \Big)\, ds\, dt \neq 0. \end{aligned} \tag{7.8}$$

Here, $\gamma_\Sigma \mathbf{V}_A^{1,2}(Q)$ is the set of restrictions of vector fields belonging to the space $\mathbf{V}_A^{1,2}(Q)$ defined in (B.22).

We prove (7.8) by contradiction. By virtue of (2.11), the solution $\widehat{\mathbf{w}}$ of Problem I satisfies $\widehat{\mathbf{w}} \in \mathbf{H}^1(\Sigma)$ and $\int_{\partial\Omega} \widehat{\mathbf{w}} \cdot \mathbf{n}\, ds = 0$ for almost every $t \in (0,T)$. As is well known, there exists a sequence $\mathbf{y}_n(t,x) \in \mathbf{C}^\infty(Q)$, $\int_{\partial\Omega} \mathbf{y}_n \cdot \mathbf{n}\, ds = 0$ a.e. $t \in (0,T)$, such that $\|\widehat{\mathbf{w}} - \mathbf{y}_n\|_{\mathbf{H}^1(\Sigma)} \to 0$ as $n \to 0$. It is well known (and the proof can be easily reproduced, e.g., from [9]) that there exists $\mathbf{z}_n \in \mathbf{V}_A^{1,2}(Q)$ such that $\mathbf{z}_n\big|_\Sigma = \mathbf{y}_n$. Hence, (7.6) and (7.8) imply that

$$\begin{aligned} 0 &= \int_\Sigma \Big(\partial_t \widehat{\mathbf{w}} \cdot \partial_t \mathbf{y}_n + \nabla_\tau \widehat{\mathbf{w}} : \nabla_\tau \mathbf{y}_n + \widehat{\mathbf{w}} \cdot \mathbf{y}_n \Big)\, ds\, dt \\ &\quad \to \int_\Sigma \Big(|\partial_t \widehat{\mathbf{w}}|^2 + |\nabla_\tau \widehat{\mathbf{w}}|^2 + |\widehat{\mathbf{w}}|^2 \Big)\, ds\, dt = M. \end{aligned}$$

This contradiction completes the proof. ∎

Appendix A. Regularity results for some auxiliary boundary value problems. We now derive some nonstandard regularity results for the Stokes problem involving the function space $\mathbf{V}_\sigma^s(\Omega)$. We will also derive similar regularity results for the adjoint boundary value problem in the form (3.14). First, we introduce an appropriate functions spaces.

A.1. Spectral function spaces. In this subsection, we will define the function space $\mathbf{V}_\sigma^s(\Omega)$ and study its properties. The tools we will need are direct integrals and intermediate spaces. These spaces will be used in the study of the regularity for the adjoint velocity in the optimality system.

Consider the operator

$$A = P(-\Delta + I) : \mathbf{V}_0^0(\Omega) \to \mathbf{V}_0^0(\Omega), \tag{A.1}$$

where $P : \mathbf{L}^2(\Omega) \to \mathbf{V}_0^0(\Omega)$ is the orthogonal projection operator onto $\mathbf{V}_0^0(\Omega)$. The domain $\mathcal{D}(A)$ of the operator A is defined by $\mathcal{D}(A) = \mathbf{V}^2(\Omega) \cap \mathbf{V}_0^1(\Omega)$, where

$$\mathbf{V}_0^1(\Omega) = \{\mathbf{v} \in \mathbf{V}^1(\Omega) \ : \ \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}. \tag{A.2}$$

Applying standard arguments for proving the solvability of the steady-state Stokes equations (see [13]), we can conclude that the operator A is a positive, self-adjoint operator whose spectrum lies in $[1,\infty)$. We wish to use a spectral decomposition theorem for self-adjoint operators. To this end we first recall the concept of a direct integral of Hilbert spaces (see [12] and [14, Chap. 1, sect. 2.3])

$$\Upsilon = \int_1^\infty \oplus H(\lambda)\, d\mu(\lambda). \tag{A.3}$$

Here, $d\mu(\lambda)$ is a nonnegative Borel measure supported in $[1,\infty)$, $H(\lambda)$ is a family of Hilbert spaces with norm and inner product denoted by $\|\cdot\|_{H(\lambda)}$ and $(\cdot,\cdot)_{H(\lambda)}$, respectively, and $H(\lambda)$ is assumed μ-measurable. The μ-measurability of $H(\lambda)$ is defined as follows: there exists a family $\mathcal{M}$ of functions f: $[1,\infty)\ni\lambda\mapsto f(\lambda)\in H(\lambda)$ satisfying

1. for all $f\in\mathcal{M}$, the function $\lambda\mapsto\|f(\lambda)\|_{H(\lambda)}$ is μ-measurable;
2. if a function $g:[1,\infty)\ni\lambda\mapsto g(\lambda)\in H(\lambda)$ is such that $\lambda\mapsto\big(f(\lambda),g(\lambda)\big)_{H(\lambda)}$ is μ-measurable for each $f\in\mathcal{M}$, then $g\in\mathcal{M}$; and
3. there exists a sequence $\{f_i\}_{i=1}^\infty\subset\mathcal{M}$ such that for every $\lambda\in[1,\infty)$ the set $\{f_i(\lambda)\}_{i=1}^\infty$ is dense in $H(\lambda)$.

In other words, $\mathcal{M}$ is the set of μ-measurable functions taking values in $H(\lambda)$. The space (A.3) is the set of functions $f\in\mathcal{M}$ for which

$$\|f\|_\Upsilon^2=\int_1^\infty\|f(\lambda)\|_{H(\lambda)}^2\,d\mu(\lambda).$$

The scalar product in Υ is defined by the formula

$$(f,g)_\Upsilon=\int_1^\infty(f(\lambda),g(\lambda))_{H(\lambda)}^2d\mu(\lambda).$$

In [12], it was proved that Υ is a Hilbert space.

By virtue of the spectral decomposition theorem (see [12]) for the operator (A.1), there exists a direct integral of Hilbert spaces (A.3) and a unitary operator $U:\mathbf{V}_0^0(\Omega)\to\Upsilon$ which maps $\mathcal{D}(A)$ into

$$\Upsilon_A=\{f\in\Upsilon:\lambda f\in\Upsilon\},\qquad\|f\|_{\Upsilon_A}^2=\int_1^\infty\lambda^2\|f(\lambda)\|_{H(\lambda)}^2d\mu(\lambda)$$

and satisfies $U(A\mathbf{v})=\lambda(U\mathbf{v})$. For $s\in[0,2]$, we introduce the spaces

$$\begin{aligned}\mathbf{V}_\sigma^s(\Omega)=\Big\{&\mathbf{v}\in\mathbf{V}_0^0(\Omega)\ :\\ &U\mathbf{v}=\int_1^\infty\oplus\widehat{\mathbf{v}}(\lambda)\,d\mu(\lambda),\ \int_1^\infty\lambda^s\|\widehat{\mathbf{v}}(\lambda)\|_{H(\lambda)}^2d\mu(\lambda)<\infty\Big\}\end{aligned}\tag{A.4}$$

with norms

$$\|\mathbf{v}\|_{\mathbf{V}_\sigma^s(\Omega)}^2=\int_1^\infty\lambda^s\|\widehat{\mathbf{v}}(\lambda)\|_{H(\lambda)}^2d\mu(\lambda).\tag{A.5}$$

