The Capture of Infinity

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Introduction

For thousands of years, philosophers have considered the idea of infinity, and after thinking about it, decided there was something so wrong with it that it couldn't be regarded as a proper concept.

This is a real shame, since we all have an intuitive feeling for what infinity is...or ought to be. However, our intuitive ideas are a little disorganized, and allow infinity so seem like a contradictory subject.

It was only in the 1870's that Georg Cantor, working entirely on his own, captured the idea of infinity in a way that was mathematically and logically correct, and corresponded to our intuitive notions.

I will start by trying to convince you that infinity can't possible exist, and then, I hope, show you how Cantor was able to sort things out!

COUNTING: The Creation of Numbers

IN THE BEGINNING, there was a pair of curly brackets $\{\ \ldots\}$.

These brackets were to enclose any list of things to count...
except that theren't any things.

Thus, the list of things was $\{\ \}$, that is, a total of 0 things.
And thus the number 0 was created.

Thus, the list of things that existed was $\{\ 0\ \}$, a total of 1 thing,
and the number 1 was created.

The list of things was now $\{0,\ 1\}$, a total of 2 things,
creating the number 2, and so on and on...

This is where all the numbers came from.

COUNTING: A Number Counts a Set

When we count a set of objects, we are actually comparing them to a set of numbers which are labeled in such a way that we know where we stopped.

I will continue to pretend that we start counting at 0. So if we see seven sheep, I will count by creating the list { 0, 1, 2, 3, 4, 5, 6 }. It makes sense that this list or set should have a name, and I propose that its name be **7**.

In a way, then, we could also write

$$\{0, 1, 2, 3, 4, 5, 6\} = 7$$

In this way, numbers help us to create sets of every needed size, and a name for each set. The name tells us how big the set is, and is also the next number, the one we were just about to get to, the one that is just bigger than every number in the set.

I seem to be describing something obvious, but in a roundabout way. However, my description might help when we realize that we can make a new list, that is, the list of all the numbers we've made so far:

$$\{0, 1, 2, 3, 4, ...\}$$

And now I ask a simple question...what should we call this list? We should give it a name that "counts" the objects, and this name should also be the "next" number.

In other words,

$$\{0, 1, 2, 3, 4, ...\} = \infty$$

This is how natural it is to invent infinity.

COUNTING: Ideas About Infinity

As soon as we think of infinity, we begin to puzzle ourselves!

1. You can't get something from nothing;
2. You can never actually do an infinite number of things.
3. There are an infinite number of moments in a second;
4. The whole must be greater than any of its parts.
5. Adding something to a quantity makes it strictly larger.
6. Infinity is "how many" counting numbers there are;
7. There is no end to the counting numbers;
8. Infinity is like "the last counting number";
9. Any two infinite things are the same size.
10. An infinite number of zeros is still zero;
11. A line one inch long is made up of an infinite number of points of zero length;
12. There are more points inside a square than on a line segment.

These ideas cannot all be true!

"If any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all."

Bertrand Russell, mathematician and philosopher.

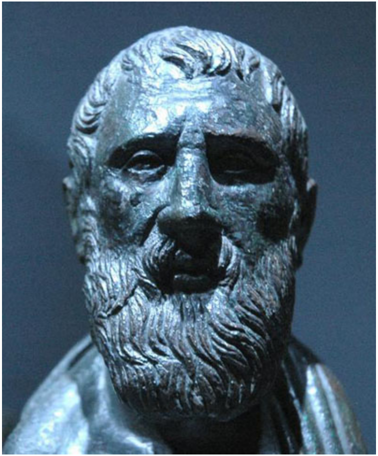
It seems that, before we will be allowed to talk about infinity, we will need to come up with a good definition.

In mathematics, a good definition identifies a crucial property of an object, from which all its other properties will follow.

PARADOXES: Zeno

Paradox: something that contradicts your belief or knowledge.

The oldest paradoxes about infinity are associated with a Greek philosopher named Zeno of Elea.



PARADOXES: Zeno Cheese

Zeno pointed out that we believe that we can slice a piece of cheese into two pieces which we can put back together to get the original piece. But then we can also slice into 4, or 8, or as many pieces as we like. There seems no reason that the cheese could not be cut into an infinite number of equal pieces.

Then how big is each piece?

If the pieces have a nonzero size, then since there are infinitely many of them, the cheese itself must have an infinite size!

