An Overview of Flow Optimization Using Sensitivities

https://people.sc.fsu.edu/~jburkardt/presentations/flow_optimization.pdf

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John Burkardt
Mathematics Department
Iowa State University, Ames, Iowa

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1 The Navier Stokes Equations

The steady incompressible flow of a viscous fluid in a two dimensional region may be completely described by three state functions: the horizontal velocity $u(x, y)$, vertical velocity $v(x, y)$, and pressure $p(x, y)$.

Because they represent the behavior of a physical fluid, the functions u, v, and p obey certain physical laws. Given our assumptions about the problem, we will find it appropriate to assume that these flow functions satisfy the Navier Stokes equations for stationary incompressible viscous flow at every point (x, y) within the flow region Ω.

These equations may be written as:

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + R\,(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) = 0 \quad (1)$$

$$-(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + R\,(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) = 0 \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

The parameter R in Equations (**??**) and (**??**) is the inverse kinematic viscosity. With appropriate scaling, we can take R to be the Reynolds number for this problem. Physically, the Reynolds number is known to have a strong influence on a flow; its value determines the balance of momentum and diffusion, and controls the onset of turbulence in the flow. Mathematically, R controls the relative weight of the nonlinear terms; as R increases, the equations become harder to solve.

If we are interested in the exact size and form of the influence of R on the solution (u, v, p), then we can differentiate the state equations (and any

boundary conditions) with respect to R, interchange the order of differentiation where desired, to arrive at the first order sensitivity equations. The horizontal momentum equation becomes:

$$\begin{aligned} -(\frac{\partial^2 u_R}{\partial x^2} + \frac{\partial^2 u_R}{\partial y^2}) \quad &+ \quad R\,(u_R \frac{\partial u}{\partial x} + u\frac{\partial u_R}{\partial x} + v_R \frac{\partial u}{\partial y} + v\frac{\partial u_R}{\partial y} + \frac{\partial p_R}{\partial x}) \\ &= \quad -(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) \end{aligned} \tag{4}$$

with a similar equation for vertical momentum, while the continuity equation becomes:

$$\frac{\partial u_R}{\partial x} + \frac{\partial v_R}{\partial y} = 0 \tag{5}$$

Note that these equations will be easier to solve than the state equations; they are linear, and the left hand side operator is identical to that of the Newton increment equations, meaning the only new coding required is for evaluating the right hand sides.

2 The Driven Cavity

To complete our problem formulation, we choose the physical region and boundary conditions known as the *driven cavity*. A viscous fluid is contained in a (two dimensional) square region with sides of length 1. The sides and bottom of the region are no-slip walls. Along the top of the region, some tangential force impels the fluid to move from left to right with a given speed U_{top}.

Peterson [**?**] chose this problem for a study of the reduced basis method. There, the focus was on using the reduced basis method to compute flow solutions for high values of R. The Newton method requires a very good starting estimate for convergence at high R, and so the usual solution method involves calculating flow solutions at a lower, but increasing sequence of values of R, with each new flow solution used as the starting estimate for the Newton iteration at the next value of R. Such a procedure can become unacceptably expensive for high R. Peterson showed that, at values of R as high as 5,000, a problem requiring as many as 463 standard basis vectors could be solved with just 5 reduced basis vectors.

3 Comparing Taylor Polynomials and Reduced Basis Solutions

The reduced basis method may be viewed as a generalization of the Taylor series approximation. The main difference is that instead of using Taylor's theorem to compute the coefficients of the basis vectors, the reduced basis method determines them by a finite element calculation.

The main limitation of a Taylor series is that the approximating power is very local, and rapidly deteriorates as some power of the distance from the point

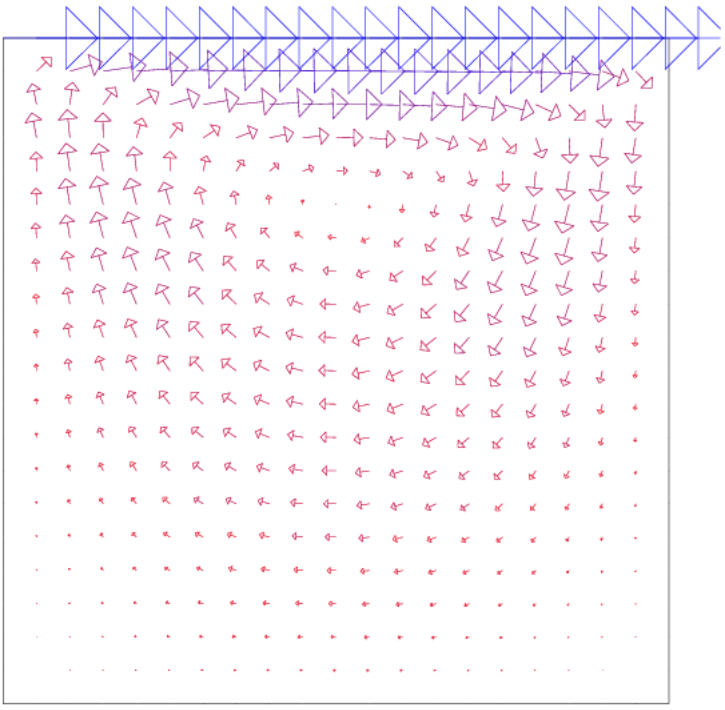

Figure 1: The computed velocity for a driven cavity with R=1.

where the series was generated. In contrast, the reduced basis method approximation seems to produce good, regular results over a much greater interval. To test these claims, we compared the power of both methods at a variety of values of R.