The definition of the spectral decomposition implies that $\mathbf{V}_\sigma^0(\Omega)=\mathbf{V}_0^0(\Omega)$, $\mathbf{V}_\sigma^2(\Omega)=\mathcal{D}(A)=\mathbf{V}^2(\Omega)\cap\mathbf{V}_0^1(\Omega)$, and, in the spaces $\mathbf{V}_\sigma^s(\Omega)$, the norms (A.5) with $s=0,2$ are equivalent to the norms of $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^2(\Omega)$, respectively. Furthermore, the following equality holds:

$$\mathbf{V}_\sigma^1(\Omega)=\mathbf{V}_0^1(\Omega)\tag{A.6}$$

and, in this space, $\|\cdot\|_{\mathbf{V}_\sigma^1(\Omega)}$ is equivalent to the $\mathbf{H}^1(\Omega)$ norm. Indeed, for each $\mathbf{v}\in\mathcal{D}(A)=\mathbf{V}_\sigma^2(\Omega)$, we have

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}^2&=-\int_\Omega\mathbf{v}\cdot\Delta\mathbf{v}\,d\mathbf{x}+\|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2\\&=\int_1^\infty(\lambda+1)\|\widehat{\mathbf{v}}(\lambda)\|_{H(\lambda)}^2\,d\mu(\lambda)\le2\|\mathbf{v}\|_{\mathbf{V}_\sigma^1(\Omega)}^2.\end{aligned}$$

Also, we obviously have $\|\mathbf{v}\|_{\mathbf{V}^1_\sigma(\Omega)} \le \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}$. Hence, the completions of $\mathbf{C}^\infty_0(\Omega) \cap \mathbf{V}^0_0(\Omega)$ under $\|\cdot\|_{\mathbf{H}^1(\Omega)}$ and $\|\cdot\|_{\mathbf{V}^1_\sigma(\Omega)}$ yield the same space, i.e., (A.6) holds.

Let X and Y be Hilbert spaces satisfying

$$X \subset Y \quad \text{and} \quad X \text{ is dense in } Y \text{ and is continuously embedded into } Y. \tag{A.7}$$

Assume that $\Lambda : Y \to Y$ is a positive, self-adjoint linear operator with domain $\mathcal{D}(\Lambda) = X$ and that $\|\cdot\|_X$ is equivalent to the norm $X \ni u \mapsto (\|u\|^2_Y + \|\Lambda u\|^2_Y)^{1/2}$. Recall that the intermediate space $[X, Y]_\theta$, $\theta \in [0, 1]$, is by definition the space $\mathcal{D}(\Lambda^{1-\theta})$ endowed with the norm

$$\|u\|^2_{[X,Y]_\theta} = \|u\|^2_Y + \|\Lambda^{1-\theta} u\|^2_Y \qquad \forall\, u \in [X, Y]_\theta\,;$$

see [14] for details.

Lemma A.1. *Suppose that X is a closed subspace of $\mathbf{H}^k(\Omega)$ with k being a positive integer, Y is a closed subspace of $\mathbf{L}^2(\Omega)$, $\|\cdot\|_X$ is equivalent to $\|\cdot\|_{\mathbf{H}^k(\Omega)}$, $\|\cdot\|_Y$ is equivalent to $\|\cdot\|_{\mathbf{L}^2(\Omega)}$, and X and Y satisfy* (A.7)*. Then, for each $\theta \in [0, 1]$, $[X, Y]_\theta$ is a closed subspace of $\mathbf{H}^{k(1-\theta)}(\Omega)$ and the norm of $[X, Y]_\theta$ is equivalent to the norm of $\mathbf{H}^{k(1-\theta)}(\Omega)$.*

Proof. Recall that, by definition, $\mathbf{H}^{k(1-\theta)}(\Omega) = [\mathbf{H}^k(\Omega), \mathbf{L}^2(\Omega)]_\theta$. Denoting by I the embedding operator we see that $I : X \to \mathbf{H}^k(\Omega)$ and $I : Y \to \mathbf{L}^2(\Omega)$ are continuous. Hence, by interpolation theorems of [14, Chap. 1, sect. 5], we deduce that $I : [X, Y]_\theta \to \mathbf{H}^{k(1-\theta)}(\Omega)$ is also embedding and

$$\|u\|_{\mathbf{H}^{k(1-\theta)}(\Omega)} \le C\|u\|_{[X,Y]_\theta} \qquad \forall\, u \in [X, Y]_\theta. \tag{A.8}$$

We consider now a bounded extension operator $L : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\mathbb{R}^3)$ (i.e., $Lu(\mathbf{x}) = u(\mathbf{x})$ for every $\mathbf{x} \in \Omega$) such that its restriction on $\mathbf{H}^k(\Omega)$ is the bounded extension operator $L : \mathbf{H}^k(\Omega) \to \mathbf{H}^k(\mathbb{R}^3)$ (such an extension can be easily constructed by the well-known Whitney formula). Clearly, LX is a closed subspace of $\mathbf{H}^k(\mathbb{R}^3)$ and we equip LX with the $\mathbf{H}^k(\mathbb{R}^3)$ norm. Analogously, LY is a closed subspace of $\mathbf{L}^2(\mathbb{R}^3)$ and we equip LY with the $\mathbf{L}^2(\mathbb{R}^3)$ norm. Denote by r the restriction operator which maps any function $f(\mathbf{x})$ defined in $\mathbb{R}^3$ into the same function f restricted to Ω. Evidently, operators $r : LX \to X$ and $r : LY \to Y$ are continuous so that interpolation theorems imply that $r : [LX, LY]_\theta \to [X, Y]_\theta$ is also continuous and

$$\|\mathbf{u}\|_{[X,Y]_\theta} = \|rL\mathbf{u}\|_{[X,Y]_\theta} \le C\|L\mathbf{u}\|_{[LX,LY]_\theta} \qquad \forall\, \mathbf{u} \in [X, Y]_\theta. \tag{A.9}$$

The Sobolev norms on LX and LY can be expressed in terms of Fourier transforms:

$$\|L\mathbf{u}\|^2_{\mathbf{H}^k(\mathbb{R}^3)} = \int_{\mathbb{R}^3} (1 + |\boldsymbol{\xi}|^2)^k |\widehat{L\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \quad \forall\, \mathbf{u} \in LX$$

and

$$\|L\mathbf{u}\|^2_{\mathbf{H}^k(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\widehat{L\mathbf{u}}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} \quad \forall\, \mathbf{u} \in LY.$$

Thus,

$$\|L\mathbf{u}\|_{[LX,LY]_\theta} = \|L\mathbf{u}\|_{\mathbf{H}^{k(1-\theta)}(\mathbb{R}^3)} \le C\|\mathbf{u}\|_{\mathbf{H}^{k(1-\theta)}(\Omega)} \qquad \forall\, \mathbf{u} \in [X, Y]_\theta, \tag{A.10}$$

where in the last step we used interpolation theorems on the extension operators $L : \mathbf{H}^k(\Omega) \to \mathbf{H}^k(\mathbb{R}^3)$ and $L : \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\mathbb{R}^3)$. Combining (A.8), (A.9), and (A.10), we prove Lemma A.1. $\square$

The following lemma is a direct consequence of Lemma A.1 by taking $k = 2$, $X = \mathbf{V}^2_\sigma(\Omega) = \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, and $Y = \mathbf{V}^0_\sigma(\Omega) = \mathbf{V}^0_0(\Omega)$.