Therefore the pieces must have zero size. But doesn't that mean the cheese itself has zero size?

Trying to be extremely logical, Zeno concluded that there is only one thing in the universe, and it cannot be divided at all!

PARADOXES: Zeno's Footrace

Zeno also considered the paradox of *Achilles and the Tortoise*.

Suppose Achilles races a tortoise which is given a small head start. Then Achilles cannot win! By the time Achilles reaches the tortoise's initial position, the tortoise has moved ahead. When he reaches *that* position, again the tortoise has moved yet further ahead. Each time Achilles thinks he has caught up, the tortoise has moved further.

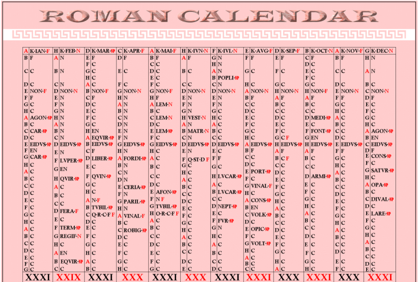
Being extremely logical, Zeno concluded not only that Achilles cannot win, but that all motion must be an illusion.

PARADOXES: The Universe Can't Be Very Old

While arguing about the age of the universe, the philosopher John Philiponous of Alexandria declared that, *as a matter of logic*, the age of the universe must be a finite number.

For let **Y** represent the age of the universe in years. Suppose that **Y** is infinite. Now we could also measure the age of the universe in months. This would be the number 12***Y**. But 12***Y** is obviously much bigger than **Y**, and yet *nothing* is bigger than infinity.

Therefore **Y** must be finite and the universe is not very old.



The philosopher Albert of Saxony considered a wooden beam that was infinitely long. We suppose only one end is infinite, with a 1 foot square cross section.

It should be obvious that if we start sawing the beam into 1-foot cubes, we can line them up left and right and make a new beam that goes to infinity in both directions.

If instead we lay them down in a spiral, we suddenly have a wooden plane that extends to infinity.

Worse yet, if we start start with one cube, then pack 8 more around that, then 18 more around that, we gradually build a cube that fills up the whole world!

Such a beam takes up some space, or all space, depending on how it was arranged. This seemed illogical.

PARADOXES: Aristotle Says Infinity Is Imaginary

The philosopher Aristotle could not find any reasonable way to speak of infinity. But he recognized that it was necessary to have some concept that allowed things like counting to go on.

But since every counting operation eventually stopped, he decided to make a distinction. An operation like counting was **potentially infinite** because, although it would stop at some finite time, we could never place a limit on the operation beforehand. Philosophy and mathematics could accept potentially infinite operations and objects.

But the **actual infinite** was a contradictory and nonsensical idea. There are no infinite objects, and infinite operations are not possible. Mathematicians should never use the word infinity. In this way, all the paradoxes could be avoided.

PARADOXES: Aristotle Says Infinity is Imaginary

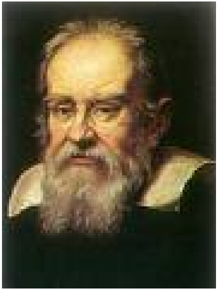
"But my argument does not anyhow rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through; for, as it is, they do not need the infinite or use it, but only require that the finite straight line shall be as long as they please. Hence it will make no difference to them for the purpose of proofs."

Aristotle.

PARADOXES: Galileo: Numbers are Real, so is Infinity

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow\ .. \\ 1 & 4 & 9 & 16 & 25 & 36\ldots \end{array}$$

Galileo matched a list of numbers to a list of their squares. Both lists can be continued indefinitely, but one is a subset of the other! Many people (not Galileo!) argued that this was nonsense, since the whole must be greater than any of its parts.



PARADOXES: Galileo: Numbers are Real, so is Infinity

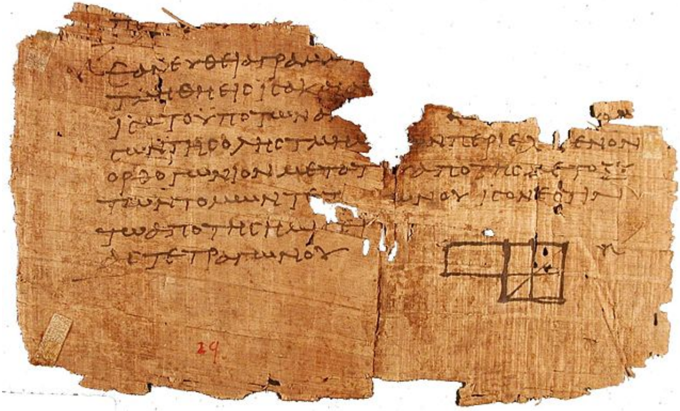
> *"So far as I see, we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally, the attributes 'equal', 'greater' and 'less' are not applicable to infinite, but only to finite, quanitities."*
>
> Galileo

Galileo says it's obvious that infinite things exist, and if they don't behave the way we expect, then maybe that's because of our limited (finite) experience.

