Calculus for the Life Sciences II

Lecture Notes – Linear Differential Equations

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Outline

Introduction

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- Examples of linear first order differential equations

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  - Arterial blood pressure

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  - Arterial blood pressure
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  - Newton's law of cooling
  - Pollution in a Lake
- Extend earlier techniques to find solutions

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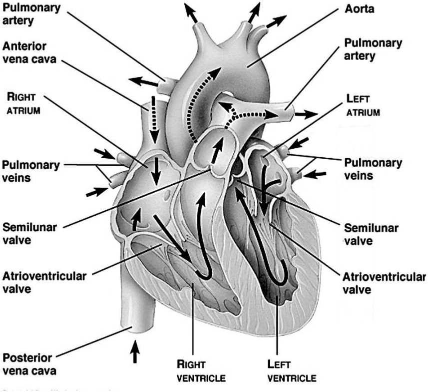
- The numbers for blood pressure reflect the force on arterial walls
- This pressure is generated by the beating of the heart

Blood Pressure

Diagram of Heart



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- The right atrium contracts, then the AV valve between the atrium and the ventricle closes

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- The main factor that changes resistance of the blood flow is change in the radius
- Blood pressure becomes a valuable tool for detecting narrowing of the blood vessels by hypertension or atherosclerosis

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- To simplify the model, we take advantage of the fact that venous pressures are very low, so we approximate the systemic flow by the equation:

$$Q_s(t) = \frac{1}{R_s}P_a(t)$$

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- Experimentally, the arterial volume, Va, is roughly equal to the compliance, C_a, times the arterial pressure

$$V_a(t) = C_a P_a(t)$$

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- Thus,

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Modeling Blood Pressure 4

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The **solution** is

$$P_a(t) = P_{sys} e^{-\frac{t}{C_a R_s}} \quad \text{for} \quad t \in [0, T]$$

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$$V = V_{sys} - V_{dia} = C_a P_{sys} - C_a P_{dia}$$

- But $V = QT$, so **compliance** satisfies

$$C_a = \frac{QT}{P_{sys} - P_{dia}}$$

Diagnosis with Model 2

Resistance: The model gives the diastolic pressure just before the next heart beat

$$P_{dia} = P_{sys} e^{-\frac{T}{C_a R_s}}$$

Solve this equation for the **resistance**, R_s

$$R_s = \frac{T}{C_a \left(\ln(P_{sys}) - \ln(P_{dia}) \right)}$$

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- Normal Person
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  - Systolic and diastolic pressures of $P_{sys} = 120$ mm Hg and $P_{dia} = 80$ mm Hg

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- Normal Person
  - Pulse of approximately 70 beats/min ($\frac{1}{T}$)
  - Cardiac output of $Q = 5.6$ (liters/min)
  - Systolic and diastolic pressures of $P_{sys} = 120$ mm Hg and $P_{dia} = 80$ mm Hg
- Compute the compliance and resistance for a normal person

$C_a = 0.002$ (liters/mm Hg) and $R_s = 17.6$ (mm Hg/liter/min)

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- Suppose an athlete has
  - A pulse of **60 beats/min (at rest)**
  - A blood pressure of **120/75**
  - A measured cardiac output of **6 liters/min**
- Find the **compliance,** C_a, and systemic **resistance,** R_s, of the arteries for this individual

Example of Athlete 2

Solution: From the formula, **compliance,** C_a

$$C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)}$$

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Solution: From the formula, **compliance,** C_a

$$C_a = \frac{QT}{P_{sys} - P_{dia}} = \frac{6.0/60}{120 - 75} = 0.00222 \text{ (liters/mm Hg)}$$

This is slightly larger than for a normal person

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The **systemic resistance,** R_s, satisfies

$$\begin{aligned} R_s &= \frac{T}{C_a\left(\ln(P_{sys}) - \ln(P_{dia})\right)} \\ &= \frac{1/60}{0.00222(\ln(120) - \ln(75))} = 15.96 \text{ (mm Hg/liter/min)} \end{aligned}$$