Although the full flow problem has three variables, the pressure is of lesser interest. We will concentrate on the velocities, (u, v), which we may also write as $\mathbf{u}$.

Let us suppose that at some parameter value R_0, we have a flow solution $\mathbf{u}(R_0)$, and the first n partial derivatives of the flow solution with respect to R, beginning with $\mathbf{u}'(R_0)$. Then for small perturbations ΔR, we may use the n term Taylor polynomial to approximate the flow solution $\mathbf{u}(R_0 + \Delta R)$.

The Taylor approximation has certain typical behaviors:

- For values of R "sufficiently near" R_0, the approximation error can be decreased by increasing n;
- For any fixed value of n, the error tends to rise as ΔR increases, as the $n + 1$ power of ΔR.

For the following table, the base parameter value $R_0 = 100$ was chosen, at which point the full basis solution $\mathbf{u}_{FL}(R_0)$ was computed, along with the first 5 Taylor vectors and reduced basis vectors. Then, for a sequence of increasing parameter values R, the full basis solution $\mathbf{u}_{FL}(R)$ was computed, as well as the Taylor approximant $\mathbf{u}_{Tay}(R)$ and the reduced basis solution $\mathbf{u}_{RB}(R)$. The relative error of the Taylor approximant was computed as the ratio

$$\frac{\|\mathbf{u}_{Tay}(R) - \mathbf{u}_{FL}(R)\|}{\|\mathbf{u}_{FL}(R)\|} \tag{6}$$

with a similar ratio used for the reduced basis. Both approximation methods were studied with the number of vectors varied from 0 to 5. The results are summarized in Table (**??**).

There are several things to note about the data in this table. First, for the Taylor data, we see that increasing n reduces the relative error for $R = 150$ and $R = 200$ but has little effect at $R = 250$ and actually begins to increase the error thereafter. This suggests that we can only make accurate Taylor approximations close to the base parameter value.

By contrast, the reduced basis data seems more robust. Even at $R = 400$, we can increase the value of n to decrease the relative error. Moreover, the error made by the reduced basis is generally less than that made by the Taylor polynomial for values of n exceeding 1.

This data tends to confirm the suspicion that the Taylor approximating space is itself good, but that the prescribed coefficients can be improved as we move away from the base parameter value.

Certain points are raised by the graphical comparison of the approximation behavior for the Taylor and reduced basis schemes. First, we note that the

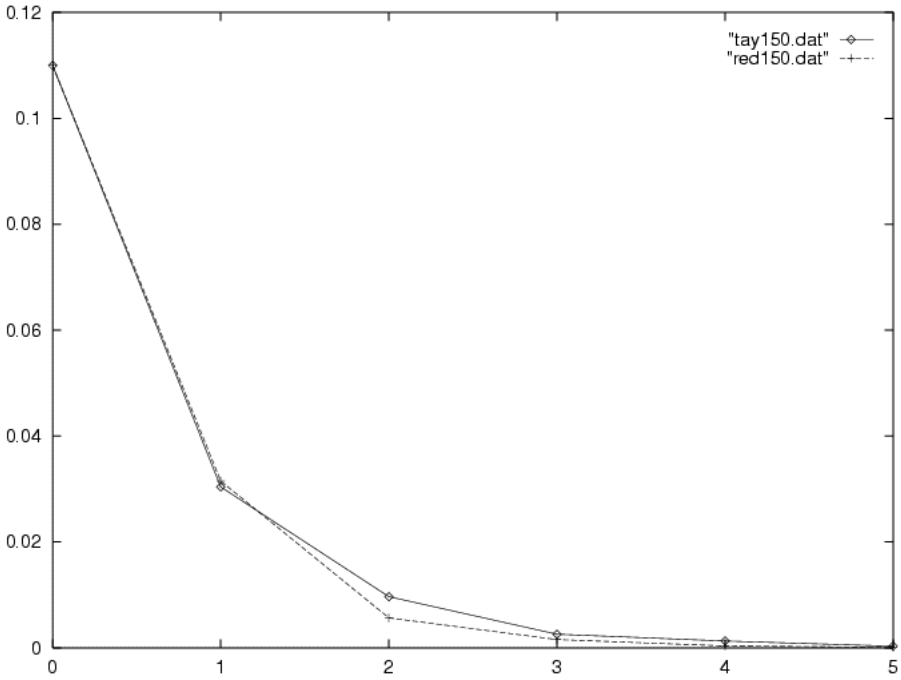

Figure 2: Re = 150.