We will return soon to this very modern outlook!

EUCLID: Prime Proof

This is the oldest surviving fragment of Euclid's geometry book, the most famous mathematics book in history. This copy was made in 100 AD.



EUCLID: Prime Proof

The ancient Greeks knew about prime numbers, that is, numbers like 2, 3, 5, 7, 11, 13, 17 which can't be divided by any other number except 1. Euclid proved a statement about them:

> *"Prime numbers are more than any assigned multitude of prime numbers."*
> Euclid

Because of Aristotle's objections, Euclid did **not** say:

> The list of all prime numbers is infinite.

Why complain about Aristotle's rule? Well, even though the two statements are equivalent, isn't it much more natural to think in terms of the second one? Even if we are forbidden to say infinity, it seems like we still think that way!

EUCLID: Turn the Statement Around

Euclid's proof is worth describing; it's simple, it's indirect, and it's a proof by contradiction. And we will need this same kind of proof when we are ready to talk about infinity.

Euler's task looks very difficult. It is a statement that says that **every** possible finite list of primes is incomplete. I know how to prove that for one particular list, but how do I do it for all possible lists?

Euclid took the roundabout approach, by considering the possible truth of the opposite of his statement:

It is possible to make a complete (and finite) list of prime numbers.

EUCLID: Consider A Finite List of Primes

OK, said Euclid, suppose there is such a list: $\mathbf{P_1}, \mathbf{P_2}, ..., \mathbf{P_N}$.

We agree every number is either a prime or is divisible by a prime (and hence is called a *"composite number"*).

I will now show that if you can make a finite list of all primes, I can define a number **Q** which is going to give us big problems.

Q is the product of all the primes on the list, plus 1:

$$Q = P_1 * P_2 * ... * P_N + 1$$

We've assumed that all primes are listed, but we see **Q** is not divisible by any prime. So every finite list of primes is incomplete.

EUCLID: Doing Math Without Infinity

Notice, again, that Euclid has just proved that there are an infinite number of primes, without ever using that word or that idea.

Moreover, he started out having to prove a statement about every possible finite list of primes, that is, he had to prove an infinite number of things.

His proof avoided this infinite task; instead, he supposed he was wrong, and that there was a single list of all primes, and he showed this could not happen.

Greek mathematics did not understand infinite objects;
Euclid was careful to drive on the finite side of the road.

DEFINE: Georg Cantor [1845-1918]

DEFINE: Sequence of Fourier Functions

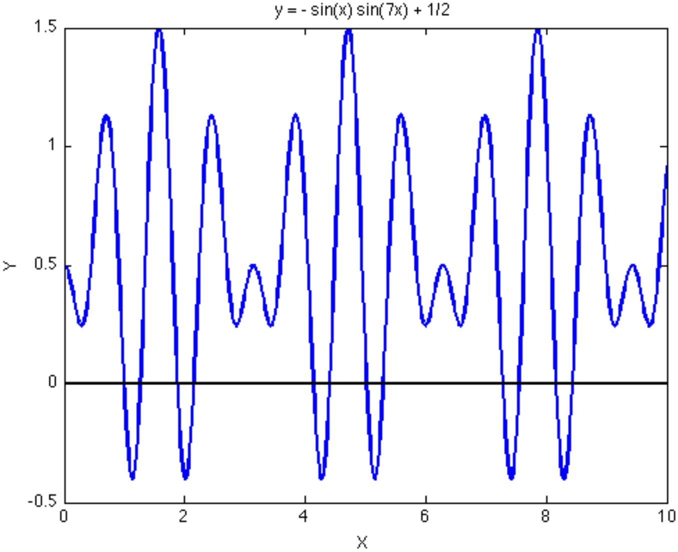
Georg Cantor was studying functions defined by a Fourier series, that is, by sums of sines and cosine functions. His work involved determining the number of times such a function could cross the x-axis over a finite interval.