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This is lower than for a normal person, which is what we would expect for someone in better condition

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- Most experiments are run so that radioactive decay is not an issue

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- ^{3}H (tritium) is used to tag certain DNA base pairs
  - Add to mutant strains of *E. coli* that are unable to manufacture one particular DNA base
  - Using antibiotics, one uses the radioactive signal to determine how much DNA is replicated under a particular set of experimental conditions
- Radioactive iodine is often used to detect or treat thyroid problems
- Most experiments are run so that radioactive decay is not an issue
  - ^{3}H has a half-life of 12.5 yrs
  - ^{131}I has a half-life of 8 days

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- ^{14}C stays at a constant concentration until the organism dies

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- This differential equation has the solution

$$R(t) = 15.3\,e^{-kt}, \quad \text{where} \quad k = \frac{\ln(2)}{5730} = 0.000121$$

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  - Patient is given medicine to supplement the loss of thyroxine

Hyperthyroidism 2

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- The remainder is excreted in the urine
- The half-life of ^{131}I is 8 days, so this isotope rapidly decays
- Still the patient must remain in a designated room for 3-4 days for this procedure, so that he or she does not irradiate the public from his or her treatment

Hyperthyroidism 3

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- Find the amount of ^{131}**I** in the thyroid (in mCi), if the patient is released four days after swallowing the radioactive cocktail
- Calculate how many mCis the patient's thyroid retains after 30 days, assuming that it was taken up by the thyroid and not excreted in the urine

Hyperthyroidism 4

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- Since the thyroid uptakes **30%** of the **120 mCi**, assume that the thyroid has **36 mCi** immediately after the procedure

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- This allows the simple model

$$\frac{dR}{dt} = -k\,R(t) \qquad \text{with} \qquad R(0) = 36 \text{ mCi}$$

Hyperthyroidism

Solution (cont): The radioactive decay model is

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Hyperthyroidism 5

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- The solution is

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Hyperthyroidism

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- Since the half-life of $^{131}\mathbf{I}$ is 8 days, after 8 days there will are 18 mCi of $^{131}\mathbf{I}$

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Hyperthyroidism

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- Thus, $k = \frac{\ln(2)}{8} = 0.0866 \text{ day}^{-1}$

Hyperthyroidism 6

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$$R(4) = 36\, e^{-4k} = \frac{36}{\sqrt{2}} = 25.46 \text{ mCi}$$

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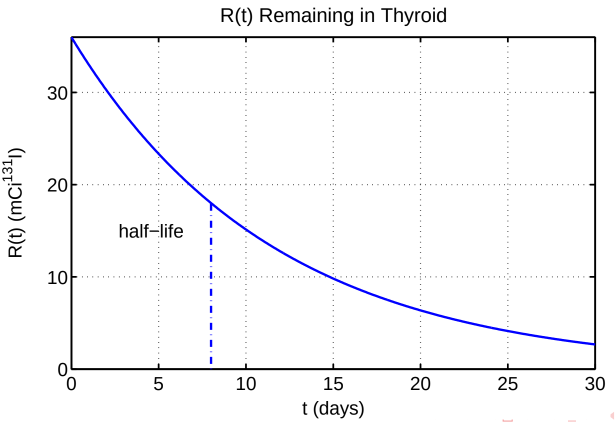
- After 30 days, we find in the thyroid
$$R(30) = 36\,e^{-30k} = 2.68 \text{ mCi}$$

Hyperthyroidism

Graph of $R(t)$



Solution of Linear Growth and Decay Models

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- Two (or more) data points are used to extrapolate back to when the murder occurred
- This property is known as **Newton's Law of Cooling**

Newton's Law of Cooling 2

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- Normal temperature of a human body when it is alive is 37°C
- Use this information to determine the approximate time that the murder occurred