LEMMA A.2. *For each* $s \in [0, 2]$*, the norms* $\|\cdot\|_{\mathbf{V}^s_\sigma(\Omega)}$ *and* $\|\cdot\|_{\mathbf{H}^s(\Omega)}$ *are equivalent on the space* $\mathbf{V}^s_\sigma(\Omega)$*.*

Let $G \subset \mathbb{R}^3$ be a bounded domain with a C^∞ boundary ∂G. We consider the problem

$$\mathbf{curl}\,\mathbf{w} = \mathbf{v} \quad \text{in } G \qquad \text{and} \qquad \mathbf{w}\cdot\mathbf{n}\big|_{\partial G} = 0. \tag{A.11}$$

LEMMA A.3. *Assume that* $\mathbf{v} \in \mathbf{H}^s(G) \cap \mathbf{V}^0(G)$*,* $s \geq 0$*, and* $\mathbf{v}\cdot\mathbf{n}\big|_{\partial\Omega} = 0$*. Then there exists a solution* $\mathbf{w} \in \mathbf{H}^{s+1}(G)$ *for the problem* (A.11) *satisfying the condition* $\mathbf{w}\big|_\Gamma = \mathbf{0}$.

The proof of this lemma is entirely analogous to that of [9, Lem. 4.3] and is omitted here.

LEMMA A.4. *For each* $s \in (0, 1/2)$*,* $\mathbf{V}^s_\sigma(\Omega) = \mathbf{H}^s(\Omega) \cap \mathbf{V}^0_0(\Omega)$*.*

Proof. By virtue of Lemma A.2 and the definition of intermediate spaces, we have

$$\mathbf{V}^s_\sigma(\Omega) = \text{closure of } \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0 \text{ in } \mathbf{H}^s(\Omega). \tag{A.12}$$

Since the topology of $\mathbf{H}^s(\Omega)$ is stronger than the $\mathbf{L}^2(\Omega)$ topology, the right-hand side of (A.12) is a subset of $\mathbf{V}^0_0(\Omega)$, so that $\mathbf{V}^s_\sigma(\Omega) \subset \mathbf{V}^0_0(\Omega) \cap \mathbf{H}^s(\Omega)$. Next we proceed to prove the reverse embedding. Let an arbitrary $\mathbf{v} \in \mathbf{H}^s(\Omega) \cap \mathbf{V}^0_0(\Omega)$ be given, and we choose a sequence $\{\mathbf{v}_n\} \subset \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0$ such that $\|\mathbf{v}_n - \mathbf{v}\|_{\mathbf{H}^s(\Omega)} \to 0$ as $n \to \infty$. Let $\rho > 0$ be a fixed, sufficiently large number satisfying (2.16). We set

$$\Omega_\rho \equiv \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| < \rho\} \cap \Omega \qquad \text{and} \qquad \Omega^c_\rho \equiv \mathbb{R}^3 \setminus \Omega_\rho.$$

Firstly, we decompose v as follows:

$$\begin{aligned} v(x) = v_1(\mathbf{x}) + v_2(\mathbf{x}), \quad \text{where} \quad &\operatorname{supp} v_1 \subset \Omega_{\rho+3},\\ &\operatorname{supp} v_2 \subset \Omega^c_{\rho+2}, \quad v_i \in \mathbf{V}^0_0 \cap \mathbf{H}^s(\Omega),\ i = 1, 2. \end{aligned} \tag{A.13}$$

Consider $w(x) \in \mathbf{H}^{s+1}(\Omega_{\rho+3})$ satisfying (A.11) with $G = \Omega_{\rho+3}$. The existence of such a vector field is established, e.g., in [9, 16]. Let $\varphi_1(\mathbf{x}) \in C^\infty(\Omega)$, $\varphi_1(\mathbf{x}) = 1$, for $|\mathbf{x}| < \rho + 1$, $\varphi_1(x) = 0$ for $|\mathbf{x}| > \rho + 2$, and $\varphi_2(\mathbf{x}) = 1 - \varphi_1(\mathbf{x})$. Then the functions

$$\begin{aligned} &v_1(\mathbf{x}) = \mathbf{curl}\,(w(\mathbf{x})\varphi_1(\mathbf{x})),\\ &v_2(\mathbf{x}) = \mathbf{curl}\,(w(x)\varphi_2(\mathbf{x})) \ \text{ for } |\mathbf{x}| < \rho + 3, \qquad v_2(\mathbf{x}) = v(\mathbf{x}) \ \text{ for } |\mathbf{x}| > \rho + 2 \end{aligned}$$

satisfy conditions (A.13).

It is enough to find sequences

$$\{v_{in}\} \subset \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega) \quad \text{such that} \quad \|v_{in} - v_i\|_{H^s(\Omega)} \to 0 \quad \text{as} \quad n \to \infty \tag{A.14}$$

for $i = 1, 2$.

In the case $i = 2$, we can take as v_{2n} the Friedrichs average

$$v_{2n}(\mathbf{x}) = n^3 \int_\Omega j(n(\mathbf{x} - \mathbf{y}))v_2(\mathbf{y})\,d\mathbf{y},$$

where $j(\mathbf{x}) \in C^\infty_0(\mathbb{R}^3)$, $j(\mathbf{x}) \geq 0$, $\operatorname{supp} j \subset \{|\mathbf{x}| \leq 1\}$, and $\int_\Omega j(\mathbf{x})\,d\mathbf{x} = 1$. Evidently, $v_{2n} \in \mathbf{V}^2 \cap \mathbf{V}^1_0(\Omega)$. Since the $v_{2n}(\mathbf{x})$ are well defined for $\mathbf{x} \in \mathbb{R}^3$, the relation (A.14) can be proved in this case with the help of the Fourier transforms.

To prove (A.14) for $i = 1$, we consider (A.11) with $G = \Omega_{\rho+3}$, and $v(\mathbf{x})$ changed to $v_1(\mathbf{x})$. By virtue of Lemma A.3, there exists a solution $w_1(\mathbf{x}) \in H^{s+1}(\Omega_{\rho+3})$ of this problem which satisfies $w_1(\mathbf{x})|_{\partial\Omega_{\rho+3}} = 0$. This equality and the condition $0 < s < 1/2$ yield that $w_1(\mathbf{x}) \in H_0^{s+1}(\Omega_{\rho+3})$. By the definition of the space $H_0^{s+1}(\Omega_{\rho+3})$, there exists a sequence $w_{1n}(x) \in C_0^\infty(\Omega_{\rho+3})$ such that $\|w_{1n} - w_1\|_{H^{1+s}(\Omega_{\rho+3})} \to 3$ as $n \to \infty$. This relation implies that sequence $v_{1n} = \mathbf{curl}\, w_{1n}$ satisfies the relation (A.14). $\square$

For $s \in [0, 1]$, we may define $\mathbf{V}_\sigma^{-s}(\Omega)$ as the closure of $\mathbf{V}_0^0(\Omega)$ with respect to the norm

$$\|\mathbf{v}\|_{\mathbf{V}_\sigma^{-s}(\Omega)} = \Big(\int_1^\infty \lambda^{-s}\|\widehat{\mathbf{v}}(\lambda)\|^2_{H(\lambda)}\, d\mu(\lambda)\Big)^{1/2} \quad \text{if} \quad U\mathbf{v} = \int_1^\infty \oplus \widehat{\mathbf{v}}(\lambda)\, d\mu(\lambda). \tag{A.15}$$

Lemma A.5. *For $s \in (0, 1/2)$, $\mathbf{V}_\sigma^{-s}(\Omega) \subset \mathbf{H}^{-s}(\Omega)$. Furthermore, the norm $\|\cdot\|_{\mathbf{V}_\sigma^{-s}(\Omega)}$ is equivalent to $\|\cdot\|_{\mathbf{H}^{-s}(\Omega)}$ on $\mathbf{V}_\sigma^{-s}(\Omega)$.*

This lemma can be proved with the help of Lemmas A.2 and A.4 just as the corresponding assertions in [6, Chap. 3, Lem. 4.5] were proved. We omit the details here.

