The Capture of Infinity

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Presentation to the Student Math Club

University of Pittsburgh

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infinity_2023_pitt.pdf

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Room 704 Thackeray

Introduction

School kids in a big number battle will finally cry out "infinity!" and figure they have won ∴ only to have their opponent respond "infinity plus 1"!

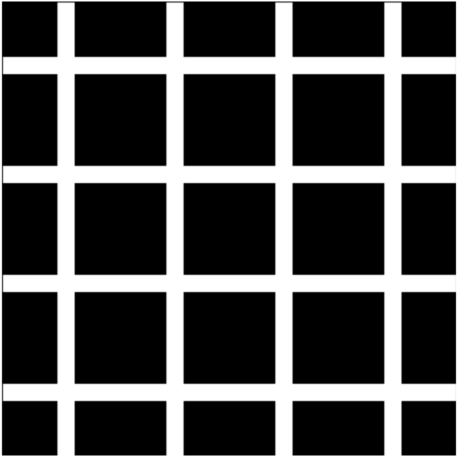
Is infinity really a number like the ones we are comfortable with?

Or is infinity simply a vague concept of endlessness?

Can we add 1 to it? And if we can, is that bigger, or a new kind of infinity, or just the same old infinity as before?

And if infinity plus 1 equals infinity, does that imply by subtraction, that 1 equals 0?

I know what infinity is until I focus on it!



You can see the gray dots at the corners ...
as long as you don't look at them!
Does infinity fall apart when we try to look at it seriously?

In search of a definition

Kids may think they know something about infinity, but philosophers, mathematicians, and scholars have been struggling to understand it or outlaw it for thousands of years.

"If any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all."

Bertrand Russell, mathematician and philosopher.

This is the story of how Georg Cantor, working alone, discovered the definition of infinity that mathematicians use today.

Can things be infinitely divisible?

Zeno considered the process of cutting a piece of cheese into more and more tiny pieces. He assumed the pieces would never get too small to cut further, so presumably, we could create an infinite number of tiny cheese pieces. Then how big is each piece?

If each piece has a nonzero size, the original cheese must be infinitely large, which cannot be true.

Therefore the pieces must have zero size, hence the cheese must have zero size, which cannot be true.

To be extremely logical, Zeno concluded that it must be the case that there is only one thing in the universe, and it cannot be divided at all!

Achilles cannot catch a tortoise!

Suppose Achilles races a tortoise which is given a small head start. By the time Achilles reaches the tortoise's initial position, the tortoise has moved some nonzero distance ahead.

So Achilles must now run to that position, but this gives the tortoise time to move yet further ahead, and so on.

Achilles faces an infinite number of mini-races he can never finish.

Zeno concluded that all motion must be an illusion.

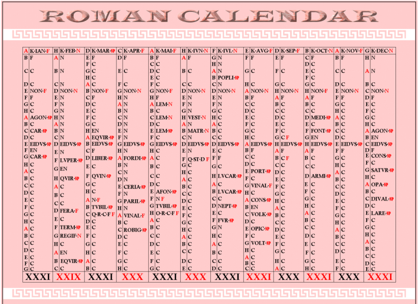
The universe can't be infinitely old

The philosopher John Philiponous of Alexandria declared that, *as a matter of logic*, the age of the universe must be a finite number.

For suppose **Y**, the age of the universe in years, is infinite. But if we measure the age of the universe in months, we get 12***Y**, which is obviously much bigger than **Y**

But *nothing* is bigger than infinity.

Therefore **Y** must be finite and the universe is not very old.



A block of wood can't be infinitely long

The philosopher Albert of Saxony imagined a wooden beam that was infinitely long at one end, with a 1 foot square cross section.

If we saw the beam into 1-foot cubes, we can line them up left and right and make a new beam that goes to infinity in *both* directions.

If instead we lay them on the floor in a spiral, we have a wooden plane that extends to infinity.