Cantor observed that a sequence of such functions could be defined for which the number of crossings grew...to infinity, we would say. He also wanted to consider whether there was a "limit function" that his sequence was tending to.

Cantor felt that it was artificial and stupefying to avoid using the concept of infinity when it so clearly seemed to be the right way to discuss his problem.

He understood that historically, infinity had not been trusted because it had not been well defined, and he wondered if, nonetheless, a definition was possible.

DEFINE: A Fourier Function



DEFINE: What Are We Counting?

Since infinity involved counting things, Cantor felt he had to start by describing the things he would count as a set.

Intuitive Notion of a Set

"A set is any group of objects that can be thought of as a whole."
Georg Cantor

This definition is as important as starting geometry with a definition of **points**.

Small sets could be described as a list:

$$\mathbf{A} = \{1, 2, 4, 97\}$$

Or a set could be described by a property of its elements:

$$\mathbf{B} = \{\mathbf{d} \,|\, \mathbf{d} \text{ is a divisor of } 36\} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

DEFINE: Equal versus Equal in Size

Now it was pretty clear that two sets were equal if, and only if, they contained the exact same things. (Order didn't matter, and you couldn't list things twice.)

If everything in set **A** was in set **B**, Cantor said that **A** was a subset of **B**, and he wrote this as $A \subseteq B$. So one way to express the equality of two sets was to write $A \subseteq B$ **and** $B \subseteq A$.

However, if $A \subseteq B$ but B contained at least one element not in A, the A was said to be a proper subset of B.

Two sets might not be equal, in the sense that they contained different elements, but they still might be equal in size, having the same <u>number</u> of elements.

Cantor tried to think of a way to express this without having to define numbers. He thought about the simple idea that two sets would be equal in size if it was possible to match elements of the two sets in pairs, with nothing left over.

DEFINE: Comparing Set Sizes

Cantor wrote $\#A$ to symbolize the size of the set A.

He wrote $\#A \leq \#B$ if it was possible to match every element of A to a distinct element of B.

Therefore, he could write that A and B were equal in size, that is, $\#A = \#B$ if, and only if, both $\#A \leq \#B$ and $\#B \leq \#A$.

Thus, it wasn't necessary to count the elements of the two sets. Instead, the elements themselves could be paired directly.

So far, we only seem to have made simple ideas more complicated!

DEFINE: Ready for Infinity?

We said a set could be defined by listing its entries, so it's easy to want to "define" the natural numbers by:

$$\mathbb{N} = \{0, 1, 2, 3, ...\}$$

But what does . . . really mean? Mathematically, the following statement is preferable:

Axiom of Infinity

There is a set S which contains the empty set, and such that, if e is an element of S, then {e,{e}} is also an element of S.

Each natural number n can be represented by the set of all natural numbers that are less than it. We represent 0 by {}. We represent 1 by $\{0\} = \{\{\}\}$. We represent 2 by {0,1} and so on. Now we recognize that the set S can be regarded as a form of $\mathbb{N}$.

DEFINE: Is ∞ the "Next" Natural Number?

If a set of (consecutive) natural numbers can be regarded as defining the "next" natural number, then we can see that we could be tempted to write

$$\{0, 1, 2, 3, \ldots\} = \infty$$

So ∞ is the smallest "number" bigger than all natural numbers.

We must be careful, because we know that the arithmetic used with regular integers won't work if we throw ∞ in. So let's hold off on this way of looking at things for now.

Instead, we are now prepared to go back to a question raised by the paradox of Galileo.

DEFINE: A Smaller Infinity Than the Smallest Infinity?

We've defined the natural numbers $\mathbb{N}$ as the "smallest" infinite set. If you stop the sequence 0, 1, 2, 3, ..., early, you get a finite set.

But, as Galileo pointed out, consider the set of even numbers $\mathbb{E}$:

- $\mathbb{E}$ is a proper subset of $\mathbb{N}$;
- $\mathbb{E}$ is infinite in size.

So, although $\mathbb{N}$ is the smallest infinite set, we apparently have a set $\mathbb{E}$ which is also infinite (undeniable!), and smaller than $\mathbb{N}$ (are we sure about this?).

