MATHEMATICAL MODELING OF ATMOSPHERIC AND OCEANIC FLOWS

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Lecture 10

§4.5: Dimensionless Numbers

- $$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f\, v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

§4.5: Dimensionless Numbers

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- $$\frac{U}{T} + \frac{U^2}{L} + \frac{U^2}{L} + \frac{W\, U}{H} - \Omega\, U = - \frac{P}{\rho_0\, L} + \nu \frac{U}{H^2}$$

§4.5: Dimensionless Numbers

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- $$\frac{1}{\Omega\,T} + \frac{U}{\Omega\,L} + \frac{U}{\Omega\,L} + \frac{W\,L}{U\,H}\cdot\frac{U}{\Omega\,L} - 1 = -\frac{P}{\rho_0\,\Omega\,L\,U} + \frac{\nu}{\Omega\,H^2}$$

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- $\dfrac{1}{\Omega} \lesssim T \quad\Longrightarrow\quad \dfrac{1}{\Omega\,T} \lesssim 1$
- $Ro_T = \dfrac{1}{\Omega\,T}$ temporal Rossby number

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- discard?
- NO! Ekman layer.

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- Re large

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- $\boxed{Ri = \dfrac{g\,H\,\Delta\rho}{\rho_0\,U^2}}$ Richardson number

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