Murder Example 2

Solution: From the model for Newton's Law of Cooling and the information that is given, if we set $t = 0$ to be 8:30 am, then we solve the initial value problem

$$\frac{dT}{dt} = -k(T(t) - 22) \quad \text{with} \quad T(0) = 30$$

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- This is the radioactive decay problem that we solved
- The solution is

$$z(t) = 8\,e^{-kt}$$

Murder Example

Solution (cont): From the solution $z(t) = 8\,e^{-kt}$, we have

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\begin{aligned}
z(t) &= T(t) - 22, \quad \text{so} \quad T(t) = z(t) + 22 \\
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- One hour later the body temperature is 28°C

$$T(1) = 28 = 22 + 8\,e^{-k}$$

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- Thus, $k = \ln\left(\frac{4}{3}\right) = 0.2877$

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Solution (cont): It only remains to find out when the murder occurred

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- Thus,

$$8\,e^{-kt_d} = 37 - 22 = 15 \quad \text{or} \quad e^{-kt_d} = \frac{15}{8} = 1.875$$

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- This gives $-kt_d = \ln(1.875)$ or

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- The murder occurred about 2 hours 11 minutes before the body was found, which places the time of death around **6:19 am**

Cooling Tea 1

Cooling Tea: We would like to determine whether a cup of tea cools more rapidly by adding cold milk right after brewing the tea or if you wait 5 minutes to add the milk

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- a. In the first scenario, you let the tea cool for 5 minutes, then add $\frac{1}{5}$ cup of cold milk, $5°\text{C}$

Cooling Tea

Cooling Tea (cont):

Cooling Tea

Cooling Tea (cont):

- Assume that after 2 minutes the tea has cooled to a temperature of 95°C

Cooling Tea 2

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- b. In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly

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- Assume that after 2 minutes the tea has cooled to a temperature of 95°C
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- Mix in the milk, assuming that the temperature mixes perfectly in proportion to the volume of the two liquids
- b. In the second case, add $\frac{1}{5}$ cup of cold milk, 5°C, immediately and mix it thoroughly
- **Find how long until each cup of tea reaches a temperature of 70°C**

Cooling Tea 3

Solution of Cooling Tea: Find the rate constant k for

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

Cooling Tea

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- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$

Cooling Tea

Solution of Cooling Tea: Find the rate constant k for

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 100 \quad \text{and} \quad T(2) = 95$$

- Let $z(t) = T(t) - 20$, so $z(0) - T(0) - 20 = 80$
- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 80$$

Cooling Tea

Solution of Cooling Tea: Find the rate constant k for

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- Since $z'(t) = T'(t)$, the initial value problem becomes

$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 80$$

- The solution is

$$z(t) = 80\,e^{-kt} = T(t) - 20$$

Cooling Tea

Solution of Cooling Tea: Find the rate constant k for

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$$\frac{dz}{dt} = -k\, z(t), \qquad z(0) = 80$$

- The solution is

$$z(t) = 80\, e^{-kt} = T(t) - 20$$

- Thus,

$$T(t) = 80\, e^{-kt} + 20$$

Cooling Tea

Solution (cont): The solution is

$$T(t) = 80\,e^{-kt} + 20$$

Cooling Tea 4

Solution (cont): The solution is

$$T(t) = 80\, e^{-kt} + 20$$

- Since $T(2) = 95$,

$$95 = 80e^{-2k} + 20 \qquad \text{or} \qquad e^{2k} = \frac{80}{75}$$

Cooling Tea 4

Solution (cont): The solution is

$$T(t) = 80\,e^{-kt} + 20$$

- Since $T(2) = 95$,

$$95 = 80e^{-2k} + 20 \qquad \text{or} \qquad e^{2k} = \frac{80}{75}$$

- $k = \dfrac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227$

Cooling Tea

Solution (cont): The solution is

$$T(t) = 80\,e^{-kt} + 20$$

- Since $T(2) = 95$,

$$95 = 80e^{-2k} + 20 \qquad \text{or} \qquad e^{2k} = \frac{80}{75}$$

  - $k = \dfrac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227$
- Find the temperature at 5 min

$$T(5) = 80e^{-5k} + 20 = 88.1°\text{C}$$

Cooling Tea

Solution (cont): The solution is

$$T(t) = 80\, e^{-kt} + 20$$

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$$95 = 80e^{-2k} + 20 \qquad \text{or} \qquad e^{2k} = \frac{80}{75}$$