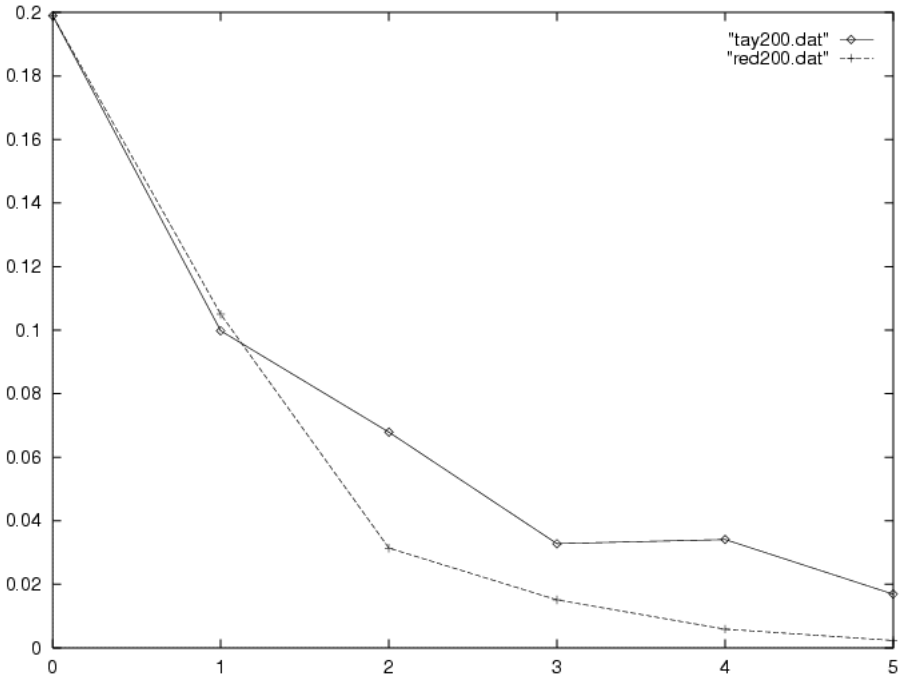

Figure 3: Re = 200.

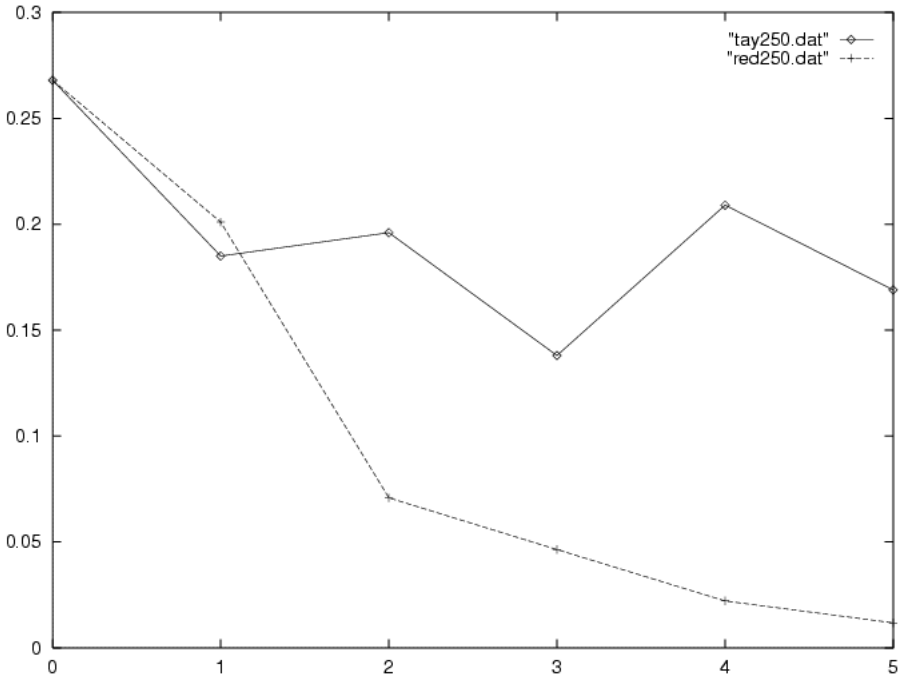

Figure 4: Re = 250.