A.2. Regularity results for the Stokes problem. For $s \in [0, 1]$, we may define the operator $P = P_s : \mathbf{H}^{-s}(\Omega) \to \mathbf{V}_\sigma^{-s}(\Omega)$ by the formula

$$\langle P\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}_0^0(\Omega)} = \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{L}^2(\Omega)} \qquad \forall\, \mathbf{v} \in \mathbf{V}_\sigma^s(\Omega), \tag{A.16}$$

where $\langle\cdot,\cdot\rangle_{\mathbf{V}_0^0(\Omega)}$ and $\langle\cdot,\cdot\rangle_{\mathbf{L}^2(\Omega)}$ denote the duality generated by the scalar products on $\mathbf{V}_0^0(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively. Note that for $s = 0$, P coincides with the orthogonal projection operator from $\mathbf{L}^2(\Omega)$ to $\mathbf{V}_0^0(\Omega)$; see the statement immediately following (A.1).

We first consider the Stokes problem with a vanishing forcing term and an inhomogeneous initial value $\mathbf{q}_0$. Note that $P\nabla r = \mathbf{0}$ for $\nabla r \in \mathbf{H}^{-s}(\Omega)$ and $P\partial_t\mathbf{q} = \partial_t\mathbf{q}$ for $\partial_t\mathbf{q}$ in $L^2(0, T; \mathbf{V}_\sigma^{-s}(\Omega))$; thus, we may write the Stokes problem as

$$\partial_t\mathbf{q}(t, \mathbf{x}) + A\mathbf{q} = \mathbf{0} \quad \text{in } Q \qquad \text{and} \qquad \mathbf{q}|_{t=0} = \mathbf{q}_0 \quad \text{in } \Omega, \tag{A.17}$$

where $A = P(-\Delta + I)$. Since we look for $\mathbf{q}$ in the space $L^2(0, T; \mathbf{V}_\sigma^{-s+2}(\Omega))$ and $-s + 2 \geq 1$, a solution $\mathbf{q}$ to (A.17) automatically satisfies

$$\operatorname{div}\mathbf{q} = 0 \qquad \text{and} \qquad \mathbf{q}|_\Sigma = \mathbf{0}. \tag{A.18}$$

As was mentioned in Lemma 3.6, the existence and uniqueness of a solution $\mathbf{q} \in L^2(0, T; \mathbf{V}_\sigma^1(\Omega)) \cap H^1(0, T; \mathbf{V}_\sigma^{-1}(\Omega))$ to (A.17) is well known. The following regularity result holds.

Lemma A.6. *Assume that $\mathbf{q}_0 \in \mathbf{V}_\sigma^s(\Omega)$, $s \geq 0$. Then a solution $\mathbf{q}$ of the problem* (A.17) *satisfies the estimate*

$$\begin{aligned}\int_t^\infty \Big(\|\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^{\nu+s+1}(\Omega)} + \|\partial_t\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^{\nu+s-1}(\Omega)}\Big)\, d\tau \\ = \|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}_\sigma^{\nu+s}(\Omega)} \leq e^{-\nu}\Big(\frac{\nu}{2}\Big)^\nu t^{-\nu}\|\mathbf{q}_0\|^2_{\mathbf{V}_\sigma^s(\Omega)}\end{aligned} \tag{A.19}$$

for every $\nu \geq 0$.

Proof. We apply to (A.17) the unitary operator U introduced in the definition of the spectral decomposition of A as direct integrals. Then (A.17) reduces to

$$\int_1^\infty \oplus\Big(\partial_t\widehat{\mathbf{q}}(t, \lambda) + \lambda\widehat{\mathbf{q}}(t, \lambda)\Big)\, d\mu(\lambda) = 0$$

and

$$\int_1^\infty \oplus\Big(\widehat{\mathbf{q}}(t,\lambda)|_{t=0} - \widehat{\mathbf{q}}_0(\lambda)\Big)\, d\mu(\lambda) = 0.$$

By the definition of direct integrals, the integrands belong to different Hilbert spaces $H(\lambda)$ for different λ. This allows us to deduce that for almost every λ (with respect to the μ-measure),

$$\partial_t \widehat{\mathbf{q}}(t,\lambda) + \lambda \widehat{\mathbf{q}}(t,\lambda) = \mathbf{0} \quad \text{in } Q \qquad \text{and} \qquad \widehat{\mathbf{q}}(t,\lambda)|_{t=0} = \widehat{\mathbf{q}}_0(\lambda) \quad \text{in } \Omega. \tag{A.20}$$

Equalities (A.20) are understood as equalities in the Hilbert spaces $H(\lambda)$. More precisely, the first equality is considered in $L^2(0,T;H(\lambda))$. Equalities (A.20) imply

$$\widehat{\mathbf{q}}(t,\lambda) = e^{-\lambda t}\widehat{\mathbf{q}}_0(\lambda). \tag{A.21}$$

Using definitions (A.5) and (A.15) of the norms for $\mathbf{V}^s_\sigma(\Omega)$, we obtain

$$\|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{\nu+s}_\sigma(\Omega)} = \int_1^\infty e^{-2\lambda t}\lambda^{\nu+s}\|\widehat{\mathbf{q}}_0(\lambda)\|^2_{H(\lambda)}\, d\mu(\lambda). \tag{A.22}$$

By solving a simple extremal problem we obtain that, for each $t>0$,

$$\max_{\lambda\in[1,\infty)} (e^{-2\lambda t}\lambda^\nu) = e^{-\max\{2t,\nu\}}\big[\max\{1,\nu/(2t)\}\big]^\nu \le e^{-\nu}\Big(\frac{\nu}{2}\Big)^\nu t^{-\nu}. \tag{A.23}$$

From (A.22) and (A.23) we deduce that

$$\|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{\nu+s}_\sigma(\Omega)} \le e^{-\nu}\Big(\frac{\nu}{2}\Big)^\nu t^{-\nu}\|\mathbf{q}_0\|^2_{\mathbf{V}^s_\sigma(\Omega)}. \tag{A.24}$$

Integrating (A.22) in t and applying (A.22) to the result, we have

$$\begin{aligned}&\int_t^\infty \|\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^{\nu+s+1}_\sigma(\Omega)}\, d\tau \\ &\quad = \frac{1}{2}\int_1^\infty e^{-2\lambda t}\lambda^{\nu+s}\|\widehat{\mathbf{q}}_0(\lambda)\|^2_{H(\lambda)}\, d\mu(\lambda) = \frac{1}{2}\|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{\nu+s}_\sigma(\Omega)}.\end{aligned} \tag{A.25}$$

Differentiating (A.21) with respect to t and repeating the arguments used in deriving (A.25), we are led to

$$\begin{aligned}&\int_t^\infty \|\partial_t\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^{\nu+s-1}_\sigma(\Omega)}\, d\tau \\ &\quad = \int_t^\infty\int_1^\infty e^{-2\lambda\tau}\lambda^{\nu+s+1}\|\mathbf{q}_0\|^2_{\mathbf{V}^s_\sigma(\Omega)}\, d\mu(\lambda)\, d\tau = \frac{1}{2}\|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{\nu+s}_\sigma(\Omega)}.\end{aligned} \tag{A.26}$$

Then inequalities (A.24)–(A.26) imply (A.19). □

We next consider the Stokes problem with nonzero forcing and the zero initial value:

$$\partial_t\mathbf{q}(t,\mathbf{x}) + A\mathbf{q} = \mathbf{h}(t,\mathbf{x}) \quad \text{in } Q \qquad \text{and} \qquad \mathbf{q}|_{t=0} = \mathbf{0} \quad \text{in } \Omega. \tag{A.27}$$

