MATHEMATICAL GAMES
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Source: *Scientific American*, Vol. 207, No. 4 (October 1962), pp. 130-143
Published by: Scientific American, a division of Nature America, Inc.
Stable URL: https://www.jstor.org/stable/10.2307/24936722

MATHEMATICAL GAMES

A collection of puzzles involving numbers, logic and probabilities

by Martin Gardner

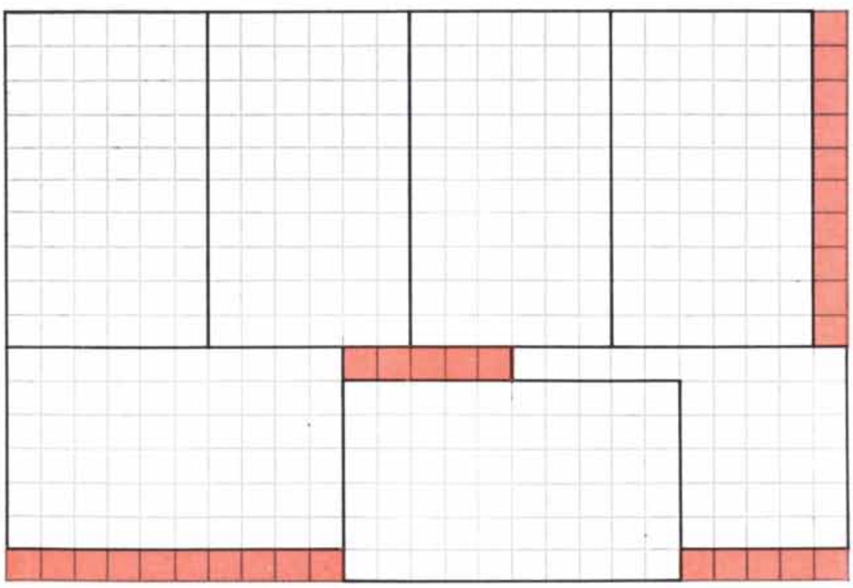
Every eight months or so this column presents a selection of short problems of various types. The nine problems that follow can for the most part be solved by anyone who "thinks mathematically." The only one that calls for knowledge of mathematics beyond the high school level is the third, Leo Moser's problem concerning a family chess tournament, but even here the probability theory involved is elementary. Answers to all problems will be given next month.

1.

A sheet of legal-sized paper, 8½ by 12½ inches, has an area of 106¼ square inches. Seven file cards of the three-by-five-inch size have a combined area of 105 square inches. Obviously it is not possible to cover the large sheet completely with the seven cards, but what is the largest area that *can* be covered? The cards must be placed flat, and they may not be folded or cut in any way. They may overlap the edges of the sheet, however, and it is not necessary for their sides to be parallel with the sides of the sheet. The illustration below shows how the seven cards can be arranged to cover an area of 98¾ square inches. This is not the maximum.

Everyone in the family, young and old, will enjoy working on this puzzle. If the required materials are not handy, a sheet of cardboard can be cut to the 8½-by-12½-inch size, and the seven three-by-five rectangles can be cut from paper. It is a good plan to rule the large sheet into half-inch squares so that the area left exposed can be computed quickly.

The problem has at least two elements of surprise. It was first posed last December by Jack Halliburton in *Recreational Mathematics Magazine,* and the first surprise was the discovery by readers of an answer that exceeded Halliburton's. The second surprise was the more recent discovery, by Stephen Barr, of a still better solution. Barr's solution will be disclosed here next month for



How much of the paper can be covered with seven file cards?

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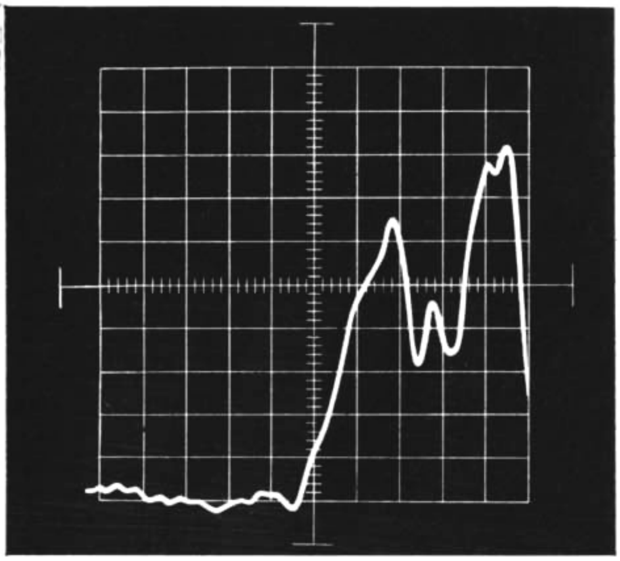
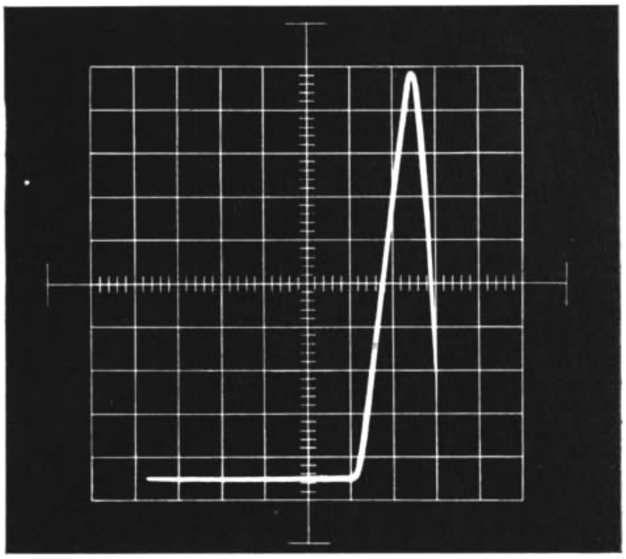


This is an example of the force-time relationship of impact printing as the striking head of an output printing device comes into contact with its platen.