  - $k = \frac{\ln\left(\frac{80}{75}\right)}{2} = 0.03227$
- Find the temperature at 5 min

$$T(5) = 80e^{-5k} + 20 = 88.1^\circ\text{C}$$

- Now mix the $\frac{4}{5}$ cup of tea at 88.1°C with the $\frac{1}{5}$ cup of milk at 5°C, so

$$T_+(5) = 88.1\left(\frac{4}{5}\right) + \left(5\frac{1}{5}\right) = 71.5^\circ\text{C}$$

Cooling Tea

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

$$T_+(5) = 71.5^\circ\text{C}$$

Cooling Tea

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

$$T_+(5) = 71.5^\circ\text{C}$$

- The new initial value problem is

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(5) = 71.5^\circ\text{C}$$

Cooling Tea

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

$$T_+(5) = 71.5^\circ\text{C}$$

- The new initial value problem is

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(5) = 71.5^\circ\text{C}$$

- With the same substitution, $z(t) = T(t) - 20$,

$$\frac{dz}{dt} = -kz, \qquad z(5) = 51.5$$

Cooling Tea 5

Solution (cont): For the first scenario, the temperature after adding the milk after 5 min satisfies

$$T_+(5) = 71.5^\circ\text{C}$$

- The new initial value problem is

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(5) = 71.5^\circ\text{C}$$

- With the same substitution, $z(t) = T(t) - 20$,

$$\frac{dz}{dt} = -kz, \qquad z(5) = 51.5$$

- This has the solution

$$z(t) = 51.5e^{-k(t-5)} = T(t) - 20$$

Cooling Tea

Solution (cont): For the first scenario, the temperature satisfies

$$T(t) = 51.5e^{-k(t-5)} + 20$$

Cooling Tea 6

Solution (cont): For the first scenario, the temperature satisfies

$$T(t) = 51.5e^{-k(t-5)} + 20$$

- To find when the tea is 70°C, solve

$$70 = 51.5e^{-k(t-5)} + 20$$

Cooling Tea

Solution (cont): For the first scenario, the temperature satisfies

$$T(t) = 51.5e^{-k(t-5)} + 20$$

- To find when the tea is 70°C, solve

$$70 = 51.5e^{-k(t-5)} + 20$$

- Thus,

$$e^{k(t-5)} = \frac{51.5}{50}$$

Cooling Tea

Solution (cont): For the first scenario, the temperature satisfies

$$T(t) = 51.5e^{-k(t-5)} + 20$$

- To find when the tea is 70°C, solve

$$70 = 51.5e^{-k(t-5)} + 20$$

- Thus,

$$e^{k(t-5)} = \frac{51.5}{50}$$

- It follows that $k(t-5) = \ln(51.5/50)$, so

$$t = 5 + \frac{\ln(51.5/50)}{k} = 5.92 \text{ min}$$

Cooling Tea

Solution (cont): For the second scenario, we mix the tea and milk, so

$$T(0) = 100\left(\tfrac{4}{5}\right) + 5\left(\tfrac{1}{5}\right) = 81^\circ\text{C}$$

Cooling Tea

Solution (cont): For the second scenario, we mix the tea and milk, so

$$T(0) = 100\left(\tfrac{4}{5}\right) + 5\left(\tfrac{1}{5}\right) = 81^\circ\text{C}$$