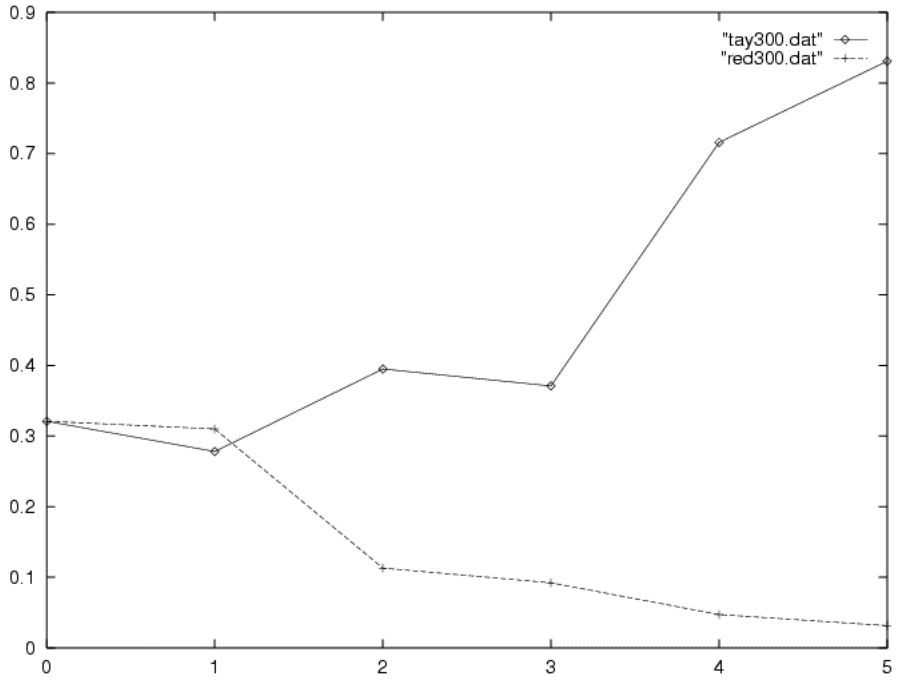

Figure 5: Re = 300.

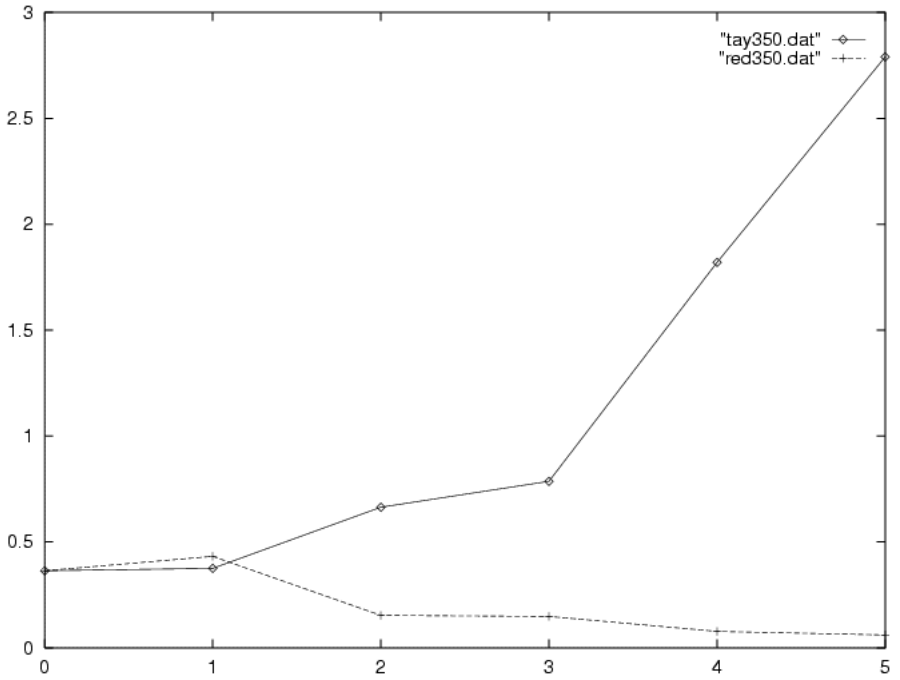

Figure 6: Re = 350.

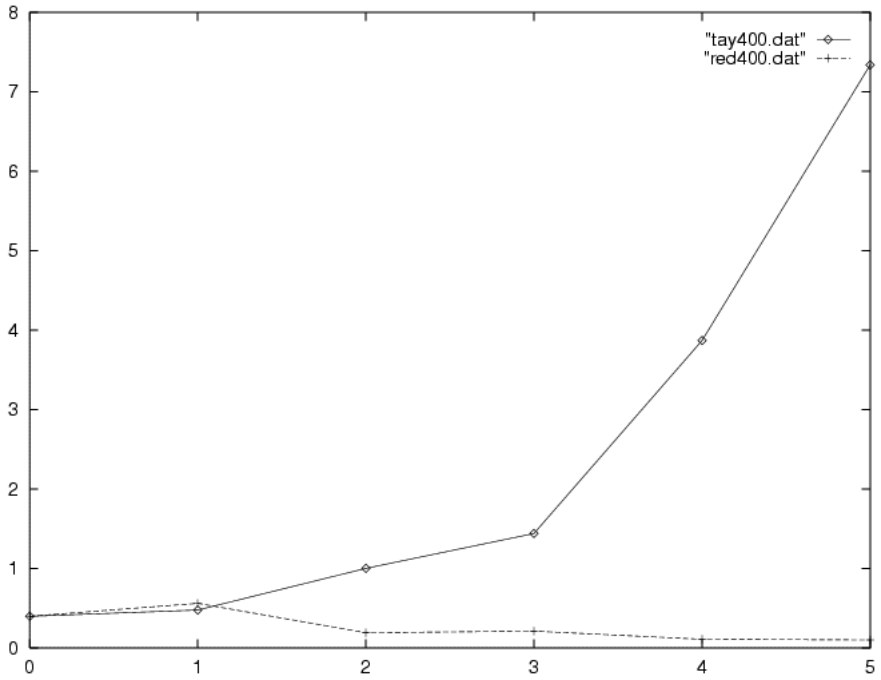

Figure 7: Re = 400.