After modifications were made in the printing mechanism, the force-time relationship reflected a cleaner impact, producing a sharper impression from the printer.

As computers go faster, so must the machines which work with them. The tremendous speed at which these mechanisms function magnifies traditional problems of wear, elasticity, and timing. To meet rigorous new standards of performance, IBM engineers are applying the techniques of mechanical analysis to the development of high-speed mechanisms.

Through a combination of experimental measurements and mathematical analyses, IBM engineers attempt to determine exactly what is taking place at any given instant during the machine cycle. For example, our engineers recently were given the assignment of increasing the operating speed of the output printing element in a data processing system by as much as twenty-five percent without major redesign. To do so, they progressively varied mass, spring forces, and the elasticity of several components, and measured resulting changes in the system's physical characteristics. By translating these experimental results into mathematical terms, they were able to arrive at a solution to the problem which might have been impossible to obtain using only trial-and-error methods.

The exact description of the way in which many variables interact during the machine cycle requires advanced measurement and analysis techniques. For example, the impact curves shown above were produced by a mechanical structure with several degrees of freedom—a system extremely complicated to calculate. IBM engineers chose to alter one structural member slightly to produce an overriding, dominant frequency. They could then analyze the structure's operation mathematically.

Mechanical analysis is helping IBM engineers to create mechanisms with fundamentally new designs. For instance, in a revolutionary new typewriter, the IBM Selectric,™ they were able to design a mechanism to compensate automatically for wear and elasticity in the print-selection system. In an important new development in disk storage, they used a hydraulic actuator to position magnetic read-write heads which float on air bearings. And they have developed a high-speed printer, controlled by a computer's program, which prints by means of engraved type suspended on a chain which moves past the paper at high speed. Mechanisms with advanced designs like these are helping to make it possible to take full advantage of the immense capabilities of the computer.

If you have been searching for an opportunity to make important contributions in mechanical analysis, space, programming systems, or any of the other fields in which IBM scientists and engineers are finding answers to basic questions, please contact us. IBM is an Equal Opportunity Employer. Write to: Manager of Professional Employment, IBM Corp., Dept. 659X, 590 Madison Avenue, N. Y. 22, N. Y.

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    E E O          P P P
      O O            P P
  -------        -------
  E O E O        P P P P
  E O O        P P P P
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O O O O O      P P P P P
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Two unorthodox cryptarithms

the first time. There is as yet no proof that Barr's solution is the maximum, so perhaps some reader of this department will find a third level of surprise.

2.

Six Hollywood stars form a social group that has very special characteristics. Every two stars in the group either mutually love each other or mutually hate each other. There is no set of three individuals who mutually love one another. Prove that there is at least one set of three individuals who mutually hate each other. The problem leads into a fascinating new field of graph theory, "blue-empty chromatic graphs," the nature of which will be explained when the answer is given next month.

3.

A certain mathematician, his wife and their teen-age son all play a fair game of chess. One day when the son asked his father for $10 for a Saturday night date, his father puffed on his pipe a moment and replied:

"Let's do it this way. Today is Wednesday. You will play a game of chess tonight, another tomorrow and a third on Friday. Your mother and I will alternate as opponents. If you win two games in a row, you get the money."

"Whom do I play first, you or Mom?"

"You may have your choice," said the mathematician, his eyes twinkling.

The son knew that his father played a stronger game than his mother. To maximize his chance of winning two games in succession, should he play father-mother-father or mother-father-mother?

Leo Moser, a mathematician at the University of Alberta, is responsible for this amusing question in elementary probability theory. Of course you must prove your answer, not just guess.

4.

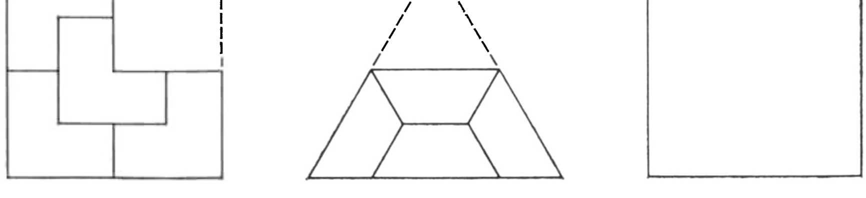
In most cryptarithms a different letter is substituted for each digit in a simple arithmetical problem. The two remarkable cryptarithms shown above are unorthodox in their departure from this practice, but each is easily solved by logical reasoning and each has a unique answer.

In the multiplication problem at the left in the illustration, newly devised by Fitch Cheney of the University of Hartford, each *E* stands for an even digit, each *O* for an odd digit. The fact that every even digit is represented by *E* does not mean, of course, that all the even digits are the same. For example, one *E* may stand for 2, another for 4, and so on. Zero is considered an even digit. The reader is asked to reconstruct the numerical problem.