Worse yet, if we start start with one cube, then pack 8 more around that, then 18 more around that, and so on, we gradually build a cube that fills the universe!

Albert concluded that no object can be infinitely long.

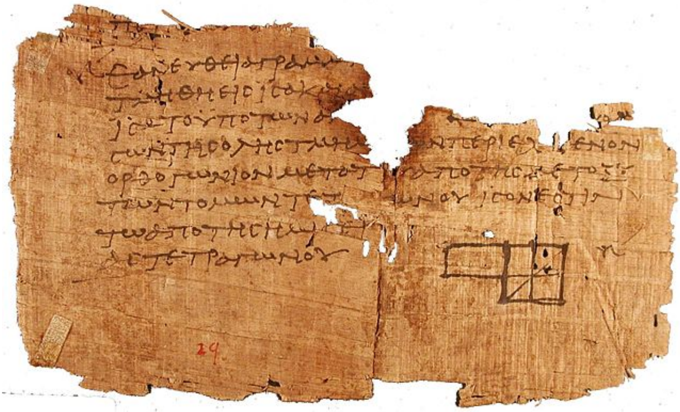
Aristotle banished infinity

Aristotle allowed **potentially infinite** processes, which had no fixed limit, but were guaranteed to end sometime; actually infinite processes were meaningless to him.

> *"But my argument does not anyhow rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through; for, as it is, they do not need the infinite or use it, but only require that the finite straight line shall be as long as they please. Hence it will make no difference to them for the purpose of proofs."*
>
> Aristotle.

Euclid's Prime Proof

This is the oldest surviving fragment of Euclid's geometry book, the most famous mathematics book in history. This copy is almost two thousand years old.



Don't Say "Infinite"

Euclid's prime number theorem says

> *"Prime numbers are more than any assigned multitude of prime numbers."*
> Euclid

Because Aristotle's objected to actual infinity, Euclid did **not** say:

> The list of all prime numbers is infinite.

That's how we think, but we're not allowed to say "infinite"!

Prove that the Opposite Can't Be True

Euclid's proof is worth describing; it's simple, it's indirect, and it's a proof by contradiction. And we will need this same kind of proof when we are ready to talk about infinity.

Euler's task looks difficult. It is a statement that **every** possible finite list of primes is incomplete. I know how to prove that for one particular list, but how do I do it for all possible lists?

Euclid considered the possible truth of the opposite statement:

It is possible to make a complete (and finite) list of prime numbers.

Suppose a complete finite list of primes

OK, said Euclid, suppose there is such a list: $P_1, P_2, ..., P_N$.

Then I can define a number Q which gives us big problems.

Q is the product of all the primes on the list, plus 1:

$$Q = (P_1 * P_2 * ... * P_N) + 1$$

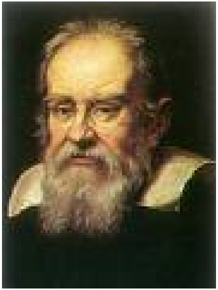
Dividing Q by any prime gives us a remainder of 1. So our list is incomplete! Either Q is prime, or it has a prime factor we didn't list. If we add the missing prime to our list, the same argument can be used again. No finite list of primes can ever be complete.

Greek mathematics was not comfortable with infinite objects; Euclid was careful to drive on the finite side of the road.

Nonetheless, infinity is...

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow\ldots \\ 1 & 4 & 9 & 16 & 25 & 36\ldots \end{array}$$

Galileo matched a list of numbers to a list of their squares. Both lists can be continued indefinitely, but one is a subset of the other! Many people (not Galileo!) argued that this was nonsense, since the whole must be greater than any of its parts.



Our mind can apprehend things beyond physical experience

> *"So far as I see, we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally, the attributes 'equal', 'greater' and 'less' are not applicable to infinite, but only to finite, quanitities."*
>
> Galileo

Galileo says our definitions are based on our experience of the finite world.