- The new initial value problem is

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 81^\circ\text{C}$$

Cooling Tea

Solution (cont): For the second scenario, we mix the tea and milk, so

$$T(0) = 100\left(\tfrac{4}{5}\right) + 5\left(\tfrac{1}{5}\right) = 81^\circ\text{C}$$

- The new initial value problem is

$$\frac{dT}{dt} = -k(T(t) - 20), \qquad T(0) = 81^\circ\text{C}$$

- With $z(t) = T(t) - 20$,

$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 61$$

Cooling Tea

Solution (cont): For the second scenario, we mix the tea and milk, so

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- With $z(t) = T(t) - 20$,

$$\frac{dz}{dt} = -k\,z(t), \qquad z(0) = 61$$

- This has the solution

$$z(t) = 61e^{-kt} = T(t) - 20$$

Cooling Tea

Solution (cont): For the second scenario, the solution is

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- To find when the tea is 70°C, solve

$$70 = 61 e{-}kt + 20$$

- Thus,

$$e^{kt} = \frac{61}{50}$$

- Since $kt = \ln\left(\frac{61}{50}\right)$,

$$t = \frac{\ln(61/50}{k} = 6.16 \text{ min}$$

Cooling Tea

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$$T(t) = 61\, e^{-kt} + 20$$

- To find when the tea is 70°C, solve

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- Thus,

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- Since $kt = \ln\left(\frac{61}{50}\right)$,

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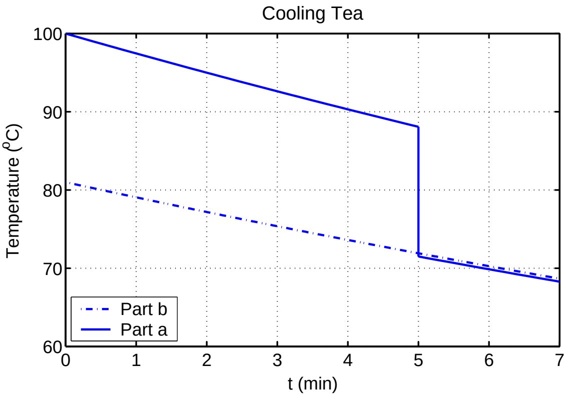
- Waiting to pour in the milk for 5 minutes, saves about 15 seconds in cooling time

Graph of Cooling Tea



Solution of General Linear Model 1

Solution of General Linear Model Consider the Linear Model

$$\frac{dy}{dt} = a\,y + b \qquad \text{with} \qquad y(t_0) = y_0$$

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Rewrite equation as

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Make the substitution $z(t) = y(t) + \frac{b}{a}$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(t_0) = y_0 + \frac{b}{a}$

$$\frac{dz}{dt} = a\,z \qquad \text{with} \qquad z(t_0) = y_0 + \frac{b}{a}$$

Solution of General Linear Model 2

Solution of General Linear Model The shifted model is

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The solution is

$$y(t) = \left(y_0 + \frac{b}{a}\right) e^{a(t-t_0)} - \frac{b}{a}$$

Example of Linear Model 1

Example of Linear Model Consider the Linear Model

$$\frac{dy}{dt} = 5 - 0.2\,y \qquad \text{with} \qquad y(3) = 7$$

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Rewrite equation as

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Example of Linear Model 1

Example of Linear Model Consider the Linear Model

$$\frac{dy}{dt} = 5 - 0.2\,y \qquad \text{with} \qquad y(3) = 7$$

Rewrite equation as

$$\frac{dy}{dt} = -0.2(y - 25)$$

Make the substitution $z(t) = y(t) - 25$, so $\frac{dz}{dt} = \frac{dy}{dt}$ and $z(3) = -18$