Lemma A.7. *Assume* $\mathbf{h}\in L^2(0,T;\mathbf{V}^{-s}_\sigma(\Omega))$, $s\in[0,1]$. *Then the solution* $\mathbf{q}$ *of the problem* (A.27) *satisfies the equality*

$$\int_0^T \Big(\|\partial_t\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{-s}_\sigma(\Omega)} + \|\mathbf{q}(t,\cdot)\|^2_{\mathbf{V}^{2-s}_\sigma(\Omega)}\Big)\, dt = \int_0^T \|\mathbf{h}(t,\cdot)\|^2_{\mathbf{V}^{-s}_\sigma(\Omega)}\, dt. \tag{A.28}$$

Proof. As in the proof of Lemma A.6, we reduce the problem (A.27) to an ordinary differential equation problem for the spectral decomposition $\widehat{\mathbf{q}}(t,\lambda)$ of $\mathbf{q}(t,\mathbf{x})$:

$$\partial_t\widehat{\mathbf{q}}(t,\lambda)+\lambda\widehat{\mathbf{q}}(t,\lambda)=\widehat{\mathbf{h}}(t,\lambda)\quad\text{in } Q \tag{A.29}$$

and

$$\widehat{\mathbf{q}}(t,\lambda)|_{t=0}=\mathbf{0} \tag{A.30}$$

for μ-almost-every $\lambda\in[1,\infty)$, where $\widehat{\mathbf{h}}$ is the spectral decomposition of $\mathbf{h}$. We extend $\widehat{\mathbf{h}}$ in t by zero outside the interval $[0,T]$ and extend $\widehat{\mathbf{q}}$ by zero for $t<0$ and by $e^{-\lambda t}\int_0^T e^{\lambda\tau}\mathbf{h}(\tau,\lambda)\,d\tau$ for $t>T$. This integral defines the solution of (A.29) for $t>T$ if $\mathbf{h}=0$ for $t>T$. We still denote the extended functions by $\widehat{\mathbf{q}}$ and $\widehat{\mathbf{h}}$, respectively. By virtue of (A.30), the extended functions satisfy (A.29)–(A.30) for $t\in\mathbb{R}$. Applying the Fourier transform in t, i.e.,

$$\widetilde{\mathbf{q}}(\tau,\lambda)=\int_{\mathbb{R}}e^{-i\tau t}\widehat{\mathbf{q}}(t,\lambda)\,dt\qquad\text{and}\qquad\widetilde{\mathbf{h}}(\tau,\lambda)=\int_{\mathbb{R}}e^{-i\tau t}\widehat{\mathbf{h}}(t,\lambda)\,dt,$$

to (A.29) we obtain

$$\widetilde{\mathbf{q}}(\tau,\lambda)=\frac{\widetilde{\mathbf{h}}(\tau,\lambda)}{i\tau+\lambda}\quad\text{and}\quad i\tau\widetilde{\mathbf{q}}(\tau,\lambda)=\frac{i\tau\widetilde{\mathbf{h}}(\tau,\lambda)}{i\tau+\lambda}.$$

Taking the $\|\cdot\|_{\mathbf{V}_\sigma^{-s}(\Omega)}$ norms yields

$$\begin{aligned}\|\widetilde{\mathbf{q}}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^{2-s}(\Omega)}&+\|i\tau\widetilde{\mathbf{q}}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^{-s}(\Omega)}\\ &=\int_1^\infty\left(\frac{\lambda^{2-s}}{|i\tau+\lambda|^2}+\frac{\lambda^{-s}|i\tau|^2}{|i\tau+\lambda|^2}\right)\|\widetilde{\mathbf{h}}(\tau,\lambda)\|^2_{H(\lambda)}\,d\lambda\\ &=\int_1^\infty\lambda^{-s}\|\widetilde{\mathbf{h}}(\tau,\lambda)\|^2_{H(\lambda)}\,d\lambda=\|\widetilde{\mathbf{h}}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^{s}(\Omega)}.\end{aligned} \tag{A.31}$$

Applying Parseval's equality to both ends of (A.31) we obtain (A.28). (Note that standard uniqueness arguments applied to (A.27) imply that we have obtained estimates precisely for the solution of (A.27).) □

A.3. Regularity estimate for the solution of the adjoint boundary value problem. We will apply the regularity results for the Stokes problem to derive regularity estimates for the solution $\mathbf{q}$ of the adjoint boundary value problem (3.14). (3.14) can be rewritten as

$$\partial_t\mathbf{q}(t,\mathbf{x})+A\mathbf{q}=L\mathbf{q}\quad\text{in } Q\qquad\text{and}\qquad\mathbf{q}|_{t=0}=\mathbf{q}_0\quad\text{in }\Omega, \tag{A.32}$$

where

$$L\mathbf{q}=P\big(\mathbf{q}+[(\mathbf{v}_0+\widehat{\mathbf{w}})\cdot\nabla]\mathbf{q}-(\nabla\widehat{\mathbf{w}})^*\mathbf{q}\big). \tag{A.33}$$

Indeed, we consider the first equation in (A.32) in the space $L^2(0,T;\mathbf{V}_\sigma^{-s}(\Omega))$, $s\in[0,1]$. Since the definition (A.16) implies that P is a projection and we look for $\partial_t\mathbf{q}$ in $L^2(0,T;\mathbf{V}_\sigma^{-s}(\Omega))$, we have that $P\partial_t\mathbf{q}=\partial_t\mathbf{q}$. The equality $A=P(-\Delta+I)$ follows from the definitions (A.1) and (A.16) of the operators A (on $\mathbf{V}_0^0(\Omega)$) and P and the definition of the space $\mathbf{V}_\sigma^{-s}(\Omega)$ (see (A.15)). Relations (A.18) are built into the space $L^2(0,T;\mathbf{V}_\sigma^{-s+2}(\Omega))$, $-s+2\ge 1$. Note that the existence and uniqueness of the solution was established in Lemma 3.6. We have the following regularity results for the adjoint boundary value problem.

PROPOSITION A.8. *Assume that* $\mathbf{q}_0\in\mathbf{V}_\sigma^s(\Omega)$ *for an* $s\in[0,1/2)$. *Let* $\mathbf{q}$ *be the unique solution of the problem* (A.32)–(A.33) *satisfying the inequality* (3.15). *Then the following estimate holds:*

$$\int_t^\infty \big(\|\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^2(\Omega)}+\|\partial_t\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}_\sigma^0(\Omega)}\big)\,d\tau\le C_0t^{s-1}\|\mathbf{q}_0\|^2_{\mathbf{V}_\sigma^s(\Omega)}, \tag{A.34}$$

where the constant C_0 *depends only on* $\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}(Q)}$ *and* $\|\mathbf{v}_0\|_{\mathbf{C}^2(\overline{\Omega})}$.

