PARTIAL SOLUTION TO "IMAGE DEBLURRING: I CAN SEE CLEARLY NOW"

By James G. Nagy and Dianne P. O'Leary

Problem 1. Show that

$$\min_{\mathbf{f}} \left\{ \|\mathbf{g} - \boldsymbol{K}\mathbf{f}\|_2^2 + \alpha^2 \|\mathbf{f}\|_2^2 \right\} \tag{1}$$

is equivalent to the linear least-squares problem,

$$\min_{\mathbf{f}} \left\| \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K} \\ \alpha \boldsymbol{I} \end{bmatrix} \mathbf{f} \right\|_2^2 . \tag{2}$$

Answer: Observe that if $\mathbf{y}$ is a $p \times 1$ vector and $\mathbf{z}$ is a $q \times 1$ vector, then

$$\left\| \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \right\|_2^2 = \sum_{i=1}^{p} y_i^2 + \sum_{i=1}^{q} z_i^2 = \|\mathbf{y}\|_2^2 + \|\mathbf{z}\|_2^2 .$$

Therefore,

$$\left\| \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K} \\ \alpha \boldsymbol{I} \end{bmatrix} \mathbf{f} \right\|_2^2 = \left\| \begin{bmatrix} \mathbf{g} - \boldsymbol{K}\mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 = \|\mathbf{g} - \boldsymbol{K}\mathbf{f}\|_2^2 + \|\alpha \mathbf{f}\|_2^2 = \|\mathbf{g} - \boldsymbol{K}\mathbf{f}\|_2^2 + \alpha^2 \|\mathbf{f}\|_2^2 .$$

Problem 2. Show that if $\boldsymbol{K}$ has a singular value decomposition $\boldsymbol{K} = \boldsymbol{U}\Sigma\boldsymbol{V}^T$, then Equation 2 can be transformed into the equivalent least-squares problem,

$$\min_{\mathbf{f}} \left\| \begin{bmatrix} \hat{\mathbf{g}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \Sigma \\ \alpha \boldsymbol{I} \end{bmatrix} \hat{\mathbf{f}} \right\|_2^2 , \tag{3}$$

where $\hat{\mathbf{f}} = \boldsymbol{V}^T\mathbf{f}$ and $\hat{\mathbf{g}} = \boldsymbol{U}^T\mathbf{g}$.

Answer: First note that

$$\boldsymbol{Q} \equiv \begin{bmatrix} \boldsymbol{U}^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}^T \end{bmatrix}$$

is an orthogonal matrix since $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$, and recall that the two-norm of a vector is invariant under multiplication by an orthogonal matrix: $\|\boldsymbol{Q}\mathbf{z}\|_2 = \|\mathbf{z}\|_2$. Therefore,

$$\begin{aligned} \left\| \begin{bmatrix} \mathbf{g} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K} \\ \alpha \boldsymbol{I} \end{bmatrix} \mathbf{f} \right\|_2^2 &= \left\| \begin{bmatrix} \mathbf{g} - \boldsymbol{K}\mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 \\ &= \left\| \boldsymbol{Q} \begin{bmatrix} \mathbf{g} - \boldsymbol{U}\Sigma\boldsymbol{V}^T\mathbf{f} \\ \alpha \mathbf{f} \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \boldsymbol{U}^T\mathbf{g} - \Sigma\boldsymbol{V}^T\mathbf{f} \\ \alpha \boldsymbol{V}^T\mathbf{f} \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \hat{\mathbf{g}} - \Sigma\hat{\mathbf{f}} \\ \alpha \hat{\mathbf{f}} \end{bmatrix} \right\|_2^2 \\ &= \left\| \begin{bmatrix} \hat{\mathbf{g}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \Sigma \\ \alpha \boldsymbol{I} \end{bmatrix} \hat{\mathbf{f}} \right\|_2^2 . \end{aligned}$$

Problem 3. Determine a formula for the solution to Equation 3. Hint: you should set the derivative of the minimization function to zero and solve for $\hat{\mathbf{f}}$.