In the multiplication problem at the right, each *P* stands for a prime digit (2, 3, 5 or 7). This charming problem was first proposed some 25 years ago by Joseph Ellis Trevor, a chemist at Cornell University. It has since become a classic of its kind.

5.

If one-fourth of a square is taken from its corner, is it possible to dissect the remaining area into four congruent (same size and shape) parts? Yes, it can be done in the manner shown at the left in the illustration below. Similarly,



Three dissection puzzles

an equilateral triangle with one-fourth of its area cut from a corner, as in the center figure in the illustration, can also be divided into four congruent parts. These are typical of a large variety of geometric puzzles. Given a certain geometric figure, the task is to cut it into a specified number of identical shapes that completely fill the larger figure.

Can the square at the right in the illustration be dissected into five congruent parts? Yes, and the answer is unique. The pieces can be any shape, however complex or bizarre, provided that they are identical in shape and size. An asymmetric piece may be "turned over"; that is, it is considered identical with its mirror image. The problem is annoyingly intractable until suddenly the solution strikes like lightning.

6.

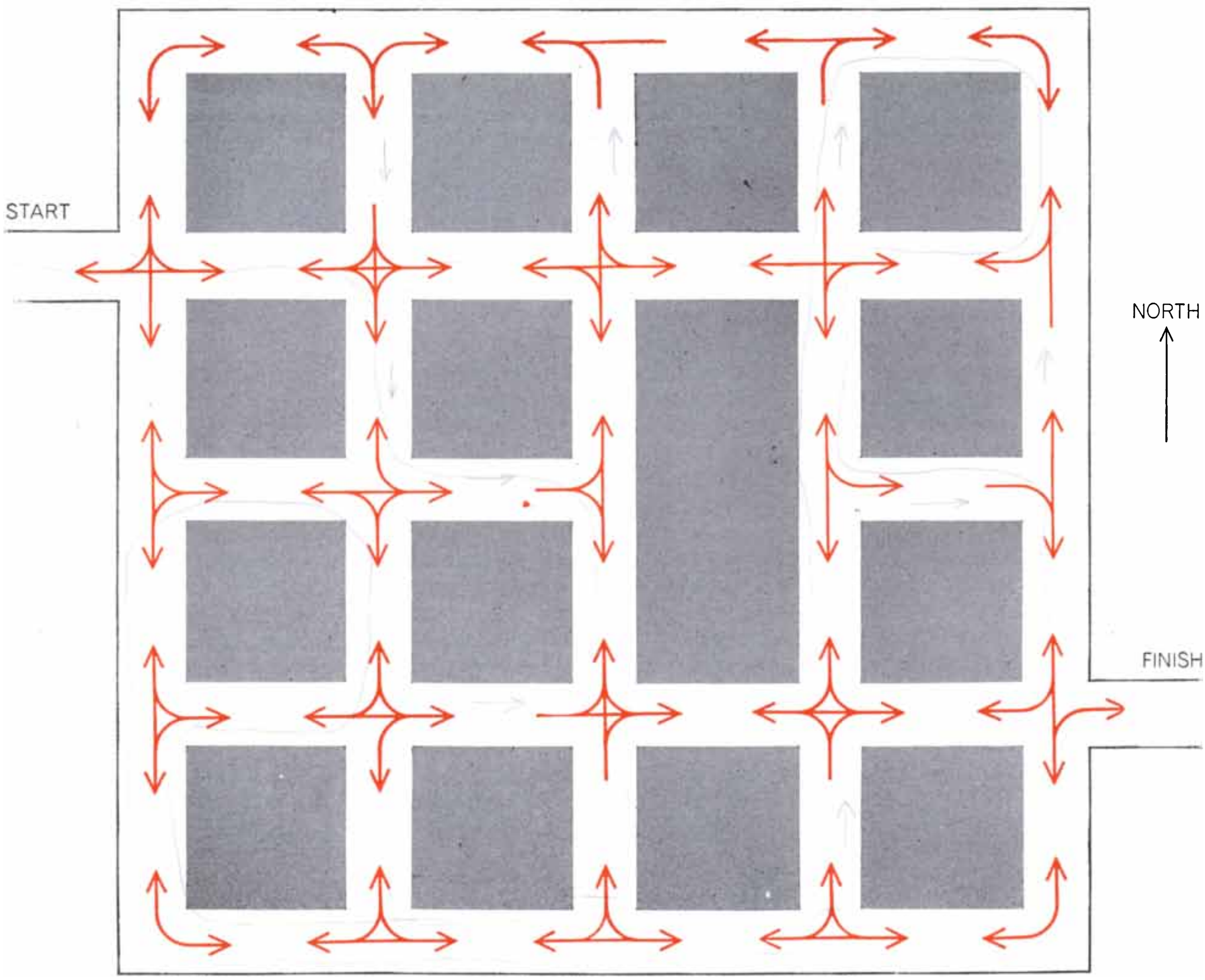
Robert Abbott, author of the recently published book *Four Card Games*, provided the curious street map reproduced below, accompanied by the following story:

"Because the town of Floyd's Knob, Indiana, had only 37 registered automobiles, the mayor thought it would be safe to appoint his cousin, Henry Stables, who was the town cutup, as its traffic commissioner. But he soon regretted his decision. When the town awoke one morning, it found that a profusion of signs had been erected establishing numerous one-way streets and confusing restrictions on turns.

"The citizens were all for tearing down these signs until the police chief, another cousin of the mayor, made a surprising discovery. Motorists passing through town became so exasperated that sooner or later they made a prohibited turn. The police chief found that the town was making even more money from these violations than from its speed trap on an outlying country road.