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

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Thus,

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Example of Linear Model 2

Example of Linear Model The substituted model is

$$\frac{dz}{dt} = -0.2\,z \qquad \text{with} \qquad z(3) = -18$$

Thus,

$$z(t) = -18\,e^{-0.2(t-3)} = y(t) - 25$$

The solution is

$$y(t) = 25 - 18\,e^{-0.2(t-3)}$$

Pollution in a Lake 1

Pollution in a Lake: Introduction

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- Here we examine a very simplistic model for pollution of a lake

Pollution in a Lake

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- Here we examine a very simplistic model for pollution of a lake
- The model illustrates some basic elements from which more complicated models can be built and analyzed

Pollution in a Lake 2

Pollution in a Lake: Problem set up

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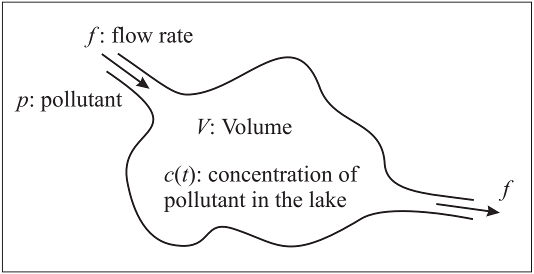
Pollution in a Lake

Pollution in a Lake: Problem set up

- Consider the scenario of a new pesticide that is applied to fields upstream from a clean lake with volume V
- Assume that a river receives a constant amount of this new pesticide into its water, and that it flows into the lake at a constant rate, f
- This assumption implies that the river has a constant concentration of the new pesticide, p
- Assume that the lake is well-mixed and maintains a constant volume by having a river exiting the lake with the same flow rate, f, of the inflowing river

Newton's Law of Cooling

Diagram for Lake Problem Design a model using a linear first order differential equation for the concentration of the pesticide in the lake, $c(t)$



Pollution in a Lake 4

Differential Equation for Pollution in a Lake

Pollution in a Lake

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Pollution in a Lake 4

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- The amount leaving has the same flow rate, f
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
- The product $f\,c(t)$ gives the amount of pollutant leaving the lake per unit time

Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

Pollution in a Lake 5

Differential Equations for Amount and Concentration of Pollutant

- The change in **amount of pollutant** satisfies the model

$$\frac{da(t)}{dt} = f\,p - f\,c(t)$$

Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

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Pollution in a Lake

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Pollution in a Lake

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- Dividing the above differential equation by the volume V,

$$\frac{dc(t)}{dt} = \frac{f}{V}(p - c(t))$$

- If the lake is initially clean, then $c(0) = 0$

Pollution in a Lake

Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

Pollution in a Lake

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- Make the substitution, $z(t) = c(t) - p$, so $z'(t) = c'(t)$

Pollution in a Lake (6)

Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

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- This DE should remind you of Newton's Law of Cooling with f/V acting like k and p acting like T_e
- Make the substitution, $z(t) = c(t) - p$, so $z'(t) = c'(t)$
- The initial condition becomes $z(0) = c(0) - p = -p$

Pollution in a Lake 6

Soluton of the Differential Equation: Rewrite the differential equation for the concentration of pollutant as

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- Make the substitution, $z(t) = c(t) - p$, so $z'(t) = c'(t)$
- The initial condition becomes $z(0) = c(0) - p = -p$
- The initial value problem in $z(t)$ becomes,

$$\frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p$$

Pollution in a Lake

Soluton of the Differential Equation (cont): Since

$$\frac{dz(t)}{dt} = -\frac{f}{V}z(t), \quad \text{with} \quad z(0) = -p$$

Pollution in a Lake

Soluton of the Differential Equation (cont): Since

$$\frac{dz(t)}{dt} = -\frac{f}{V} z(t), \quad \text{with} \quad z(0) = -p$$

- The solution to this problem is

$$z(t) = -p\, e^{-\frac{f\,t}{V}} = c(t) - p$$

Pollution in a Lake

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- This is exactly what you would expect, as the entering river has a concentration of p

Example: Pollution in a Lake 1

Example: Pollution in a Lake Part 1

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- Suppose that you begin with a 10,000 m^3 clean lake