But our minds can directly apprehend infinity by contemplating numbers.

Georg Cantor [1845-1918]

Cantor encountered infinity while looking at approximation using Fourier series and counting the zeros in the initial terms.

What Are We Counting?

Since infinity involved counting abstract things, Cantor started by describing the things he would count as a set.

Intuitive Notion of a Set

"A set is any group of objects that can be thought of as a whole."
Georg Cantor

This is like starting geometry with a definition of **points**.

Small sets could be described as a list:

$$A = \{1, 2, 4, 97\}$$

or by a property of its elements:

$$B = \{d \mid d \text{ is a divisor of } 36\} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

but it takes a little more courage to say:

$$\mathbb{N} = \{n \mid n \text{ is a natural number}\} = \{1, 2, 3, \ldots\}$$

Equal versus Equal in Size

Two sets were equal if, and only if, they contained the same things.

Two sets might not be equal, but might be equal in size, having the same number of elements.

Cantor tried to express this without numbers. He decided two sets would be equal in size if it was possible to match elements of the two sets in pairs, with nothing left over.

Cantor wrote $\#A$ to symbolize the size of the set A.

He wrote $\#A \leq \#B$ if it was possible to match every element of A to a distinct element of B.

Therefore, he could write that A and B were equal in size, that is, $\#A = \#B$ if, and only if, both $\#A \leq \#B$ and $\#B \leq \#A$.

Ready for Infinity?

We said a set could be defined by listing its entries, so it's easy to want to "define" the natural numbers by:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

But what does ... really mean? Mathematically, the following statement is preferable:

Axiom of Infinity

There is a set S which contains the empty set, and such that, if e is an element of S, then {e,{e}} is also an element of S.

Each natural number n can be represented by the set of all natural numbers that are less than it. We represent 0 by $\{\}$. We represent 1 by $\{0\} = \{\{\}\}$. We represent 2 by {0,1} and so on. Now we recognize that the set S can be regarded as a form of $\mathbb{N}$.

A Smaller Infinity Than the Smallest Infinity?

We've defined the natural numbers $\mathbb{N}$ as the "smallest" infinite set. If you stop the sequence 0, 1, 2, 3, ..., early, you get a finite set.

But, as Galileo pointed out, consider the set of even numbers $\mathbb{E}$:

- $\mathbb{E}$ is a proper subset of $\mathbb{N}$;
- $\mathbb{E}$ is infinite in size.

So, although $\mathbb{N}$ is the smallest infinite set, we apparently have a set $\mathbb{E}$ which is also infinite (undeniable!), and smaller than $\mathbb{N}$ (are we sure about this?).

Remember how we defined the size of a set! Two sets are equal in size if their elements can be matched:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow \ldots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12\ldots \end{array}$$

A Definition of an Infinite Set

Cantor realized that removing some elements from an infinite set might not make it smaller; this property defined an infinite set!

> *"An infinite set is precisely any set that can be placed in 1-to-1 correspondence with a proper subset of itself."*
> Georg Cantor

The definition guarantees that an infinite set will remain infinite if:

- you add one item;
- you add an infinite number of items;
- you subtract one, or finitely many items;
- you **carefully** subtract an infinite number of items.

$1 + \aleph_0 = \aleph_0$

Cantor used the Hebrew letter $\aleph_0$ ("aleph-null") to count the natural numbers.

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$
$$\#\mathbb{N} = \aleph_0.$$

Some count this way: 1, 2, 3, ... and others count 0, 1, 2, 3, ...

Call these sets $\mathbb{N}^+$ and $\mathbb{N}_0$. Is it true that $\mathbb{N}^+ \sim \mathbb{N}_0$?
Yes, because here is a matching:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6... \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow .. \\ 0 & 1 & 2 & 3 & 4 & 5... \end{array}$$

This is puzzling result #1 in transfinite arithmetic:

$$1 + \aleph_0 = \aleph_0$$

So does $1 = 0$?