Remember how we defined the size of a set! Two sets are equal in size if their elements can be matched:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow \ldots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12\ldots \end{array}$$

DEFINE: A Definition of an Infinite Set

Cantor realized that infinite sets did not follow all the rules that finite sets did. Removing some elements from an infinite set was not guaranteed to make it smaller!

Cantor realized that this property defined an infinite set!

> *"An infinite set is precisely any set that can be placed in 1-to-1 correspondence with a proper subset of itself."*
> Georg Cantor

With this statement, Cantor declared that many of the paradoxes about infinity were, in fact, simply mistaken attempts to describe infinity using ideas about finite sets.

We'll shorten "a 1-to-1 correspondence" to simply *a matching*. We symbolize a matching between sets **A** and **B** by

$$A \sim B$$

DEFINE: Strange Rules of Infinity

It's important that we agree that this definition "captures" infinity, that is, that what we are defining here has a close relationship with the intuitive concept of infinity.

The definition guarantees that an infinite set will remain infinite if:

- you add one item;
- you add an infinite number of items;
- you subtract one item;
- you **carefully** subtract an infinite number of items.

TRANSFINITE: $1 + \aleph_0 = \aleph_0$

Cantor decided not to use the symbol ∞ to denote the "size" (cardinal number) of the counting numbers. Instead, he picked the Hebrew letter $\aleph_0$ ("aleph-null").

$$\begin{aligned} \mathbb{N} =& \{0, 1, 2, 3, \ldots\} \\ \#\mathbb{N} =& \aleph_0. \end{aligned}$$

Some count this way: 1, 2, 3, ... and others count 0, 1, 2, 3, ...

Call these sets $\mathbb{N}^+$ and $\mathbb{N}_0$. Is it true that $\mathbb{N}^+ \sim \mathbb{N}_0$?
Yes, because here is a matching:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow\ .. \\ 0 & 1 & 2 & 3 & 4 & 5\ldots \end{array}$$

This is puzzling result #1 in transfinite arithmetic:

$$\color{red}{1 + \aleph_0 = \aleph_0}$$

So why doesn't the equation

$$1 + \aleph_0 = \aleph_0$$

imply

$$1 = 0?$$

In Cantor's arithmetic, you can't directly add or subtract cardinals.

You first find sets A and B so that $A \sim B$
then you can write $\#A = \#B$!

To "prove" that $1 = 0$, we'd have to match a set of 0 elements and a set of 1 element.

TRANSFINITE: $\aleph_0 = \aleph_0 + 1 + \aleph_0$

The set of integers is $\mathbb{Z} =$ ..., -3, -2, -1, 0, 1, 2, 3, ...
Since $\mathbb{N} \subset \mathbb{Z}$, we know $\#\mathbb{N} \leq \#\mathbb{Z}$. But is a matching possible?

$$\begin{array}{ccccccc} \ldots -3 & -2 & -1 & 0 & 1 & 2 & 3\ldots \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow\,.. \\ \ldots 7 & 5 & 3 & 1 & 2 & 4 & 6\ldots \end{array}$$

Therefore, $\mathbb{N} \sim \mathbb{Z}$ and so $\#\mathbb{N} = \#\mathbb{Z}$ or:

$$\aleph_0 = \aleph_0 + 1 + \aleph_0 = \aleph_0 + (1 + \aleph_0) = \aleph_0 + \aleph_0 = 2 \times \aleph_0$$

So puzzling result $\#2$ in transfinite arithmetic is:

$$2 \times \aleph_0 = \aleph_0$$

The set of all positive fractions or "rational numbers" $\mathbb{Q}^+$. is surely bigger than the positive numbers $\mathbb{N}^+$. $\mathbb{Q}^+$ can be thought of as *infinitely many* copies of the counting numbers. Shouldn't this be a much bigger set, then?

Let us imagine the fractions laid out in a table. In the first row are the fractions with 1 as a denominator, and in the second those with 2 and so on.

Our task of "counting" these fractions can be thought of as requiring us to imagine writing all the numbers in this table, in such a way that every fraction is "scheduled" to be written at a definite, finite time. We can do this by zigzagging!