Example: Pollution in a Lake 1

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- Suppose that you begin with a 10,000 m^3 clean lake
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- Form the differential equation describing the concentration of pollutant in the lake at any time t and solve it
- Find out how long it takes for this lake to have a concentration of 2 ppm

Example: Pollution in a Lake 2

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

Example: Pollution in a Lake 2

Solution: This example follows the model derived above, so the differential equation for the concentration of pollutant is

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) \quad \text{with} \quad c(0) = 0$$

- Since $V = 10,000$, $f = 100$, and $p = 5$,

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$$z(t) = -5e^{-0.01t} = c(t) - 5$$

Example: Pollution in a Lake 3

Solution (cont): The concentration of pollutant in the lake is

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$$2 = 5 - 5e^{-0.01t)}$$

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$$e^{-0.01t)} = \frac{3}{5} \quad \text{or} \quad e^{0.01t)} = \frac{5}{3}$$

- Solving this for t, we obtain

$$t = 100\ln\left(\frac{5}{3}\right) = 51.1 \text{ days}$$

Example: Pollution in a Lake

Example: Pollution in a Lake Part 2

Example: Pollution in a Lake 4

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- Suppose that when the concentration reaches 4 ppm, the pesticide is banned
- For simplicity, assume that the concentration of pesticide drops immediately to zero in the river
- Assume that the pesticide is not degraded or lost by any means other than dilution
- Find how long until the concentration reaches 1 ppm

Example: Pollution in a Lake 5

Solution: The new initial value problem becomes

$$\frac{dc}{dt} = -0.01(c(t) - 0) = -0.01c(t) \quad \text{with} \quad c(0) = 4$$

Example: Pollution in a Lake 5

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- To find how long it takes for the concentration to return to 1 ppm, solve the equation

$$1 = 4e^{-0.01t} \quad \text{or} \quad e^{0.01t} = 4$$

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- Solving this for t

$$t = 100\ln(4) = 138.6 \text{ days}$$

Pollution in a Lake: Complications

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- The next section shows numerical methods to handle more complicated models

Example: Lake Pollution with Evaporation 1

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- Suppose that a new industry starts up river from a lake at $t = 0$ days, and this industry starts dumping a toxic pollutant, $P(t)$, into the river at a rate of 7 g/day, which flows directly into the lake

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- The lake is situated in a hot area and loses 50 m^3/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m^3/day through a river

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- The lake is situated in a hot area and loses 50 m^3/day of water to evaporation (pure water with no pollutant), while the remainder of the water exits at a rate of 950 m^3/day through a river
- Assume that all quantities are well-mixed and that there are no time delays for the pollutant reaching the lake from the river

Example: Lake Pollution with Evaporation 2

Example: Lake Pollution with Evaporation (cont) Part a

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- Write a differential equation that describes the concentration, $c(t)$, of the pollutant in the lake, using units of mg/m^3

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- If a concentration of only 2 mg/m^3 is toxic to the fish population, then find how long until this level is reached

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- Write a differential equation that describes the concentration, $c(t)$, of the pollutant in the lake, using units of mg/m^3
- Solve the differential equation
- If a concentration of only 2 mg/m^3 is toxic to the fish population, then find how long until this level is reached
- If unchecked by regulations, then find what the eventual concentration of the pollutant is in the lake, assuming constant output by the new industry

Example: Lake Pollution with Evaporation 3

Solution: Let $P(t)$ be the amount of pollutant

The change in amount of pollutant =

Amount entering - Amount leaving

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- The **amount entering** is the constant rate of pollutant dumped into the river, which is given by $k = 7000$ mg/day
- The **amount leaving** is given by the concentration of the pollutant in the lake, $c(t)$ (in mg/m^3), times the flow of water out of the lake, $f = 950$ m^3/day