So why doesn't the equation

$$1 + \aleph_0 = \aleph_0$$

imply

$$1 = 0?$$

In Cantor's arithmetic, you can't directly add or subtract cardinals.

You first find sets A and B so that $A \sim B$
then you can write $\#A = \#B$!

To "prove" that $1 = 0$, we'd have to match a set of 0 elements and a set of 1 element.

$\aleph_0 = \aleph_0 + 1 + \aleph_0$

The set of integers is $\mathbb{Z} =$ ..., -3, -2, -1, 0, 1, 2, 3, ...
Since $\mathbb{N} \subset \mathbb{Z}$, we know $\#\mathbb{N} \leq \#\mathbb{Z}$. But is a matching possible?

$$\begin{array}{ccccccc} \ldots -3 & -2 & -1 & 0 & 1 & 2 & 3\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow \ldots \\ \ldots 7 & 5 & 3 & 1 & 2 & 4 & 6\ldots \end{array}$$

Therefore, $\mathbb{N} \sim \mathbb{Z}$ and so $\#\mathbb{N} = \#\mathbb{Z}$ or:

$$\aleph_0 = \aleph_0 + 1 + \aleph_0 = \aleph_0 + (1 + \aleph_0) = \aleph_0 + \aleph_0 = 2 \times \aleph_0$$

So puzzling result $\#2$ in transfinite arithmetic is:

$$2 \times \aleph_0 = \aleph_0$$

Cantor Counts the Fractions

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	~~$\frac{2}{2}$~~	$\frac{2}{3}$	~~$\frac{2}{4}$~~	$\frac{2}{5}$	~~$\frac{2}{6}$~~	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	~~$\frac{3}{3}$~~	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	~~$\frac{4}{2}$~~	$\frac{4}{3}$	~~$\frac{4}{4}$~~	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	~~$\frac{6}{2}$~~	~~$\frac{6}{3}$~~	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

$\aleph_0 = \aleph_0 * \aleph_0$

So now we know that $\mathbb{Q}^+ \sim \mathbb{N}^+$, and therefore $\#\mathbb{Q}^+ = \#\mathbb{N}^+$

Now the elements of the set $\mathbb{Q}^+$ are pairs of elements of $\mathbb{N}^+$

In this case, we say that $\mathbb{Q}^+$ is the *product* of $\mathbb{N}^+$ with itself; this is symbolized by $\mathbb{Q}^+ = \mathbb{N}^+ \otimes \mathbb{N}^+$

The number of elements in a product is simply the product of the number of elements in each factor, so $\#\mathbb{Q}^+ = \#\mathbb{N}^+ \times \#\mathbb{N}^+$

So puzzling result #3 in transfinite arithmetic is:

$$\aleph_0 \times \aleph_0 = \aleph_0$$

It's beginning to seem like all infinities are the same size!

Are These Denumerable?

A set is *denumerable* if it can be matched with $\mathbb{N}$.

- all possible (finite) English sentences.
- all possible (finite) books in English.
- all possible (finite) musical pieces.
- all possible images on a computer screen.
- all possible (pixellated and finite) movies.
- the number of times you can write an “X” of any size on one sheet of paper without crossing any others (don’t ask about ”O”!)

Are the points on a line denumerable?

It would be enough to count just the real numbers between 0 and 1, symbolized by $\mathbb{R}[0,1]$. If they could be counted, so could $\mathbb{R}$.

So a solution to the problem could be thought of as **a numbered list** of all the elements in $\mathbb{R}[0,1]$, our 1 inch line.

How to count real numbers?

Turn the counting numbers around, with a decimal point in front.

$\mathbb{N}$	$\mathbb{R}[0,1]$
0	.0
1	.1
2	.2
...	...
128	.821
129	.921
130	.031
...	...
12345	.54321
...	...