Proof. By virtue of Theorem 2.1, we have that

$$\sup_{\mathbf{x}\in\overline{\Omega}}|\mathbf{v}_0(\mathbf{x})-\mathbf{w}_\infty|+\sum_{i=1}^3\sup_{\mathbf{x}\in\overline{\Omega}}|\partial_{x_i}\mathbf{v}_0(\mathbf{x})|+\sum_{i,j=1}^3\sup_{\mathbf{x}\in\overline{\Omega}}|\partial_{x_i}\partial_{x_j}\mathbf{v}_0(\mathbf{x})|\le C|\mathbf{v}_\infty|.$$

As indicated in Theorems 4.1 and 4.2 and by (2.22), (2.19), and (2.15), the function $\widehat{\mathbf{w}}$ satisfies

$$\widehat{\mathbf{w}}\in\mathbf{V}^{1,1/2}(Q)\subset\mathbf{C}([0,T];\mathbf{H}^1(\Omega)). \tag{A.35}$$

Therefore, for $\mathbf{q}\in L^2(0,T;\mathbf{V}_\sigma^1(\Omega))$, we have $(\mathbf{v}_0\cdot\nabla)\mathbf{q}\in L^2(0,T;\mathbf{L}^2(\Omega))$, and we obtain by Sobolev embedding theorems and the Holder inequality that

$$\begin{aligned}\Big|\int_\Omega(\widehat{\mathbf{w}}\cdot\nabla)\mathbf{q}\cdot\boldsymbol{\phi}\,dx\Big|&\le\|\widehat{\mathbf{w}}\|_{\mathbf{L}^6(\Omega)}\|\nabla\mathbf{q}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\phi}\|_{\mathbf{L}^3(\Omega)}\\&\le C\|\widehat{\mathbf{w}}\|_{\mathbf{H}^1(\Omega)}\|\nabla\mathbf{q}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Omega)},\end{aligned}$$

which, upon using (A.35), yields

$$\|(\widehat{\mathbf{w}}\cdot\nabla)\mathbf{q}\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Omega))}\le C\|\mathbf{q}\|_{L^2(0,T;\mathbf{V}_\sigma^1(\Omega))}\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}(Q)}. \tag{A.36}$$

Similarly, we obtain

$$\begin{aligned}\Big|\int_\Omega(\nabla\widehat{\mathbf{w}})^*\mathbf{q}\cdot\boldsymbol{\phi}\,dx\Big|&\le\|\nabla\widehat{\mathbf{w}}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{q}\|_{\mathbf{L}^6(\Omega)}\|\boldsymbol{\phi}\|_{\mathbf{L}^3(\Omega)}\\&\le C\|\nabla\widehat{\mathbf{w}}\|_{\mathbf{L}^2(\Omega)}\|\mathbf{q}\|_{\mathbf{H}^1(\Omega)}\|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Omega)}\end{aligned} \tag{A.37}$$

and

$$\|(\nabla\widehat{\mathbf{w}})^*\mathbf{q}\|_{L^2(0,T;\mathbf{H}^{-1/2}(\Omega))}\le C\|\mathbf{q}\|_{L^2(0,T;\mathbf{V}_\sigma^1(\Omega))}\|\widehat{\mathbf{w}}\|_{\mathbf{V}^{1,1/2}(Q)}. \tag{A.38}$$

From (A.33), (A.36), and (A.38), we deduce that

$$\|L\mathbf{q}\|_{L^2(t,T;\mathbf{V}_\sigma^{-1/2}(\Omega))}\le C\|\mathbf{q}\|_{L^2(t,T;\mathbf{V}_\sigma^1(\Omega))},\qquad t\in(0,T). \tag{A.39}$$

We decompose the solution $\mathbf{q}$ of (A.32) into

$$\mathbf{q}=\mathbf{q}_1+\mathbf{q}_2, \tag{A.40}$$

where $\mathbf{q}_1$ is the solution of (A.17) and $\mathbf{q}_2$ is the solution of (A.27) with $\mathbf{h}=L\mathbf{q}$. Then, by Lemma A.7, we have

$$\|\mathbf{q}_2\|^2_{L^2(0,T;\mathbf{V}_\sigma^{3/2}(\Omega))}+\|\partial_t\mathbf{q}_2\|^2_{L^2(0,T;\mathbf{V}_\sigma^{-1/2}(\Omega))}\le C\|L\mathbf{q}\|^2_{L^2(0,T;\mathbf{V}_\sigma^{-1/2}(\Omega))}. \tag{A.41}$$

The relations (A.40)–(A.41) and Lemma A.6 with $s+\nu = 1/2$ and (A.39) imply that, for each $t \in (0,T)$,

$$\begin{aligned}\int_t^T \Big(\|\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^{3/2}_\sigma(\Omega)} + \|\partial_t \mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^{-1/2}_\sigma(\Omega)} \Big)\, d\tau \\ \le C\Big(\|\mathbf{q}_1(t,\cdot)\|^2_{\mathbf{V}^{1/2}_\sigma(\Omega)} + \|\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^1_\sigma(\Omega))} \Big).\end{aligned} \tag{A.42}$$

Repeating arguments similar to those used in the derivation of (A.36)–(A.39), we obtain

$$\|L\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^{-\varepsilon}_\sigma(\Omega))} \le C\|\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^{3/2}_\sigma(\Omega))}, \tag{A.43}$$

where $\varepsilon > 0$ is an arbitrary small number. Also, analogous to (A.42), we have that

$$\begin{aligned}\|\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^{2-\varepsilon}_\sigma(\Omega))} + \|\partial_t \mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^{-\varepsilon}_\sigma(\Omega))} \\ \le C\Big(\|\mathbf{q}_1(t,\cdot)\|^2_{\mathbf{V}^1_\sigma(\Omega)} + \|L\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^{-\varepsilon}_\sigma(\Omega))} \Big).\end{aligned} \tag{A.44}$$

We substitute (A.43) into (A.44) and use the obtained estimate as well as the arguments which led us to (A.43) and (A.44) to deduce that (A.43) and (A.44) hold with $\varepsilon = 0$. Combining the estimates (A.42)–(A.44) (with $\varepsilon > 0$ and with $\varepsilon = 0$), we are led to

$$\begin{aligned}\int_t^T \Big(\|\mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^2_\sigma(\Omega)} + \|\partial_t \mathbf{q}(\tau,\cdot)\|^2_{\mathbf{V}^0_\sigma(\Omega)} \Big)\, d\tau \\ \le C\Big(\|\mathbf{q}_1(t,\cdot)\|^2_{\mathbf{V}^1_\sigma(\Omega)} + \|\mathbf{q}\|^2_{L^2(t,T;\mathbf{V}^1_\sigma(\Omega))} \Big).\end{aligned} \tag{A.45}$$

Applying Lemma A.6 with $\nu = 1-s$ and the estimate (3.15) to the right-hand side of (A.45), we arrive at (A.34). □

Appendix B. Orlicz spaces. In section 5.2, we needed to determine the space in which to search for the adjoint vector field for the optimality system. For this purpose, we first calculate the dual space for the function space $L^2(Q) \cap L^{6/5}(Q)$. We can consider $L^2(Q) \cap L^{6/5}(Q)$ as the Orlicz space with the N-function

$$A(t) = \max\,(t^2, t^{6/5}), \qquad t \ge 0.$$

(For the N-function as well as other notations and assertions connected with Orlicz spaces, see [1, Chap. 8].) In other words, the Orlicz space $L^2(Q) \cap L^{6/5}(Q)$ can be defined as follows:

$$\begin{aligned}L^2(Q) \cap L^{6/5}(Q) = L_A(Q) \\ \equiv \Big\{ f(t,\mathbf{x}),\ (t,\mathbf{x}) \in Q : \int_Q A(|f(t,\mathbf{x})|)\, d\mathbf{x}\, dt < \infty \Big\}.\end{aligned} \tag{B.1}$$

The norm in a general Orlicz space $L_A(Q)$ is defined as follows:

$$\|f\|_{L_A(Q)} = \inf \Big\{ k > 0 : \int_Q A\Big(\frac{|f(t,\mathbf{x})|}{k} \Big)\, d\mathbf{x}\, dt \le 1 \Big\}. \tag{B.2}$$

Define the Legendre transform

$$A^*(s) = \max_{t \ge 0} (st - A(t))$$

for $A(t) = \max(t^2, t^{6/5})$. Then $A^*(s)$ is an N-function and the Orlicz space

$$L_{A^*}(Q) = \left\{ f(t,\mathbf{x}),\ (t,\mathbf{x}) \in Q : \int_Q A^*(|f(t,\mathbf{x})|)\, d\mathbf{x}\, dt < \infty \right\} \tag{B.3}$$

is the function space dual to space (B.1) (see [1, Chap. 8]). Straightforward calculations show that

$$A^*(s) = \begin{cases} 5^5 s^6/6^6, & s \in [0, 6/5], \\ s-1, & s \in [6/5, 2], \\ s^2/4, & s \geq 2. \end{cases} \tag{B.4}$$

The norm $\|\cdot\|_{L_{A^*}(Q)}$ of the space $L_{A^*}(Q)$ is defined similarly to (B.2) with the help of the N-function $A^*(t)$.