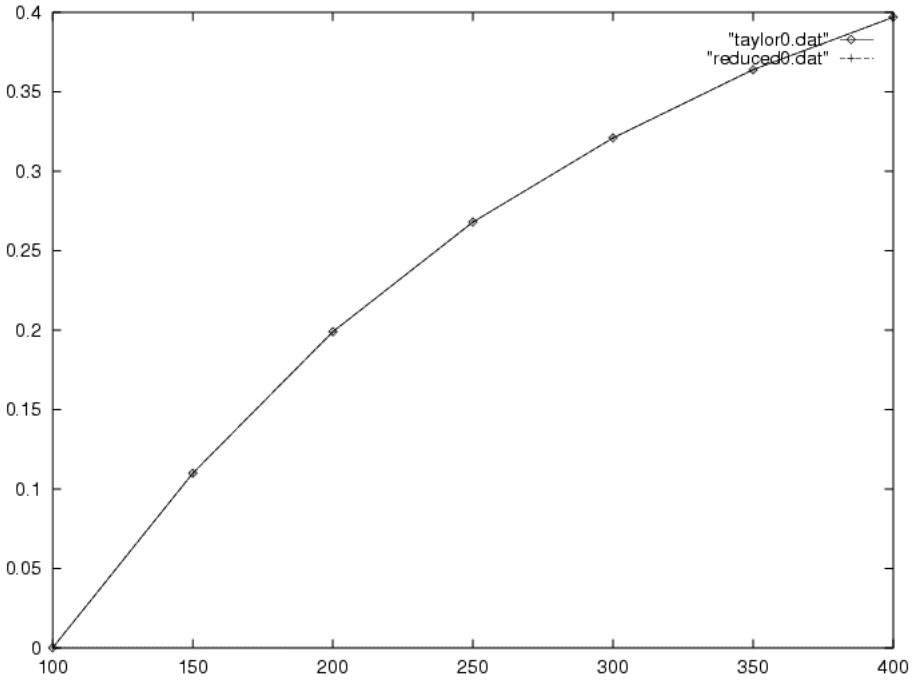

Figure 8: Order 0.

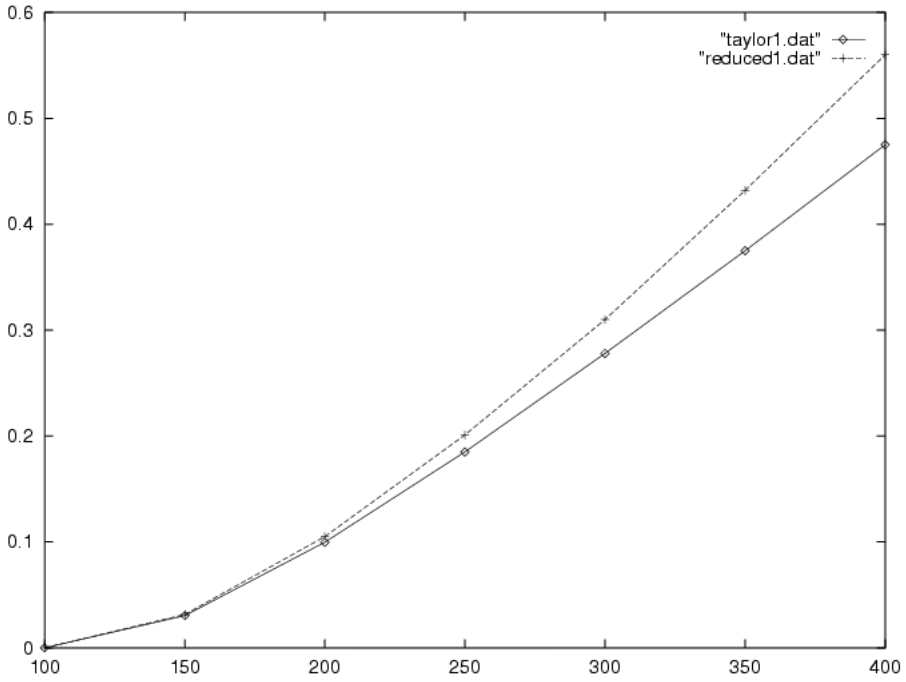

Figure 9: Order 1.