"Of course everyone was overjoyed, particularly because the next day was Saturday and Moses MacAdam, the county's richest farmer, was due to pass through town on his way to the county seat. They expected to extract a large fine from Moses, believing it to be impossible to drive through town without at least one traffic violation. But Moses had been secretly studying the signs. When Saturday morning came, he astonished the entire town by driving from his farm through town to the county seat without a single violation!

"Can you discover the route Moses took? At each intersection you must follow one of the arrows. That is, you may turn in a given direction only when there is a curved line in that direction, and you may go straight only when there is a straight line to follow. You may leave an intersection only at the head of an arrow. For instance, at the first intersection after leaving the farm, you have only two choices: to go north or to go straight. If you go straight, at the next intersection you must either go straight or turn south. True, there is a curved line to the north, but there is no arrow pointing



The traffic maze in Floyd's Knob

north, so you are forbidden to leave that intersection in a northerly direction."

7.

Every now and then a magazine runs a cover picture that contains a picture of the same magazine, on the cover of which one can see a still smaller picture of the magazine, and so on presumably to infinity. Infinite regresses of this sort are a common source of confusion in logic and semantics [see "Paradox," by W. V. Quine; SCIENTIFIC AMERICAN, April, 1962]. Sometimes the endless hierarchy can be avoided, sometimes not. The English mathematician J. E. Littlewood, commenting on this topic in one of his books, recalls three footnotes that appeared at the end of one of his papers. The paper had been published in a French journal. The notes, all in French, read:

"1. I am greatly indebted to Prof. Riesz for translating the present paper.

"2. I am indebted to Prof. Riesz for translating the preceding footnote.

"3. I am indebted to Prof. Riesz for translating the preceding footnote."

Assuming that Littlewood was completely ignorant of the French language, on what reasonable grounds did he avoid an infinite regress of identical footnotes by stopping after the third footnote?

8.

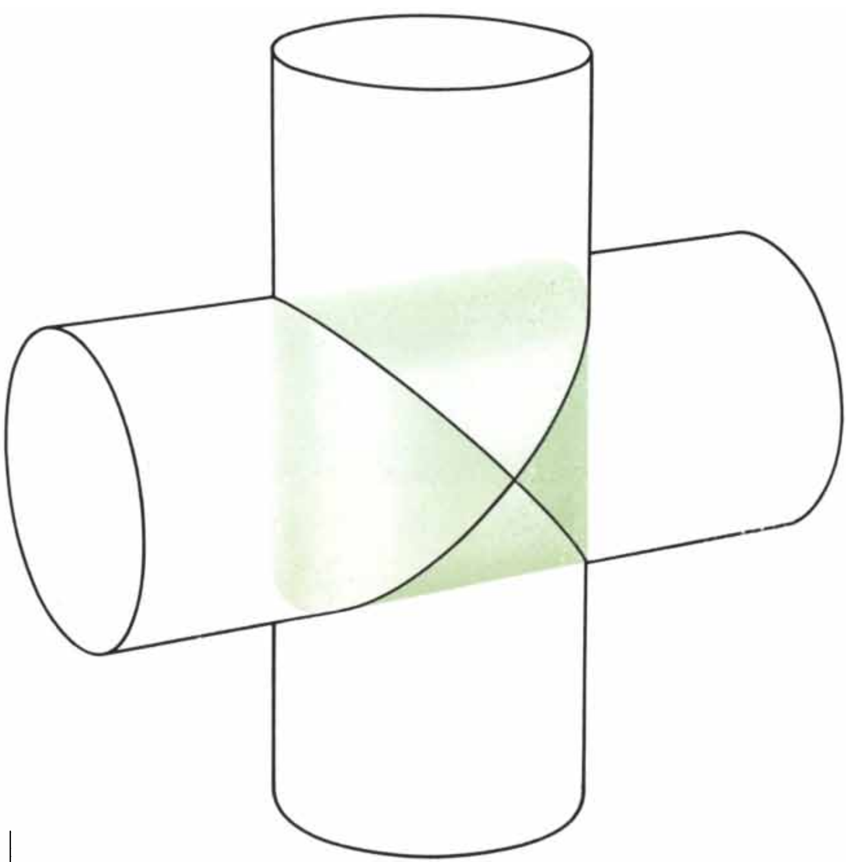
An old numerical problem that keeps reappearing in puzzle books as though it had never been analyzed before is the problem of inserting mathematical signs wherever one likes between the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 to make the expression equal 100. The digits must remain in the same sequence. There are many hundreds of solutions, the easiest to find perhaps being