Let us consider now the spaces $\mathbf{L}_A(Q)$ and $\mathbf{L}_{A^*}(Q)$ of vector functions that are defined by (B.1) and (B.3), respectively, where f is the vector function $f = (f_1(t,x), f_2(t,x), f_3(t.x))$.

Lemma B.1. *The spaces $\mathbf{L}_A(Q)$ and $\mathbf{L}_{A^*}(Q)$ are dual.*

Proof. We will prove that $(\mathbf{L}_A(Q))^* = \mathbf{L}_{A^*}(Q)$, where $(\mathbf{L}_A(Q))^*$ is the adjoint space of $\mathbf{L}_A(Q)$. Each $\mathbf{v} \in \mathbf{L}_{A^*}(Q)$ forms a linear bounded functional on $\mathbf{L}_A(Q)$ by the formula

$$l(u) = \int_Q \mathbf{v}(t,x) \cdot \mathbf{u}(t,\mathbf{x})\, d\mathbf{x}\, dt \qquad \forall \mathbf{u} \in \mathbf{L}_A(Q) \tag{B.5}$$

because of the generalized Holder inequality (see [1, Chap. 8]):

$$\int_Q \mathbf{v}(t,x) \cdot \mathbf{u}(t,\mathbf{x})\, d\mathbf{x}\, dt \leq 2\|\mathbf{v}\|_{\mathbf{L}_{A^*}(Q)} \|\mathbf{u}\|_{\mathbf{L}_A(Q)}. \tag{B.6}$$

Therefore $\mathbf{L}_{A^*}(Q) \subset (\mathbf{L}_A(Q))^*$. To prove the inverse inclusion we establish first that $\mathbf{L}_A(Q)$ is isomorphic to $L_A(Q) \times L_A(Q) \times L_A(Q)$. For this we have to prove the estimates

$$\|\mathbf{v}\|_{\mathbf{L}_A(Q)} \leq \sum_{j=1}^3 \|v_j\|_{L_A(Q)} \leq 3\|\mathbf{v}\|_{\mathbf{L}_A(Q)} \quad \forall\, \mathbf{v} = (v_1, v_2, v_3) \in \mathbf{L}_A(Q). \tag{B.7}$$

Suppose first that for each j, $v_j(t,\mathbf{x}) \neq 0$ on a set of positive Lebesque measure. Then, using the definition (B.2), the inequality $|\mathbf{v}(\mathbf{x})| \leq \sum_{j=1}^3 |v_j(\mathbf{x})|$, and the convexity of $A(\lambda)$, we obtain

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbf{L}_A(Q)} &= \inf \left\{ k > 0 : \int_\Omega A\left(\frac{|\mathbf{v}(\mathbf{x})|}{k}\right) d\mathbf{x} \leq 1 \right\} \\
&\leq \inf \left\{ k > 0 : \int_\Omega A\left(\frac{\sum_{j=1}^3 |v_j(\mathbf{x})|}{k}\right) d\mathbf{x} \leq 1 \right\} \\
&= \inf \left\{ \sum_{i=1}^3 k_i : k_i > 0,\ i = 1,2,3, \int_\Omega A\left(\sum_{j=1}^3 \frac{k_j}{k_1+k_2+k_3} \frac{|v_j(\mathbf{x})|}{k_j}\right) d\mathbf{x} \leq 1 \right\} \\
&\leq \inf \left\{ \sum_{i=1}^3 k_i : k_i > 0,\ i = 1,2,3, \sum_{j=1}^3 \frac{k_j}{k_1+k_2+k_3} \int_\Omega A\left(\frac{|v_j(\mathbf{x})|}{k_j}\right) d\mathbf{x} \leq 1 \right\} \\
&\leq \inf \left\{ \sum_{i=1}^3 k_i : k_i > 0, \int_\Omega A\left(\frac{|v_j(\mathbf{x})|}{k_j}\right) d\mathbf{x} \leq 1\ i = 1,2,3 \right\} \\
&\leq \sum_{i=1}^3 \inf \left\{ k > 0 : \int_\Omega A\left(\frac{|v_i(\mathbf{x})|}{k}\right) d\mathbf{x} \leq 1 \right\} = \sum_{i=1}^3 \|v_i\|_{L_A(Q)}.
\end{aligned}$$

This inequality and the homogeniety property of norms imply the first estimate in (B.7)

Denote $\hat{v}(t,\mathbf{x}) = \max_{j=1,2,3} |v_j(t,\mathbf{x})|$. For each $(t,\mathbf{x})$ we have

$$\frac{1}{3}|\mathbf{v}(t,\mathbf{x})| \leq \hat{v}(t,\mathbf{x}) \leq |\mathbf{v}(t,\mathbf{x})|$$

so that $A(\hat{v}(t,\mathbf{x})/k) \leq A(|\mathbf{v}(t,\mathbf{x})|/k)$ for all $k > 0$. Thus,

$$\text{(B.8)}\qquad \begin{aligned} \sum_{j=1}^{3} \|v_j\|_{L_A(Q)} &\leq 3\|\hat{v}\|_{L_A(Q)} = 3\inf\left\{k>0:\ \int_Q A(\frac{|\hat{v}(t,\mathbf{x})|}{k})\,d\mathbf{x}\,dt \leq 1\right\} \\ &\leq 3\inf\left\{k>0:\ \int_Q A\left(\frac{|\mathbf{v}(t,\mathbf{x})|}{k}\right)d\mathbf{x}\,dt \leq 1\right\} = 3\|\mathbf{v}\|_{\mathbf{L}_A(Q)}. \end{aligned}$$

The bound (B.8) implies the second estimate in (B.7).

The estimate (B.7) in the case when $v_j(t,\mathbf{x}) = 0$ almost everywhere for one or two j can be considered similarly. The case $v_j(t,\mathbf{x}) = 0$ a.e. for $j = 1,2,3$ is trivial.

By the lemma on the form of a functional on direct product of spaces (see [2, sect. 2.1.2]), each functional $\Lambda \in (L_A(Q) \times L_A(Q) \times L_A(Q))^*$ has the form

$$\text{(B.9)}\qquad \Lambda(\mathbf{u}) = l_1(u_1) + l_2(u_2) + l_3(u_3) \quad \forall\, \mathbf{u} = (u_1,u_2,u_3) \in [L_A(Q)]^3,$$

where $l_j \in (L_A(Q))^*$, $j = 1,2,3$. By [1, Chap. 8], there exist $v_j \in L_{A^*}(Q)$, $j = 1,2,3$, such that

$$\text{(B.10)}\qquad l_j(u) = \int_Q v_j(t,\mathbf{x})u(t,\mathbf{x})\,d\mathbf{x}\,dt, \quad j = 1,2,3 \quad \forall\, u \in L_A(Q).$$

Relations (B.9), (B.10), and the isomorphism $\mathbf{L}_A(Q) \cong L_A(Q) \times L_A(Q) \times L_A(Q)$ proved by (B.7) yield the embedding $\mathbf{L}_{A^*}(Q) \supset (\mathbf{L}_A(Q))^*$.

The proof of the relation $(\mathbf{L}_{A^*}(Q))^* = \mathbf{L}_A(Q)$ is similar. ∎

We will need the following assertion.