Example: Lake Pollution with Evaporation 4

Solution (cont): The conservation of amount of pollutant is given by the equation:

$$\frac{dP}{dt} = k - f\,c(t) = 7000 - 950\,c(t)$$

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- Evaporation concentrates the pollutant by allowing water to leave without the pollutant
- Divide the equation above by the volume, $V = 400,000 \text{ m}^3$

$$\left(\frac{1}{V}\right)\frac{dP(t)}{dt} = \frac{k}{V} - \frac{f}{V}c(t) = \frac{7}{400} - \frac{950}{400000}c(t)$$

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- The concentration equation is

$$\frac{dc}{dt} = \frac{7}{400} - \frac{950}{400000}c(t) = -\frac{f}{V}\left(c(t) - \frac{k}{f}\right)$$

Example: Lake Pollution with Evaporation 5

Solution (cont): The concentration equation is

$$\frac{dc}{dt} = -\frac{95}{40000}\left(c(t) - \frac{700}{95}\right)$$

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$$\frac{dz}{dt} = -\frac{95}{40000}z(t) \quad \text{with} \quad z(0) = -\frac{700}{95}$$

- The solution is

$$z(t) = -\frac{700}{95}e^{-95t/40000} = c(t) - \frac{700}{95}$$

Example: Lake Pollution with Evaporation 6

Solution (cont): The concentration equation is

$$c(t) = \frac{700}{95}\left(1 - e^{-95t/40000}\right) \approx 7.368\left(1 - e^{-0.002375t}\right)$$

Example: Lake Pollution with Evaporation 6

Solution (cont): The concentration equation is

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- If a concentration of 2 mg/m^3 is toxic to the fish population, then find when $c(t) = 2$ mg/m^3

Example: Lake Pollution with Evaporation

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- If a concentration of 2 mg/m^3 is toxic to the fish population, then find when $c(t) = 2$ mg/m^3
- Solve

$$2 = 7.368\left(1 - e^{-0.002375t}\right) \quad \text{or} \quad e^{0.002375t} \approx 1.3726$$

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- Thus, $t = \frac{\ln(1.3726)}{0.002375} \approx 133.3$ days

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- Solve

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- Thus, $t = \frac{\ln(1.3726)}{0.002375} \approx 133.3$ days
- The limiting concentration is

$$\lim_{t\to\infty} c(t) = \frac{700}{95} \approx 7.368$$

Example: Lake Pollution with Evaporation 7

Example: Lake Pollution with Evaporation (cont) Part b

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Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days

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Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days
- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation

Example: Lake Pollution with Evaporation

Example: Lake Pollution with Evaporation (cont) Part b

- Suppose that the lake is at the limiting level of pollutant and a new environmental law is passed that shuts down the industry at a new time $t = 0$ days
- Write a new differential equation describing the situation following the shutdown of the industry and solve this equation
- Calculate how long it takes for the lake to return to a level that allows fish to survive

Example: Lake Pollution with Evaporation 8

Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000}c(t) = -0.002375\, c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

Example: Lake Pollution with Evaporation 8

Solution: Now $k = 0$, so the initial value problem becomes

$$\frac{dc}{dt} = -\frac{95}{40000}c(t) = -0.002375\, c(t) \quad \text{with} \quad c(0) = \frac{700}{95}$$

- This has the solution

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- This has the solution

$$c(t) = \frac{700}{95}\,e^{-0.002375t} \approx 7.368\,e^{-0.002375t}$$

- The concentration is reduced to 2 mg/m^3 when

$$2 = 7.368\,e^{-0.002375t} \quad \text{or} \quad e^{0.002375t} = 3.684$$

Example: Lake Pollution with Evaporation

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- The concentration is reduced to 2 mg/m^3 when

$$2 = 7.368\,e^{-0.002375t} \quad \text{or} \quad e^{0.002375t} = 3.684$$

- The lake is sufficiently clean for fish when

$$t = \frac{\ln(3.684)}{0.002375} \approx 549 \text{ days}$$