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + (8 \times 9) = 100 .$$

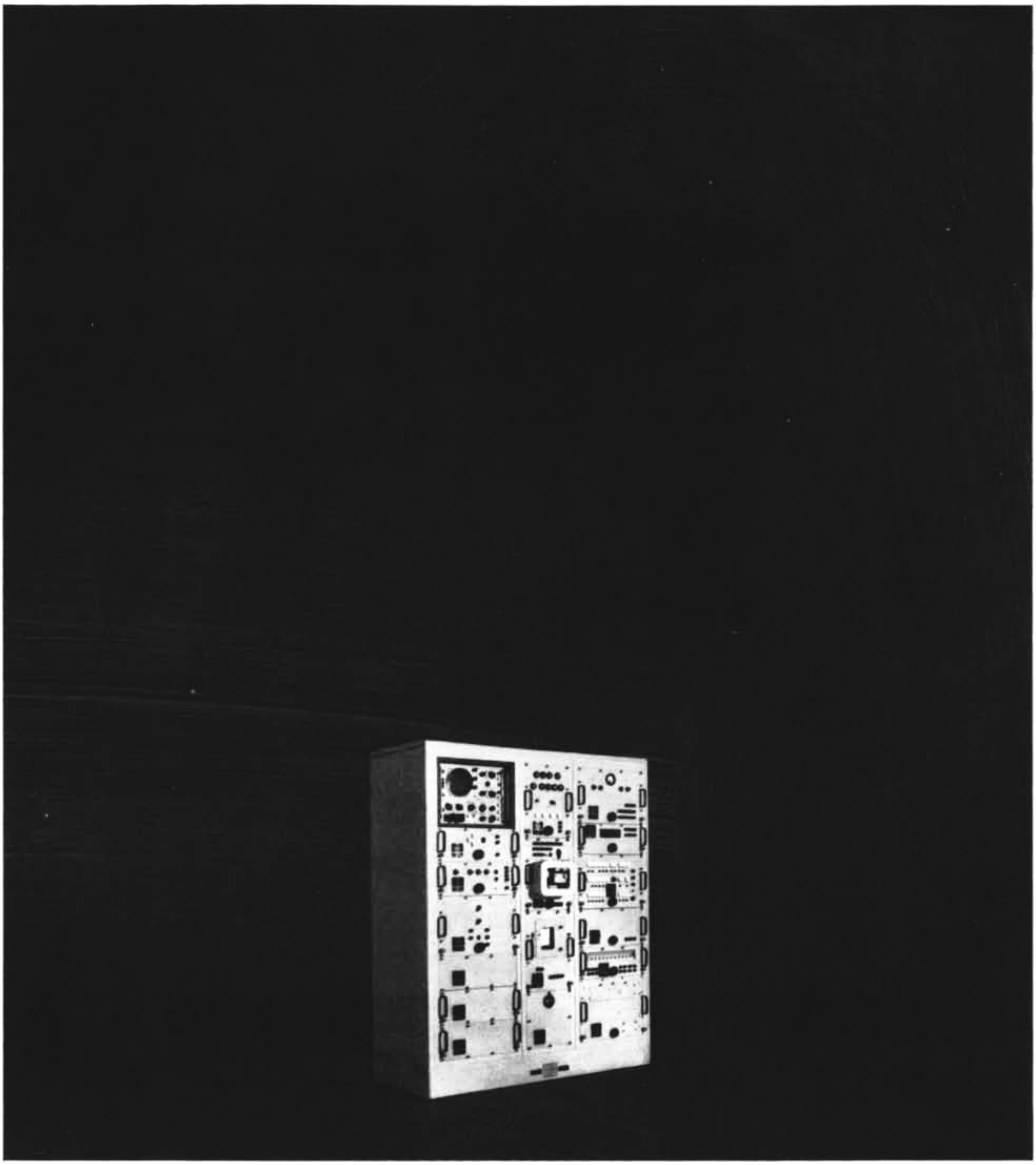
The problem becomes more of a challenge if the mathematical signs are limited to plus and minus. Here again there are many solutions, for example

$$1 + 2 + 34 - 5 + 67 - 8 + 9 = 100 ,$$
$$12 + 3 - 4 + 5 + 67 + 8 + 9 = 100 ,$$
$$123 - 4 - 5 - 6 - 7 + 8 - 9 = 100 ,$$
$$123 + 4 - 5 + 67 - 89 = 100 ,$$
$$123 + 45 - 67 + 8 - 9 = 100 ,$$
$$123 - 45 - 67 + 89 = 100 .$$

"The last solution is singularly simple," writes the English puzzlist Henry



Archimedes' problem of the crossed cylinders



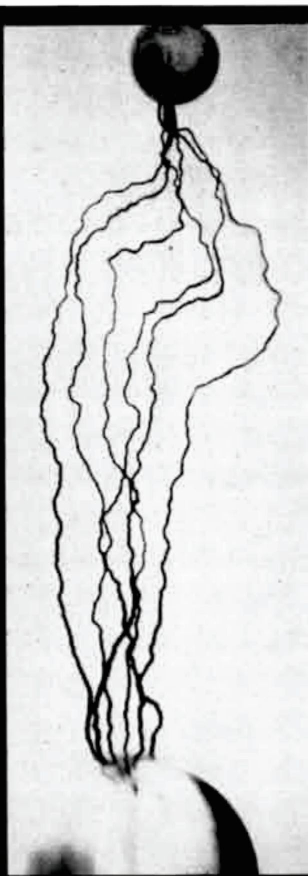
Blabbermouth

NARATE is an automatic radar test system developed for the Navy by Northrop. One of its main functions is to warn the radar operator when anything goes wrong with his equipment. But that's just the first step.

When it discovers a fault in the radar, NARATE can be programmed to find out what the trouble is, and isolate it down to the lowest removable unit. Or it can be directed to bypass the fault in order to continue the test.

NARATE has been tested at the Navy Electronics Laboratory, San Diego, for integration and compatibility with a complex air search radar. It is now installed aboard the USS Columbus for evaluation in fleet operations. Based on the proven Datico automatic test equipment concept, NARATE was developed for the Navy by the Nortronics Division of Northrop.

NORTHROP



Ernest Dudeney in the answer to Problem No. 94 in his *Amusements in Mathematics*, "and I do not think it will ever be beaten." As far as I know, Dudeney's claim has never been challenged.