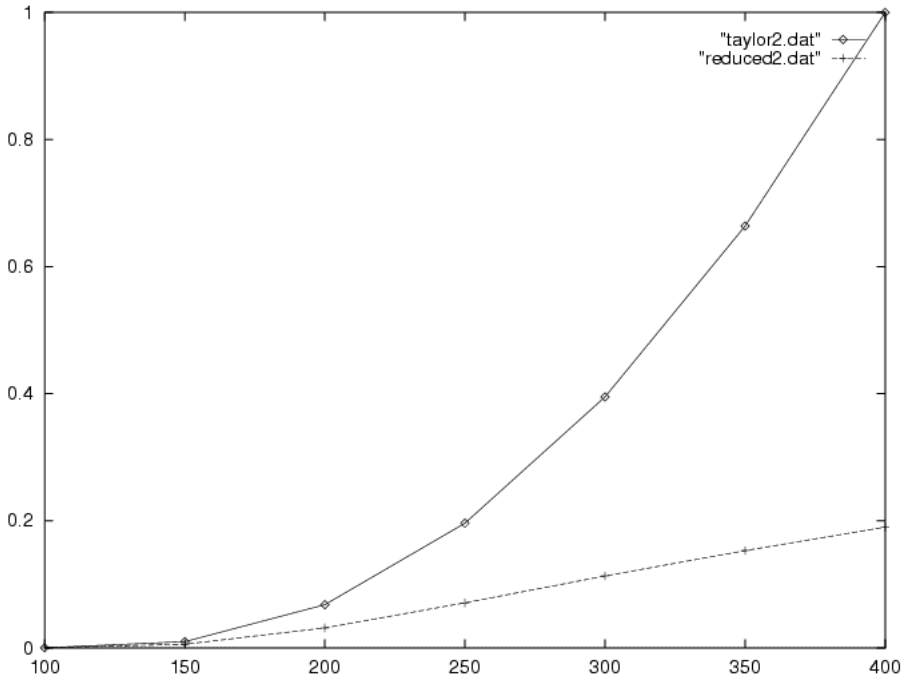

Figure 10: Order 2.

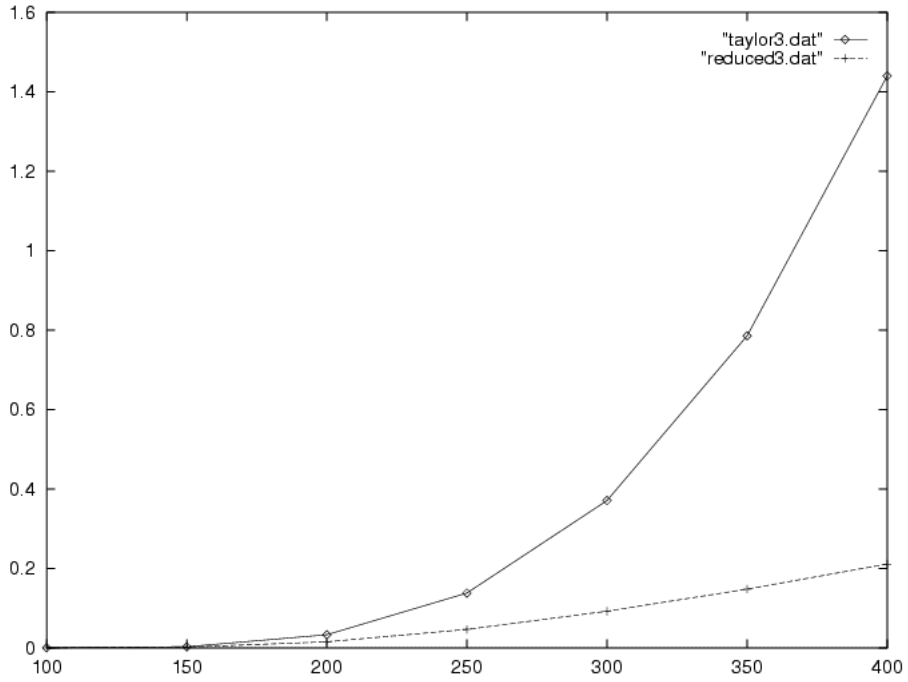

Figure 11: Order 3.

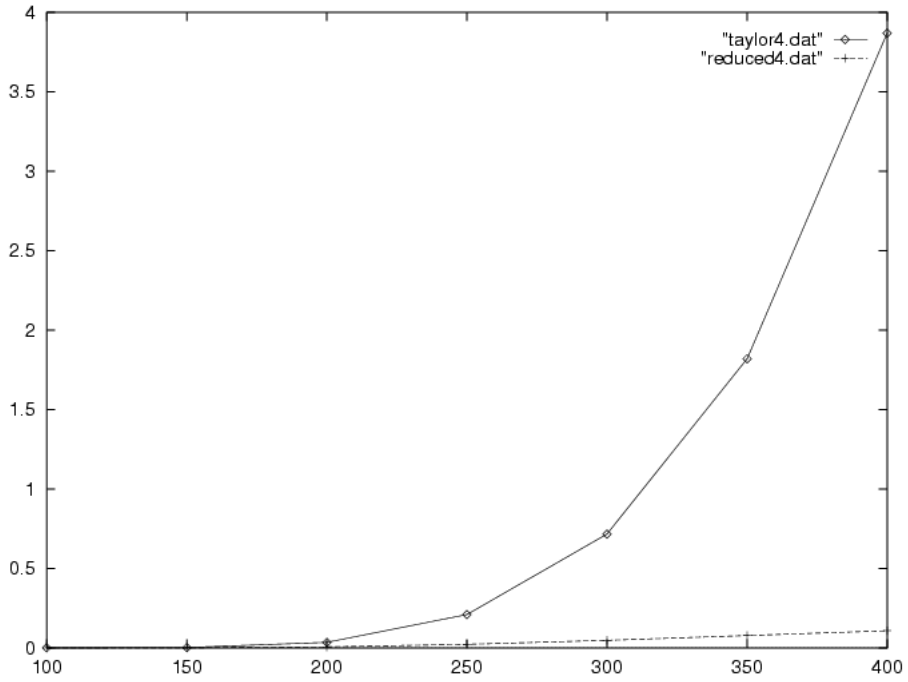

Figure 12: Order 4.