Actually, this list only includes (some) rational numbers with decimal denomintors, but not $1/3$, not $\frac{pi}{4}$.

Suppose We Had A List

So (like Euclid's primes) Cantor wondered if there could be a list of every real number in [0,1]:

$\mathbb{N}$	$\mathbb{R}[0,1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

Scan the List Diagonally

Cantor considered the number formed by the first digit of the first number, the second digit of the second number, and so on:

$\mathbb{N}$	$\mathbb{R}[0, 1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

The diagonal number is .1937683211....

Here's a Missing Number!

Cantor then increased each digit of .1937683211... by 1, (but 9's become 0's), to get the modified number .2048794322....

He pointed out this number cannot be on the list, because:

- its first digit differs from that of the number in position 1,
- its second digit differs from that of the number in position 2,
- its **n**-th digit differs from that of the number in position **n**,
- and so on.

So a numbered list of elements in [0,1] can never be complete! There is no 1-to-1 matching between $\mathbb{N}$ and $\mathbb{R}$.

> *"I see it, but I can't believe it!"*
> Georg Cantor

More Infinities than $\aleph_0$ and c?

With the discovery that $\aleph_0 < c$, Cantor realized that the statement "all infinities are the same size" was false. There is more than one size of infinite set.

$\aleph_0$ was certainly the first step into infinity (can you prove there is no infinite set strictly smaller than $\mathbb{N}$?), and c was a further step.

Was c the very "next" step after $\aleph_0$, (so that we could rename c to $\aleph_1$) or were there intermediate steps?

And were there infinite sets that were strictly larger? If so, was there a way to create them?

$c \times c = c$

Since c counts the points in a line, perhaps a bigger infinity is the number of points in a plane, symbolized by c $\times$ c.

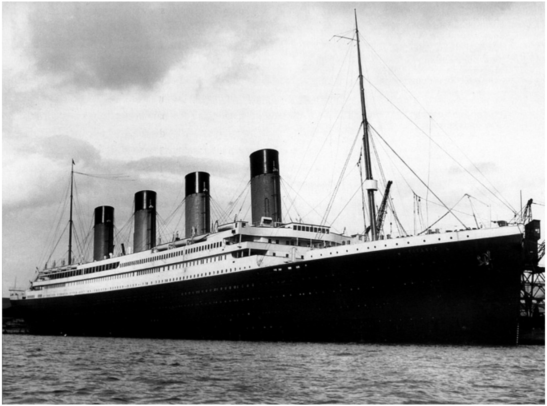
Given a point in the unit square such as $(x, y) = (.012345, .56789)$, Cantor found its matching point in the unit line by alternating the digits of x and y to get .0516273849.

So c counts the points in a line, in a plane, and in all space.

Since c counts the points in a 1 inch line segment as well, all these objects have the same number of points. The points on a 1 inch line segment (which has no area, and no volume) can be rearranged to fill an entire line, an entire plane, or all of space!

It's stranger than saying you could build the Titanic from an inch of wire!

Titanic = Nail?



Are there even bigger sets?

Cantor had achieved his goal of finding a set that was bigger than the counting numbers $\mathbb{N}$. Naturally, he wondered whether there were even larger sets.

Since the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$ were no bigger than $\mathbb{R}$, Cantor tried to think of other mathematical operations or objects that might reach a higher level.

Cantor's breakthrough came when he tried to explain why c had to be bigger than $\aleph_0$.

A Real Number is a Kind of Subset of $\mathbb{N}$

Every real number in [0,1] can be written as a binary decimal.

$$\frac{\pi}{4} = 0.785398... = 0.110010010000111111...$$

If we record the locations where a 1 occurs, we get a subset of $\mathbb{N}$.

$$\frac{\pi}{4} \to \{1, 2, 5, 8, 13, 14, 15, 16, 17, 18, ...\}$$

So the number of reals is equal to the number of subsets of $\mathbb{N}$.