In view of the popularity of this problem it is surprising that so little effort seems to have been spent on the problem in reverse form. That is, take the digits in descending order, 9 through 1, and form an expression equal to 100 by inserting the smallest possible number of plus or minus signs. The answer to this problem is still open, but next month I shall give the best solution I know.

9.

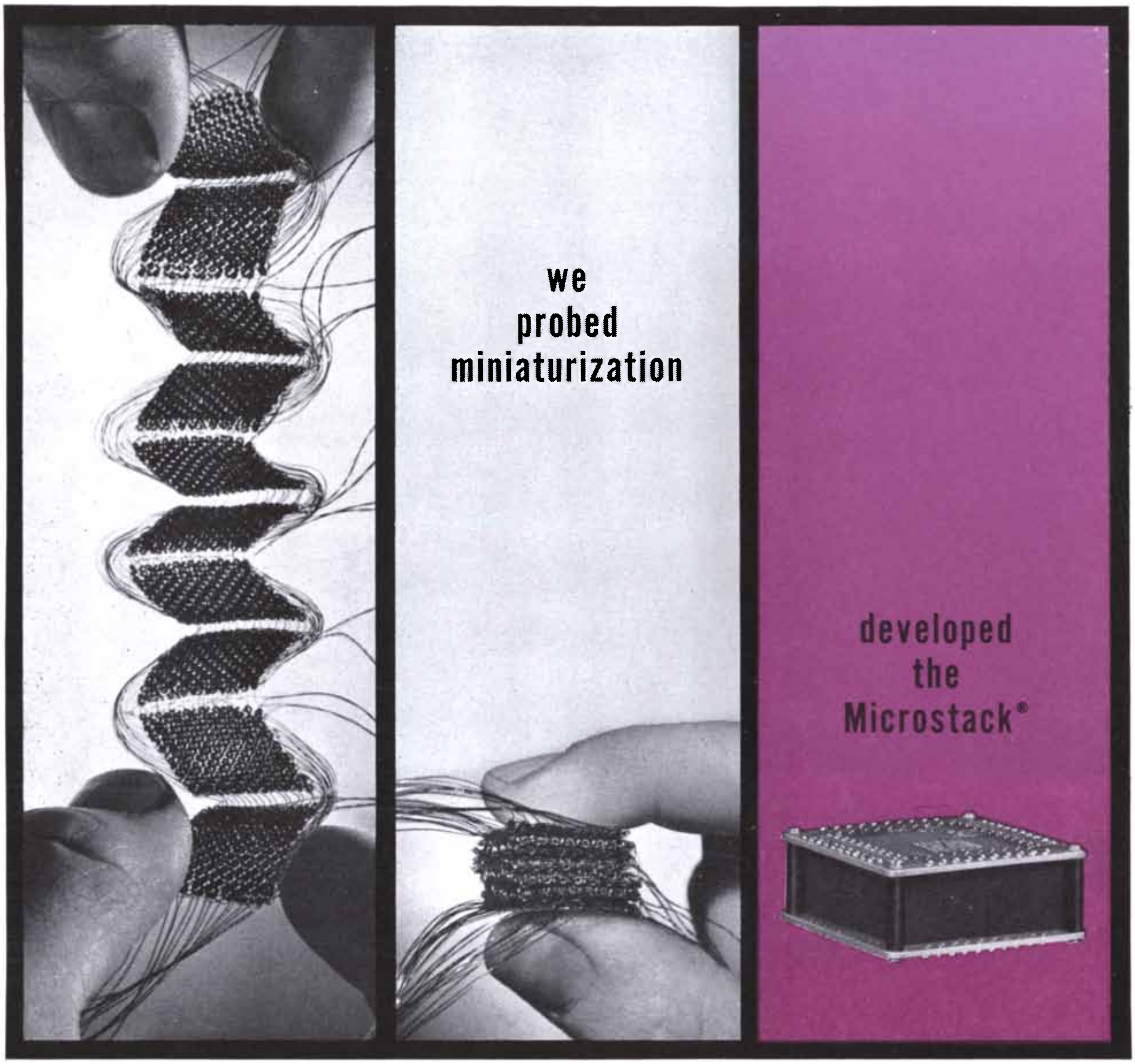
One of Archimedes' greatest achievements was his anticipation of some of the fundamental ideas of calculus. The problem illustrated on page 136 is a classic example of a problem that most mathematicians today would regard as unsolvable without a knowledge of calculus (indeed, it is found in many calculus textbooks) but that yielded readily to Archimedes' ingenious methods. The two circular cylinders intersect at right angles. If each cylinder has a radius of one unit, what is the volume of the shaded solid figure that is common to both cylinders?

No surviving record shows exactly how Archimedes solved this problem. There is, however, a startlingly simple way to obtain the answer; in fact, one need know little more than the formula for the area of a circle (pi times the square of the radius) and the formula for the volume of a sphere (four-thirds pi times the cube of the radius). It may have been the method Archimedes used. In any case, it has become a famous illustration of how calculus often can be completely side-stepped by finding a simple approach to a problem.

The six divisibility problems presented in last month's department are solved as follows:

1. To prove that a number of the form ABABAB must be evenly divisible by 7, we have only to note that such a number is the product of AB and 10101. Because 10101 is a multiple of 7, the number ABABAB must be also.