Answer: Let's write the answer for a slightly more general case: $\boldsymbol{K}$ of dimension $m \times n$ with $m \geq n$:

$$\begin{aligned} \left\| \begin{bmatrix} \hat{\mathbf{g}} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \Sigma \\ \alpha \boldsymbol{I} \end{bmatrix} \hat{\mathbf{f}} \right\|_2^2 &= \left\| \hat{\mathbf{g}} - \Sigma\hat{\mathbf{f}} \right\|_2^2 + \alpha^2 \left\| \hat{\mathbf{f}} \right\|_2^2 \\ &= \sum_{i=1}^{n} \left(\hat{g}_i - \sigma_i \hat{f}_i \right)^2 + \sum_{i=n+1}^{m} \hat{g}_i^2 + \alpha^2 \sum_{i=1}^{n} \hat{f}_i^2 . \end{aligned}$$

Setting the derivative with respect to f_i to zero, we get

$$-2\sigma_i \left(\hat{g}_i - \sigma_i \hat{f}_i \right) + 2\alpha^2 \hat{f}_i = 0 ,$$

so

$$\hat{f}_i = \frac{\sigma_i \hat{g}_i}{\sigma_i^2 + \alpha^2}, \quad i = 1, \ldots, n .$$

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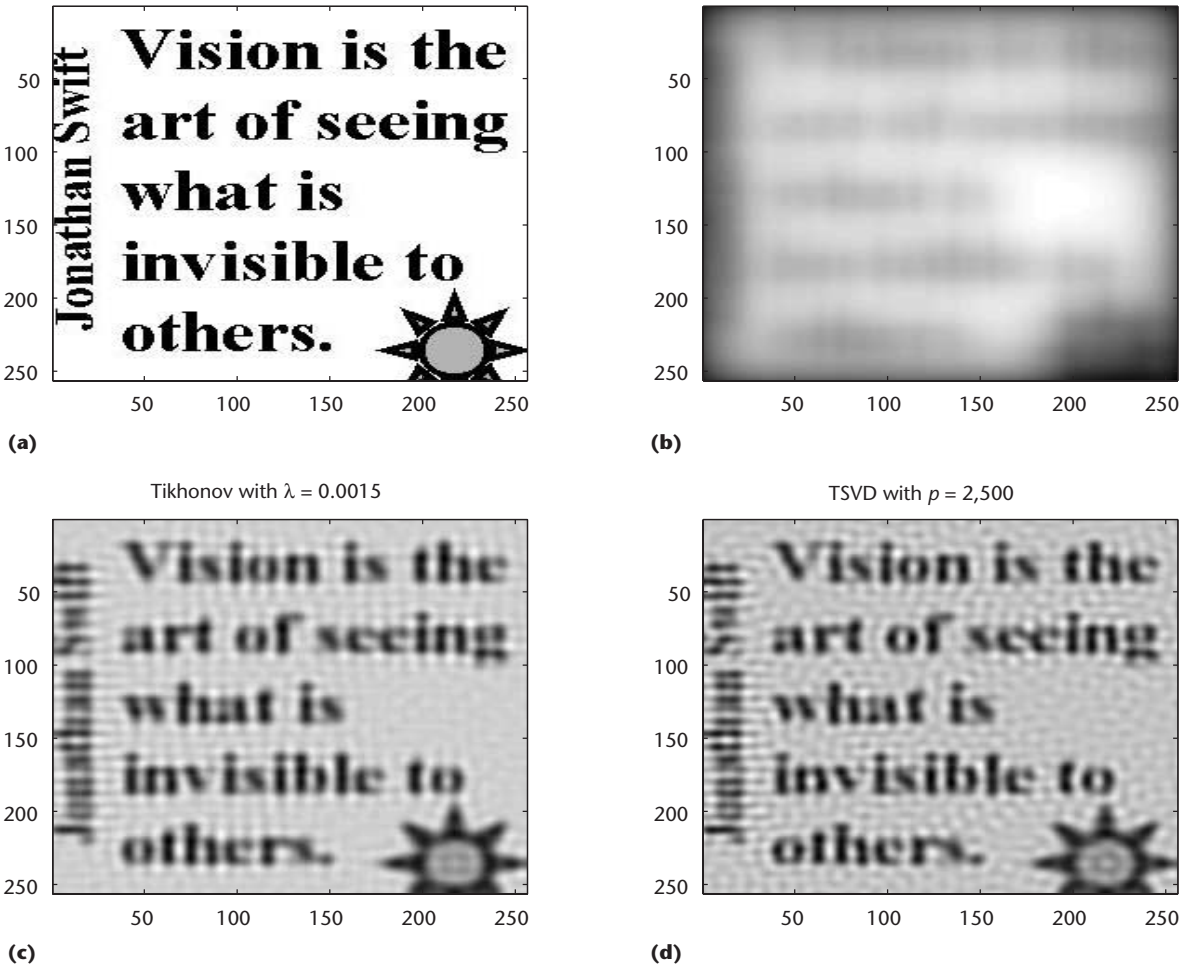