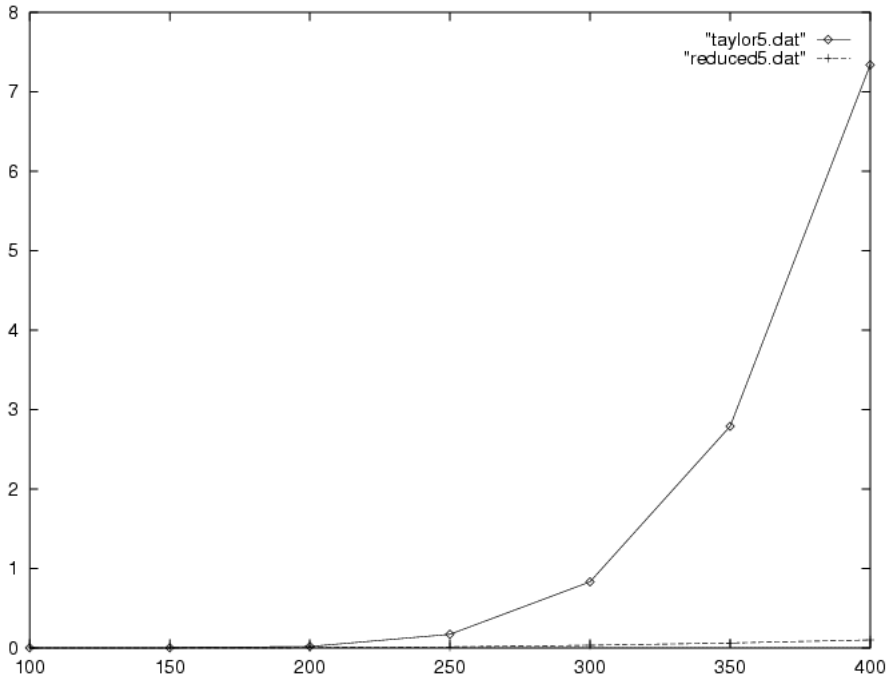

Figure 13: Order 5.

Table 1: Comparison of Taylor and Reduced Basis Approximation Errors.

Value of R:	150	200	250	300	350	400
$\|\mathbf{u}_{FL}\|$	0.266	0.271	0.274	0.278	0.280	0.282
Taylor N_{Tay}						
0	1.10e-1	1.99e-1	2.68e-1	3.21e-1	3.64e-1	3.97e-1
1	3.04e-2	9.98e-2	1.85e-1	2.78e-1	3.75e-1	4.75e-1
2	9.66e-3	6.79e-2	1.96e-1	3.95e-1	6.64e-1	1.00
3	2.56e-3	3.28e-2	1.38e-1	3.71e-1	7.86e-1	1.44
4	1.28e-3	3.41e-2	2.09e-1	7.16e-1	1.82	3.87
5	3.05e-4	1.69e-2	1.69e-1	8.31e-1	2.79	7.34
Reduced Basis N_{RB}						
0	1.10e-1	1.99e-1	2.68e-1	3.21e-1	3.64e-1	3.97e-1
1	3.15e-2	1.05e-1	2.01e-1	3.10e-1	4.32e-1	5.60e-1
2	5.64e-3	3.14e-2	7.08e-2	1.13e-1	1.53e-1	1.90e-1
3	1.56e-3	1.51e-2	4.64e-2	9.21e-2	1.48e-1	2.11e-1
4	3.60e-4	5.90e-3	2.21e-2	4.71e-2	7.75e-2	1.08e-1
5	7.86e-5	2.31e-3	1.18e-2	3.14e-2	6.07e-2	9.86e-2

Taylor scheme is only guaranteed to have good approximation properties in some "small" neighborhood of unspecified size. Yet the graphs suggest that the neighborhood of $R = 100$ extends to the right at least to $R = 200$. How can we explain such a large range for the approximation? It turns out that an answer is fairly easy to find; the solution curve for this problem is surprisingly "flat", as evidenced by the l_∞ norms of the velocity sensitivities. The zero order sensitivitity is 500 times larger than the first order, which is 100 times larger than the second order, and the norms continue to decrease with higher order. This suggests that we must take a very large step ΔR indeed before the higher order sensitivities will be multiplied by scale factors large enough to affect the zero order sensitivity.

Table 2: l_∞ Norms of Velocity Sensitivities at R=100.

Order:	Norm
0	1.00
1	1.65e-3
2	2.01e-5
3	4.08e-7
4	9.63e-9
5	4.23e-10

Note that this result is a special property of the particular Navier Stokes problem being solved. For problems which are much more nonlinear, we would

expect at least a few of the early order derivatives to be of comparable magnitude to the zero order sensitivity, and this would in turn make Taylor approximation more liable to early breakdown.