The Power Set is Strictly Bigger than the Set

Given any set A, the set of all possible subsets of A is known as its power set, written as $\mathcal{P}(\mathcal{A})$. Since a subset can be represented as a string of 0's and 1's, the power set of A is sometimes represented as 2^A.

Cantor showed that the power set of A is always strictly larger than A. It is **never** possible to find a 1-1 matching between A and $\mathcal{P}(\mathcal{A})$.

$$\#A < \#2^A$$

Therefore, **c**, the number of real numbers in the interval [0,1] was an infinite number strictly larger than $\aleph_0$.

To see how tricky Cantor had to think, here's how he argued that the power set of A can't be the same size as A.

The Power Set is Strictly Bigger than the Set

Let us suppose $A \sim \mathcal{P}(\mathcal{A})$. Then there is a 1-1 mapping $f(A) \to \mathcal{P}(\mathcal{A})$ taking each $a \in A$ to some subset $f(a) \in \mathcal{P}(\mathcal{A})$.

Now some elements a might actually also be elements of the subset $f(a)$, and some might not be. Let B be the set of all elements of A which are **not** elements of their image under $f()$.

There must be an element $b \in A$ such that $f(b) = B$.

Simple question: Is $b \in B$?

Suppose $b \in B$. Then $b \in f(b)$, so it cannot be in B.

Suppose $b \notin B$. Then $b \notin f(b)$, so it must be in B.

The idea of a perfect mapping from A to $\mathcal{P}(\mathcal{A})$ is self-contradictory. So there cannot be such a mapping.

Infinities Beyond Infinity

Cantor spent many years trying to prove that **c** had to be the "very next" infinite number after $\aleph_0$, but he could not do so.

Nonetheless, he had been able to show the existence of two kinds of infinite number, one larger than the other.

Moreover, the power set of **c** must be an infinite set strictly bigger than **c**, and this process can be continued to produce an endless sequence of bigger and bigger infinite sets.

Cantor had shown that it is not true that "all infinities are the same size". Instead, he had shown that the two infinities we knew about were simply starting steps on an endlessly rising stairway.

And it's even more complicated. Look up *inaccessible cardinals*!

Cantor's Vision

After Cantor

Cantor also worked on the related topic of ordered infinite numbers, or ordinals. There were many new surprises in that area as well.

Cantor was never able to prove his "Continuum Hypothesis", that c is the "next" infinity after $\aleph_0$ and it was left to Kurt Gödel and Paul Cohen to show that this statement was not provable from the basic laws of set theory.

Cantor's results about infinity, and his work in set theory, were controversial for years; they have since become standard, respected results.

His diagonal argument, used to showed the real numbers cannot be counted, was reused by Alan Turing in a proof about computability, and by Kurt Gödel in his famous proof that mathematical systems are incomplete.

Conclusion:

Cantor was not interested in debates about the possibility of infinite objects and activities in the **physical** world.

Cantor showed infinity was logically consistent in **mathematics**.

Cantor cleared up the paradoxes by realizing that "laws" that infinity violated were deduced by observation of finite numbers.

By inventing set theory, he was able to define a consistent model of arithmetic that included infinite values.

The key to his theory defined infinite objects as exactly those that are "equal" to some proper subset of themselves.

> *"No one will expel us from this paradise Cantor has created for us."*
> *David Hilbert*

Even the BBC gets it wrong: 19 March 2023

"However, the following century, the logician Georg Cantor took the concept of infinity and made it even more mind-bending. Some infinities, he showed, are bigger than others".

"How so? One of the simplest ways to understand why is to imagine the set of all the even numbers. This would be infinite, right? But it must be smaller than the set of all whole numbers, because it does not contain the odd numbers. Cantor proved that when you compare such sets, they contain numbers that do not match up, therefore there must be multiple sizes of infinity".

"Richard Fisher is a senior journalist for BBC Future".

"The author used ChatGPT to research trusted sources and calculate parts of this story."