Figure 1. You can see clearly now. (a) The original image, (b) blurred image, and (c), (d) the results of the two algorithms.

Problem 4. Show that the solution to the problem

$$\min_{\mathbf{f}}\left\{\|\mathbf{g}-\boldsymbol{K}\mathbf{f}\|_2^2\right\}$$

is

$$\mathbf{f}_{\ell s}=\boldsymbol{V}\,\Sigma^{\dagger}\,\boldsymbol{U}^T\mathbf{g}\equiv\sum_{i=1}^{n}\frac{\mathbf{u}_i^T\mathbf{g}}{\sigma_i}\mathbf{v}_i\ ,$$

where $\mathbf{u}_i$ is the ith column of $\boldsymbol{U}$, and $\mathbf{v}_i$ is the ith column of $\boldsymbol{V}$.

Answer: From Problems 2 and 3, with $\alpha = 0$, the solution is

$$\hat{f}_i=\frac{\sigma_i\hat{g}_i}{\sigma_i^2}=\frac{\hat{g}_i}{\sigma_i}\ .$$

Note that $\hat{g}_i=\mathbf{u}_i^T\mathbf{g}$. Now, since $\mathbf{f}=\mathbf{V}\hat{\mathbf{f}}$, we have

$\mathbf{f}=\mathbf{v}_1\hat{f}_1+\ldots+\mathbf{v}_n\hat{f}_n$, and the formula follows.

Problem 5. Write a program that takes matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and image $\boldsymbol{G}$ and computes approximations to image $\boldsymbol{F}$ using Tikhonov regularization and truncated SVD. For each of these two algorithms, experiment to find the value of the regularization parameter (α for Tikhonov or p for truncated SVD) that gives the clearest image.

Answer: See the posted program (http://computer.org/cise/homework/v5n3.htm).

One common error is zeroing out pieces of $\boldsymbol{S}_A$ and $\boldsymbol{S}_B$. This does not zero the smallest singular values, and although it is very efficient in time, it gives worse results than doing it correctly.

The data were generated by taking an original image $\boldsymbol{F}$ (posted now on the Web page), multiplying by $\boldsymbol{K}$, and then adding random noise: `G = B * F * A' + .001 * rand(256,256)`. Note that the noise prevents us from completely recovering the initial image.

In this problem, the best way to choose the regulariza-

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tion parameter is by eye. Choose a detailed part of the image and watch its resolution for various choices of the regularization parameter, probably using bisection to find the best parameter. Figure 1 shows data and two reconstructions. Although both algorithms yield images in which you can read the text, noise in the data appears in the background of the reconstructions. Some nonlinear reconstruction algorithms reduce this problem.

In many applications, you must choose the regularization parameter by automatic methods rather than by eye. If you know the noise level, then the discrepancy principle is useful: choose the parameter to make the residual $\mathbf{g} - \boldsymbol{K}\mathbf{f}$ close in norm to the noise's expected norm. If you don't know the noise level, then *generalized cross validation* and the *L-curve* are popular methods.[1]

Acknowledgments

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Reference

1. P.C. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems*, SIAM Press, 1998, Chapter 7.

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