LEMMA B.2. *Let Y be a closed subspace of a reflexive space X. Assume that the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ are equivalent. Then Y is a reflexive space.*

Proof. By virtue of the Eberlein–Shmulian theorem [18, Appendix], the reflexivity of Y is equivalent to the following property: each bounded sequence $y_n \in Y$ has a subsequence $\{y_{n_k}\}$ converging weakly to a $\widehat{y} \in Y$. Let a bounded sequence $y_n \in Y$ be given. Since the sequence $\{y_n\}$ is bounded in X and X is reflexive, there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ that converges weakly in X to a $\widehat{y} \in X$. Note that Y being closed and convex, it is sequentially weakly closed. By the Hahn–Banach theorem, the weak convergence in X of $\{y_{n_k}\} \subset Y$ implies the weak convergence in Y. Hence, Y is a reflexive space. ∎

B.1. Orlicz spaces of solenoidal fields. Now we consider solenoidal vector fields. Using the space of vector fields

$$\mathcal{V}(Q) = \{\mathbf{v}(t,\mathbf{x}) \in \mathbf{C}_0^\infty(Q) :\ \operatorname{div}\mathbf{v} = 0, \quad \operatorname{supp}\mathbf{v} \subset\subset Q\}$$

we introduce

$$\text{(B.11)}\qquad L^2(0,T;\mathbf{V}_0^0(\Omega)) = \text{closure of } \mathcal{V}(Q) \text{ in } \mathbf{L}^2(Q).$$

It is well known that $\mathbf{V}_0^0(\Omega)$ in (B.11) is the space (2.14). Define also

$$\mathbf{V}_A(Q) = \text{closure of } \mathcal{V}(Q) \text{ in } \mathbf{L}_A(Q). \tag{B.12}$$

Evidently $\mathbf{V}_A(Q) = L^2(0,T;\mathbf{V}_0^0(\Omega)) \cap \mathbf{L}^{6/5}(Q)$. We set

$$\mathbf{V}_{A^*}(Q) = \text{closure of } \mathcal{V}(Q) \text{ in } \mathbf{L}_{A^*}(Q). \tag{B.13}$$

All elements of the spaces (B.11), (B.12), and (B.13) possess the properties $\operatorname{div}\mathbf{u} = 0$ and $(\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega} = 0$ (these equalities are understood in the sense of distributions).

Denote

$$\mathbf{G}_A(Q) = \{\nabla p \in \mathbf{L}_A(Q) :\ p \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega))\}, \tag{B.14}$$

LEMMA B.3. *The relations* $(\mathbf{V}_{A^*}(Q))^* = \mathbf{V}_A(Q)$ *and* $(\mathbf{V}_A(Q))^* = \mathbf{V}_{A^*}(Q)$ *hold.*

Proof. We begin from the first identity. Each $\mathbf{u} \in \mathbf{V}_A(Q)$ defines a functional on $\mathbf{V}_{A^*}(Q)$ by (B.5) and (B.6). Hence $\mathbf{V}_A(Q) \subset (\mathbf{V}_{A^*}(Q))^*$. To prove the reverse inclusion let a functional $l \in (\mathbf{V}_{A^*}(Q))^*$ be given, and we extend it by the Hahn–Banach theorem into an $\widehat{l} \in (\mathbf{L}_{A^*}(Q))^*$. By Lemma B.1, there exists a $\mathbf{u} \in \mathbf{L}_A(Q)$ such that $\widehat{l}(\mathbf{v}) = \int_Q \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}\,dt$. Thus,

$$l(\mathbf{v}) = \int_Q \mathbf{u}\cdot\mathbf{v}\,d\mathbf{x}\,dt \qquad \forall\,\mathbf{v} \in \mathbf{V}_{A^*}(Q). \tag{B.15}$$

Note that the following decomposition is true:

$$\mathbf{L}_A(Q) = \mathbf{V}_A(Q) + \mathbf{G}_A(Q). \tag{B.16}$$

Indeed, since $\mathbf{L}_A(Q) \subset \mathbf{L}^2(Q)$, by virtue of Weyl's decomposition for each $\mathbf{u} \in \mathbf{L}_A(Q)$, there exist $\nabla p \in \mathbf{G}(Q) = \{\nabla p \in \mathbf{L}^2(Q) :\ \ p \in L^2(0,T;H^1_{\mathrm{loc}}(\Omega))\}$ and $\mathbf{w} \in L^2(0,T;\mathbf{V}_0^0(\Omega))$ such that

$$\mathbf{u} = \mathbf{w} + \nabla p. \tag{B.17}$$

To prove Weyl's decomposition, one defines $\nabla p(t,\mathbf{x})$ to be the solution of the problem

$$\int_Q \nabla p(t,\mathbf{x})\cdot\nabla q(t,\mathbf{x})\,d\mathbf{x}\,dt = \int_Q \mathbf{u}(t,\mathbf{x})\cdot\nabla q(t,\mathbf{x})\,d\mathbf{x}\,dt \qquad \forall\,\nabla q. \tag{B.18}$$

If $\nabla q \in \mathbf{L}^2(Q)$, then (B.18) defines $\nabla p \in \mathbf{L}^2(Q)$. But since $\mathbf{u} \in \mathbf{V}_A(Q) = \mathbf{L}^2(Q) \cap \mathbf{L}^{6/5}(Q)$, the right-hand side of (B.18) is a bounded functional with respect to $\nabla q \in \mathbf{L}^6(Q)$, so that $\nabla p \in \mathbf{L}^{6/5}(Q)$. Hence $\nabla p \in \mathbf{V}_A(Q)$ and, by (B.17), $\mathbf{w} \in \mathbf{V}_A(Q)$, which proves (B.16). Note that the relation $\mathbf{V}_A(Q) \cap \mathbf{G}_A(Q) = 0$ follows from the orthogonality between $\mathbf{w}$ and ∇p in $\mathbf{L}^2(Q)$.

Now we substitute (B.17) into (B.15) and take into account that $\int_Q \nabla p\cdot\mathbf{v}\,d\mathbf{x}\,dt = 0$ for every $\nabla p \in \mathbf{G}_A(Q)$ and every $\mathbf{v} \in \mathbf{V}_{A^*}(Q)$. As a result we deduce

$$l(\mathbf{v}) = \int_Q \mathbf{w}\cdot\mathbf{v}\,d\mathbf{x}\,dt \qquad \forall\,\mathbf{v} \in \mathbf{V}_{A^*}(Q) \tag{B.19}$$

with $\mathbf{w} \in \mathbf{V}_A(Q)$. Hence, we have proved the equality $(\mathbf{V}_{A^*}(Q))^* = \mathbf{V}_A(Q)$. This equality yields $(\mathbf{V}_A(Q))^* = \mathbf{V}_{A^*}(Q)$, since by Lemma B.2, the spaces $\mathbf{V}_A(Q))$ and $\mathbf{V}_{A^*}(Q)$ are reflexive. $\square$

We set

$$\mathcal{W}_0^{(2)}(Q) = \{\mathbf{v} \in L^2(0,T;\mathbf{V}^2(\Omega)) : \partial_t \mathbf{v} \in L^2(0,T;\mathbf{V}^0(\Omega)),\ \mathbf{v}\big|_{t=0} = \mathbf{0}\} \tag{B.20}$$

and

$$\mathbf{V}_A^{1,2}(Q) = \mathcal{W}_0^{(2)}(Q)) \cap \mathbf{W}_{6/5}^{1,2}(Q), \tag{B.21}$$

where $\mathbf{W}_{6/5}^{1,2}(Q)$ is Sobolev space defined by (3.7).

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