2. When the digits 1 to 7 are randomly arranged to form a number, the probability that the number is divisible by 11 is 4/35. To be divisible by 11 the digits must be arranged so that the difference between the sum of one set of alternate digits and the sum of the other set of alternate digits is either 0 or a multiple of 11. The sum of all seven



Indiana General engineers probed new ways to increase the packaging density and extend the operating temperature range of memory stacks. By using a continuous wire method of inserting drive lines through memory planes — and then folding the individual arrays — the MICROSTACK was developed. This unique stringing technique eliminates bulky frames and all solder connections between planes. As a result, a MICROSTACK slashes space requirements 90% compared to conventional stacks and offers new standards of reliability. ■ IGC MICROSTACKS are available with up to 32,768 cores in a single matrix enclosure. Individual units may be stacked to provide any desired memory capacity. Temperature-controlled MICROSTACKS operate between −55° C and +85° C. ■ **MEMORY SYSTEMS ARE OUR BUSINESS.** MICROSTACKS are typical of the advanced engineering you receive when you let IGC engineers solve your memory system problems. **Write today for facilities brochure to** Indiana General Corporation, Electronics Division, Keasbey, New Jersey.

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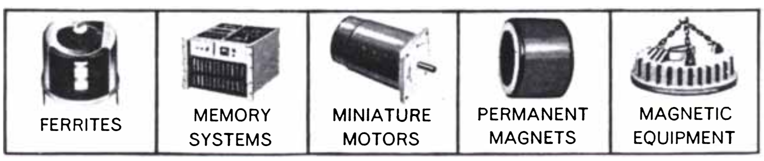


digits is 28. It is easy to find that 28 can be partitioned in only two ways that meet the 11 test: 14|14 and 25|3. The 25|3 partition is ruled out because no sum of three different digits can be as low as 3. Therefore only the 14|14 partition need be considered. There are 35 different combinations of three digits that can fall into the B positions in the number ABABABA. Of the 35, only four (167, 257, 347, 356) sum to 14. Therefore the probability that the number will be divisible by 11 is 4/35.

3. The smallest number that has a remainder of one less than the divisor, when divided by each integer from 2 to 10 inclusive, is 2519. It is amusing to note that "Professor Hoffmann," in his book *Puzzles Old and New* (1893), calls this a "difficult problem" and devotes more than two pages to solving it by a complicated application of divisibility rules. Hoffmann failed to note that each division falls just one short of being exact, so we need only to find the lowest common multiple of 2, 3, 4, 5, 6, 7, 8, 9, 10, which is 2520, then subtract 1 to get the answer.

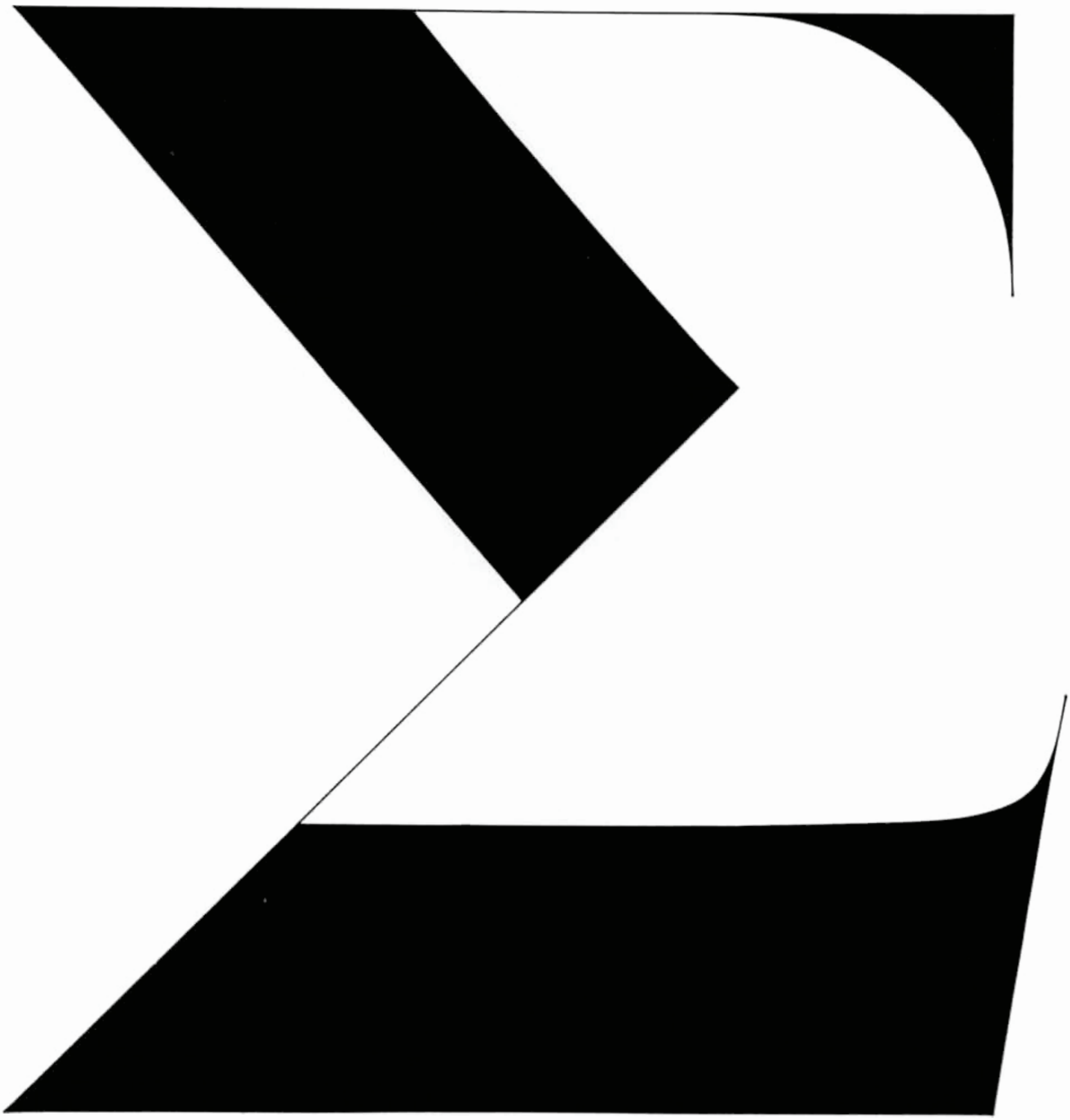
4. The problem of the cube with the missing edge of smaller cubes is equivalent to showing that a number of the form $n^3 - n$ (where n is any positive integer greater than 1) must always be evenly divisible by 6. The following is perhaps the simplest proof:

