SPACE-FILLING CURVES (SFC)

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November 2015
OVERVIEW

• Introduction

• Type of Space-Filling Curves
  1. The Peano Space-Filling Curves
  2. The Hilbert Space-Filling Curves
  3. The Sierpinski Space-Filling Curves
  4. The Lebesgue Space-Filling Curves

• Applications of Space-Filling Curves


**SURJECTIVE MAPPING:**

The function is **surjective** (onto) if every element of the codomain is mapped to by at least one element of the domain.

Surjective map from $A$ onto $B$:
\[ f(a) = b \]

If $f : A \rightarrow B$ then $f$ is said to be surjective if
\[ \forall b \in B \, \exists a \in A \, f(a) = b \]
INJECTIVE MAPPING:

An **injective** function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain.

Injective map from A onto B:
\[ f(a) = b \]

If \( f : A \rightarrow B \) then \( f \) is said to be surjective if
\[
\forall a, b \in A, f(a) = f(b) \rightarrow a = b
\]

or
\[
\forall a, b \in A, a \neq b \rightarrow f(a) \neq f(b)
\]
BIJECTIVE MAPPING:

A function is **bijective** if it is both injective and surjective. A function is bijective *if and only if* every possible image is mapped to by exactly one argument.

If \( f : A \rightarrow B \) then \( f \) is said to be bijective if
For all \( b \in B \), there’s a unique \( a \in A \)
Such that \( f(a) = b \)
SPACE-FILLING CURVE (SFC) DEFINITION:

Intuitively, a continuous curve in 2 or 3 (or higher) dimensions can be thought of as the path of a continuously moving point.
HISTORY OF THE HILBERT CURVES:

- Hilbert Curves are named after the German mathematician David Hilbert. They were first described in 1891.

- A Hilbert curve is a continuous space-filing curve. They are also fractal and are self-similar; if you zoom in and look closely at a section of a higher-order curve, the pattern you see looks just the same as itself.

- An easy way to imagine creation of a Hilbert Curve is to envisage you have a long piece of string and want to lay this over a grid of squares on a table. Your goal is to drape the string over board so that the string passes through each square of the board only once.
THE HILBERT CURVE: GEOMETRIC GENERATION:

• If $I$ can be mapped continuously on $\Omega$, then after partitioning $I$ into four congruent subintervals and $\Omega$ into four congruent subsquares, each subinterval can be mapped continuously onto one of the subsquares. This partitioning can be carried out ad infinitum.

• The subsquares must be arranged such that adjacent subintervals are mapped onto adjacent subsquares.

• Inclusion relationship: if an interval corresponds to a square, then its subintervals must correspond to the subsquares of that square.

• This process defines a mapping $f_h(I)$, called the Hilbert space-filling curve.
THE HILBERT CURVE: GEOMETRIC GENERATION:

1st iteration

2nd iteration

3rd iteration
6th Iteration
APPLICATION: 1D INTO 2D:

• We find that if we convert the distance \(d\) shown on the tape-measure to a pair of \((x,y)\) coordinates on the grid, then we find that other marks on the tape measure (close to \(d\)) have \((x,y)\) coordinates that are very close.

• We have created a useful mapping function between one-dimension and two-dimensions that does a good job of preserving locality.
• To the right you can see the rainbow painted, using a Hilbert Curve. Colors that are close to each other on the spectrum have similar (x,y) coordinates.

• For more info:
  https://www.jasondavies.com/hilbert-curve/
HIGHER DIMENSIONS:

• The characteristics of Hilbert Curves can be extended to more than two dimensions. A 1D line can be wrapped up in as many dimensions as you can imagine.

• A rendering of a curve in 3D can is shown on the left. In this shape, points on the line numerically close to one another are also close in 3D space.

• It's easy to see how this can be extended into higher dimensions.