TRANSFINITE: Cantor Counts the Fractions

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

So now we know that $\mathbb{Q}^+ \sim \mathbb{N}^+$, and therefore $\#\mathbb{Q}^+ = \#\mathbb{N}^+$

Now the elements of the set $\mathbb{Q}^+$ are pairs of elements of $\mathbb{N}^+$

In this case, we say that $\mathbb{Q}^+$ is the *product* of $\mathbb{N}^+$ with itself; this is symbolized by $\mathbb{Q}^+ = \mathbb{N}^+ \otimes \mathbb{N}^+$

The number of elements in a product is simply the product of the number of elements in each factor, so $\#\mathbb{Q}^+ = \#\mathbb{N}^+ \times \#\mathbb{N}^+$

So puzzling result #3 in transfinite arithmetic is:

$$\aleph_0 \times \aleph_0 = \aleph_0$$

It's beginning to seem like all infinities are the same size!

TRANSFINITE: Are These Denumerable?

A set is *denumerable* if it can be matched with $\mathbb{N}$.

- all possible (finite) English sentences.
- all possible (finite) books in English.
- all possible (finite) musical pieces.
- all possible images on a computer screen.
- all possible (pixellated and finite) movies.
- the number of times you can write an "X" of any size on one sheet of paper without crossing any others (**not** true for "O"!)

Thus, there would be "book 999" and "song 1234" and image "54321" and this presentation would be "presentation 192837465".

Is the whole world denumerable? As you can imagine, Cantor thought about this!

How many points are there on a line? If we identify the points with their numerical distance from an origin, it's clear we have an infinite number.

Are the points on a line denumerable?

It would be enough to count just the real numbers between 0 and 1, symbolized by $\mathbb{R}[0,1]$. If they could be counted, so could $\mathbb{R}$.

So a solution to the problem could be thought of as **a numbered list** of all the elements in $\mathbb{R}[0,1]$, our 1 inch line.

REALS: How to Count Them?

So here's one idea that almost seems to work. Turn the counting numbers around, with a decimal point in front.

$\mathbb{N}$	$\mathbb{R}[0,1]$
0	.0
1	.1
2	.2
3	.3
4	.4
...	...
128	.821
129	.921
130	.031
...	...
12345	.54321
...	...

REALS: How (NOT) to Count Them

Unfortunately, this list only includes some real numbers. Actually, it only includes rational numbers. And it doesn't even include all of them. It only includes rational numbers with a denominator that is 10, 100, 1000 and so on.

Ask yourself where, on this list, does the number $\frac{1}{3}$ occur?
It never shows up!

Where on this list does the nonterminating decimal $\frac{pi}{4}$ occur?
Nowhere!

So this brilliant idea actual is a dead end.

Eventually, Cantor wondered if a complete list was impossible.

REALS: Suppose We Had A List

So Cantor imagined that someone could produce a numbered list containing absolutely every real number in [0,1]:

$\mathbb{N}$	$\mathbb{R}[0,1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

REALS: Scan the List Diagonally

Cantor considered the number formed by the first digit of the first number, the second digit of the second number, and so on:

$\mathbb{N}$	$\mathbb{R}[0,1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

The diagonal number is .1937683211....

REALS: Here's a Missing Number!

Cantor then increased each digit of .1937683211... by 1, (but 9's become 0's), to get the modified number .2048794322....

He pointed out this number cannot be on the list, because:

- its first digit differs from that of the number in position 1,
- its second digit differs from that of the number in position 2,
- its **n**-th digit differs from that of the number in position **n**,
- and so on.

So a numbered list of elements in [0,1] can never be complete! There is no 1-to-1 matching between $\mathbb{N}$ and $\mathbb{R}$.

"I see it, but I can't believe it!"
Georg Cantor

With the discovery that $\aleph_0 < c$, Cantor realized that the statement "all infinities are the same size" was false. There is more than one size of infinite set.

$\aleph_0$ was certainly the first step into infinity (can you prove there is no infinite set strictly smaller than $\mathbb{N}$?), and **c** was a further step.

Was **c** the very "next" step after $\aleph_0$, (so that we could rename **c** to $\aleph_1$) or were there intermediate steps?

And were there infinite sets that were strictly larger? If so, was there a way to create them?

REALS: $c \times c = c$

Since $\mathbb{R}$ counts the points in a line, perhaps a bigger infinity is the number of points in a plane, symbolized by $\mathbb{R} \times \mathbb{R}$.

But Cantor found a matching between points in a line and a plane.

Given a point in the unit square such as $(x, y) = (.012345, .56789)$, we can construct its matching point in the unit line by alternating the digits of x and y to get .0516273849.

With some adjustments, you can match all points in the plane with the points in a line.