It is not so easy to answer a similar question about the reduced basis method. If the Taylor approximation breaks down after a while, why does the Taylor space itself still provide a good approximation? We see from the graphs that, at least for the basis solution at R=100, the reduced basis method is able to produce approximations still valid at much higher Reynolds numbers than the Taylor method; moreover, the low order approximations can be improved by increasing the approximation order N_{RB}, which is not true for the Taylor method after about R=200.

Researchers have been able to show analytically why the reduced basis method produces a good result when quite near the base solution, but have not been able to address the question of why the approximation can be extended over such a wide range.

4 Flow Optimization Using a Reduced Basis

The problem of flow optimization requires the determination of parameter values that specify a flow which in turn produces the "best" value of some special scalar quantity. For our purposes, we may pose this problem as the search for the parameter value λ^* for which the corresponding flow $(u, v, p)(\lambda^*)$ produces the minimal value of the *cost function* $J(u, v, p, \lambda)$. We may regard (u, v, p) as determined by λ, and consider the equivalent cost function $\mathcal{J}(\lambda)$.

To efficiently solve this problem using optimization software, it is necessary to evaluate both $\mathcal{J}$ and the partial derivatives such as $\mathcal{J}_u$ and $\mathcal{J}_\lambda$ for many argument values. This in turn requires that for many values of λ, the state and sensitivity equations for (u, v, p) be solved.

An optimization problem was set up and solved using the standard finite element basis, and then again using the reduced basis. The formulation began with the driven cavity problem. The parameter, as before, was the Reynolds number R. The cost functional was the integral of the square of the difference between the computed and desired flow profiles along the vertical bisector of the region:

$$\mathcal{J}(\lambda) = \int_{x=0.5} (u(\lambda) - u^*)^2 + (v(\lambda) - v^*)^2 \, dy \tag{7}$$

The desired flow profile data (u^*, v^*) was generated by setting $R^* = 100$ and solving for the corresponding flow. The optimization procedure was then begun with a starting guess $R_0 = 1$. Table **??** lists the sequence of iterates produced during this optimization.

The optimization was repeated using the reduced basis. A full solution was computed at $R = 1$, the reduced basis was set up there, and then the reduced problem was used to evaluate the cost functional. By this means, the optimization code was able to reach $R = 95$, at which point it signaled that no further progress could be made. This was taken to be a signal that the reduced

Table 3: Optimization Results for Full System.

Step	R	$J(R)$
0	1.0000	4.99e-3
5	1.0111	4.99e-3
10	6.1167	4.49e-3
15	100.0002	1.68e-14
	Convergence	

basis should be regenerated. The optimization was restarted from this point, and quickly converged to the desired minimizing value of R.

Table 4: Optimization Results for Reduced System.

Step	R	$J(R)$
0	1.0000	4.99e-3
5	1.0350	4.98e-3
10	12.0375	3.93e-3
15	95.30091	1.99e-5
20	95.30003	1.99e-5
	Restart	
21	95.30003	1.00e-5
25	95.30050	1.00e-5
30	95.51940	9.14e-6
31	95.77294	8.13e-6
32	97.47383	2.89e-6
33	99.63274	6.10e-8
34	99.99644	5.79e-12
	Convergence	

A cursory comparison of the tables shows that the reduced basis method took more steps than the full method. However, the reduced basis method steps are drastically cheaper; there are only 5 nonlinear equations to solve for a state solution, and only a 5 by 5 matrix to invert during the Newton iterations. In contrast, the full system involved 3803 equations, with a matrix of 3803 rows and a bandwidth of 385.

In fact, the major cost for the reduced basis method was in starting and restarting, where the full basis system had to solved.

5 Conclusions

The reduced basis method is fairly easy to implement in a pre-existing finite element code, since the reduced basis method itself can be viewed as a finite element method with a special basis.

When approximating points along a solution curve, the reduced basis method can be much more robust than the Taylor method, producing acceptable solutions far away from the starting point, which can be improved by increasing the degree of the approximation.

The reduced basis method can also be applied to optimization problems, where the primary computational cost is that of repeated system solutions. By steeply cutting this cost, the reduced basis method can solve the same problem much more quickly. To check the accuracy of the result, however, the reduced basis should be regenerated at the optimization candidate, and the optimization restarted once.

For the simple driven cavity problem studied here, the generation of the appropriate reduced basis vectors was easy. This may not be the case when other parameters are varied. Such parameters might control the magnitude or shape of the boundary condition function, or of a source term in the equations. In such cases, a low-order reduced basis system could be generated by finite differences. A second approach would be simply to generate the reduced basis vectors associated with the parameter R. These still represent possible behaviors of the fluid, and may yield a usable approximation to other flows, although without the approximating power available in the usual Taylor space.

References

[1] Janet Peterson, *The Reduced Basis Method for Incompressible Viscous Flows*, SIAM Journal of Scientific and Statistical Computing, volume 10, 1989, pages 777-786.