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

The expression to the right of the second equal sign reveals that the number $(n^3 - n)$ is the product of three consecutive integers. In any set of three consecutive integers, it is easy to see that one integer must be divisible exactly by 3 and that at least one integer must be even. (These two properties may, to be sure, unite in the *same* integer, e.g., 17, *18*, 19.) Since 2 and 3 are factors of the product of the three consecutive integers, the product must be divisible by 2×3, or 6.

5. The remainder, when 3 to the power of 123456789 is divided by 7, is 6. The short cut here is that successive powers of 3, when divided by 7, have remainders that repeat endlessly the six-digit cycle 3, 2, 6, 4, 5, 1. Divide 123456789 by 6 to obtain a remainder of 3, then note the third digit in the cycle. It is 6, the answer to the problem.

6. The problem asked for a set of four different digits, excluding 0, that could not be arranged to make a four-digit number divisible by 7. Of the 126 different combinations of four digits, only three work: 1238, 1389 and 2469.




This is the source of new
scientific measuring accuracy

PROGRESS

in scientific measurement depends in part on continuing improvement of the oscilloscope, a basic measuring tool that visually displays repetitive or fleeting electrical phenomena.

Dramatic improvements in circuitry have made the oscilloscope a versatile and indispensable tool. Yet, refined and sophisticated as scope circuitry may be, the limitations of conventional cathode ray tubes can create reading errors which only improved CRT design can correct. Hewlett-Packard has combined both electronic and *manufacturing* achievements to produce oscilloscope CRTs that come closer than ever before to eliminating human reading error.

Three basic improvements have been made: a 50% greater picture size on a high-frequency scope, removal of parallax viewing error, and the elimination of reflected glare.

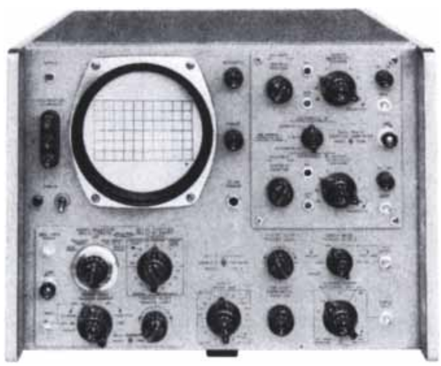
For high speed (50 MC) oscillography, hp has produced an entirely new cathode ray display tube. This new 12,000 volt CRT displays high speed electrical signals on a full 6 cm x 10 cm calibrated screen (50% larger than previously available for high frequencies). Deflection defocusing has been eliminated, and the sensitivity of the CRT has been doubled. Less amplification is needed to swing the electron beam so simpler, more reliable driving circuits can be used. The 50% larger display that stays in focus at full deflection significantly reduces the reading error.

Second, hp developed a manufacturing technique to eliminate a parallax viewing error which, on most oscilloscopes sold today, can be as great as 5% unless every portion of the trace is viewed from precisely in front of that part of the screen. This parallax error was caused by a separation as great as ¼″ between the trace, falling on the inside of the CRT face, and the square-centimeter-scribed scaling graticule, placed on the outside surface of the CRT face. hp completely eliminated parallax error by placing the graticule on the inside of the tube, in the same plane as the trace. Readout is now identical from any angle.

Finally, glare and reflections from ambient light have always plagued scope users. Viewing screens blocked out reflected light but limited viewing to one person at a time. Now, an etching process on the surface of the safety glass face plate on hp CRTs eliminates glare, at the same time preserving clear viewing.

Illustrated here is the electron gun against a background of flame from the gas jets of the assembly fixture.

Hewlett-Packard oscilloscope technology includes the continuing improvement of cathode ray tube design and construction for new scientific measuring accuracy.



hp 175A 50 MC Universal Oscilloscope

is the first hp scope to incorporate the new 12 Kv no-parallax CRT as standard equipment, although the no-parallax feature is available as standard or as an optional extra on other hp scopes. The 175A offers a 50 MC main vertical amplifier; plug-in versatility for dual trace viewing to 40 MC, high sensitivity, sweep delay, time marker, X-Y recorder output; simple calibration and maintenance; preset automatic trigger over entire bandwidth; new hp modular packaging for bench and rack mount in a single instrument. $1,325.00 (Basic instrument. Versatile plug-ins optional at extra cost.)

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