So $c \times c = c$.

The same trick means that $c \times c \times c = c$.

REALS: A stronger version of Albert's Paradox

So c counts the points in a line, in a plane, and in all space.

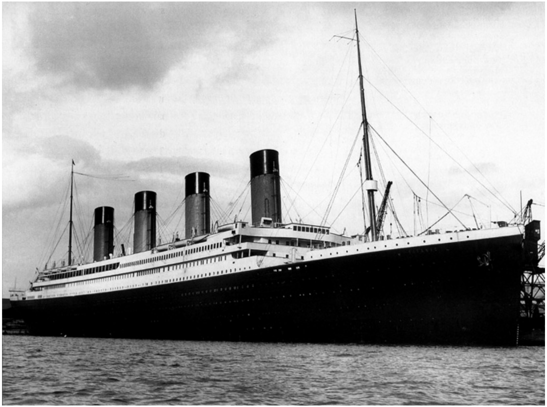
Since c counts the points in a 1 inch line segment as well, all these objects have the same number of points.

Remember Albert of Saxony's paradox, where an infinite piece of wood can be used to fill space?

Cantor has shown something more shocking. The points on a 1 inch line segment (which has no area, and no volume) can be rearranged to fill an entire line, an entire plane, or all of space!

It's stranger than saying you could build the Titanic from an inch of wire!

REALS: Titanic = Nail?



REALS: Are there even bigger sets?

Cantor had achieved his goal of finding a set that was bigger than the counting numbers $\mathbb{N}$. Naturally, he wondered whether there were even larger sets.

Since the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$ were no bigger than $\mathbb{R}$, Cantor tried to think of other mathematical operations or objects that might reach a higher level.

Cantor's breakthrough came when he tried to explain why c had to be bigger than $\aleph_0$.

REALS: A Real Number is a Kind of Subset of $\mathbb{N}$

Every real number in [0,1] can be written as a binary decimal.

$$\frac{\pi}{4} = 0.785398... = 0.110010010000111111...$$

If we record the locations where a 1 occurs, we get a subset of $\mathbb{N}$.

$$\frac{\pi}{4} \to \{1, 2, 5, 8, 13, 14, 15, 16, 17, 18, ...\}$$

So the number of reals is equal to the number of subsets of $\mathbb{N}$.

REALS: The Power Set is Strictly Bigger than the Set

Given any set A, the set of all possible subsets of A is known as its power set, written as $\mathcal{P}(\mathcal{A})$.

Cantor was able to show that, for any set A, the power set is always strictly larger in size. It is **never** possible to find a 1-1 matching between A and $\mathcal{P}(\mathcal{A})$.

Therefore, **c**, the number of real numbers in the interval [0,1] was an infinite number strictly larger than $\aleph_0$.

Cantor spent many years trying to prove that **c** had to be the "very next" infinite number after $\aleph_0$, but he could not do so.

Nonetheless, he had been able to show the existence of two kinds of infinite number, one larger than the other.

Moreover, the power set of **c** must be an infinite set strictly bigger than **c**, and this process can be continued to produce an endless sequence of bigger and bigger infinite sets.

Cantor had shown that it is not true that "all infinities are the same size". Instead, he had shown that the two infinities we knew about were simply starting steps on an endlessly rising stairway.

REALS: Cantor's Vision

CONCLUSION:

Cantor also worked on the related topic of ordered infinite numbers, or ordinals. There were many new surprises in that area as well.

Cantor was never able to prove his "Continuum Hypothesis", that $\mathbf{c}$ is the "next" infinity after $\aleph_0$ and it was left to Kurt Gödel and Paul Cohen to show that this statement was not provable from the basic laws of set theory.

Cantor's results about infinity, and his work in set theory, were controversial for years; they have since become standard, respected results.

His diagonal argument, used to showed the real numbers cannot be counted, was reused by Alan Turing in a proof about computability, and by Kurt Gödel in his famous proof that mathematical systems are incomplete.

CONCLUSION:

Cantor gave infinity a logical base as a mathematical subject.

Cantor cleared up the paradoxes by realizing that the "laws" that infinity violated were laws that defined finite numbers.

He found the key to his theory when he defined infinite objects as exactly those that are "equal" to a proper subset of themselves.

The mathematician Hilbert judged Cantor's work as follows:

> *"No one will expel us from this paradise Cantor has created for us."*
> *David Hilbert*