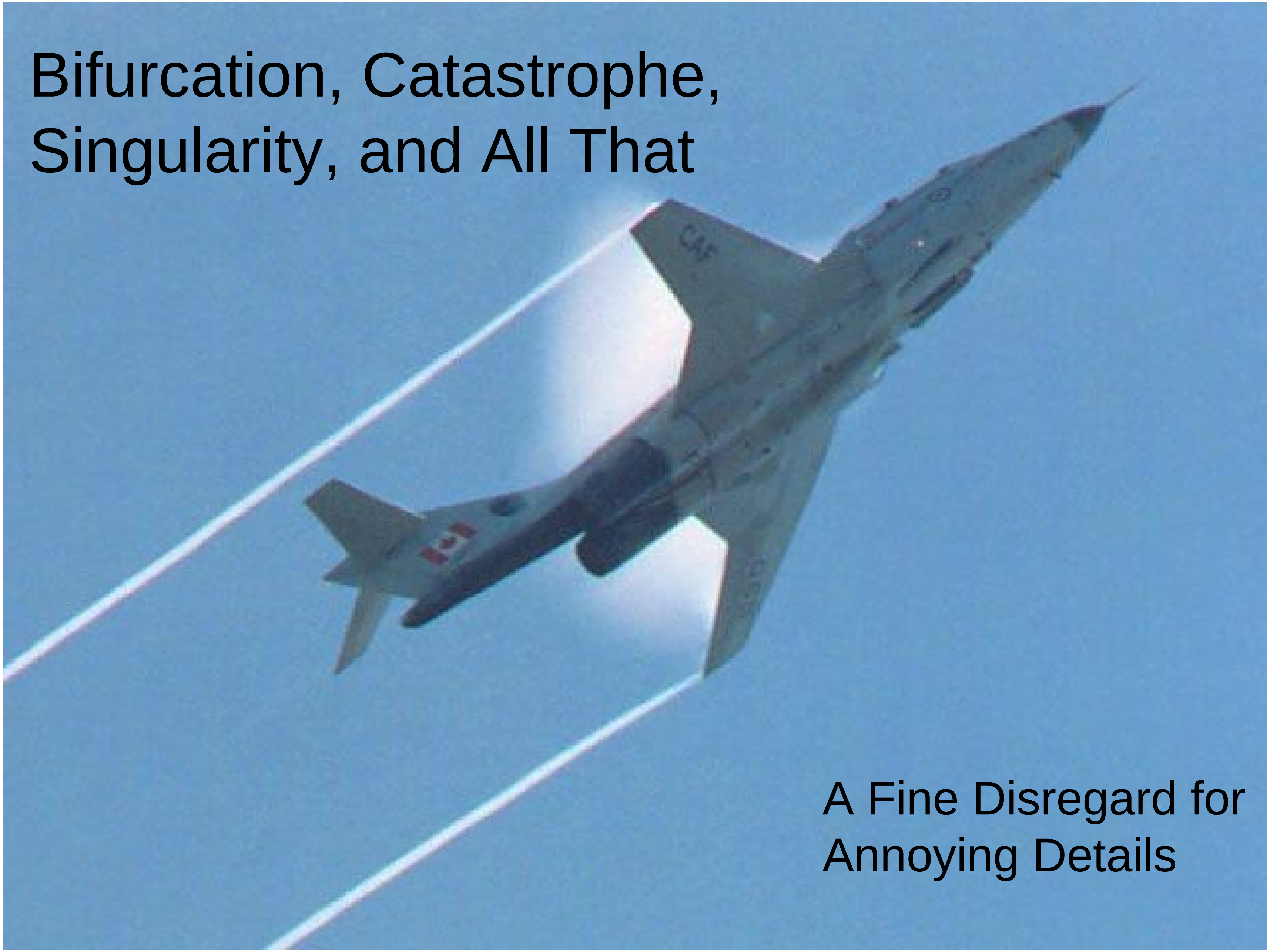
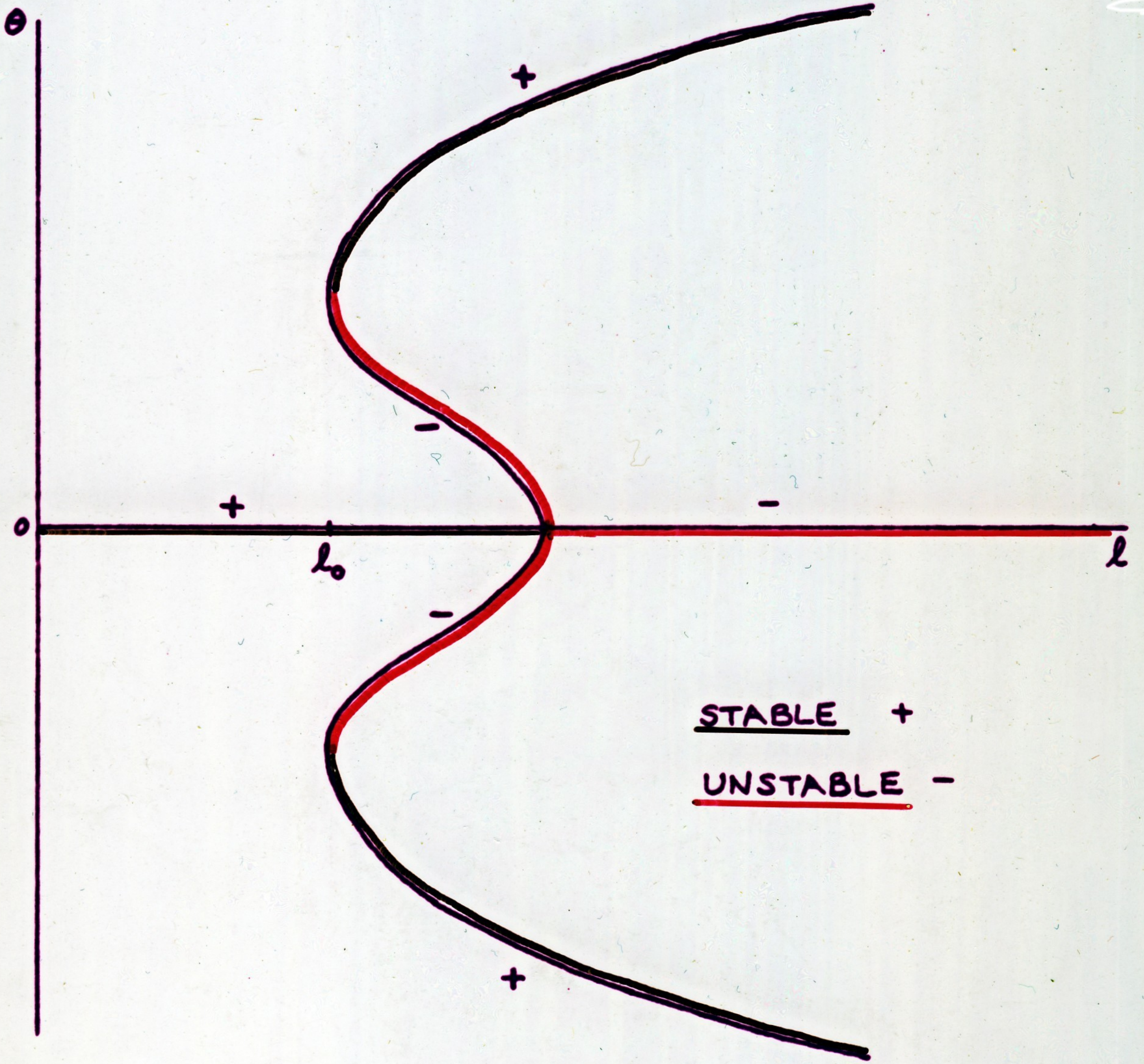


Bifurcation, Catastrophe, Singularity, and All That

A Fine Disregard for
Annoying Details

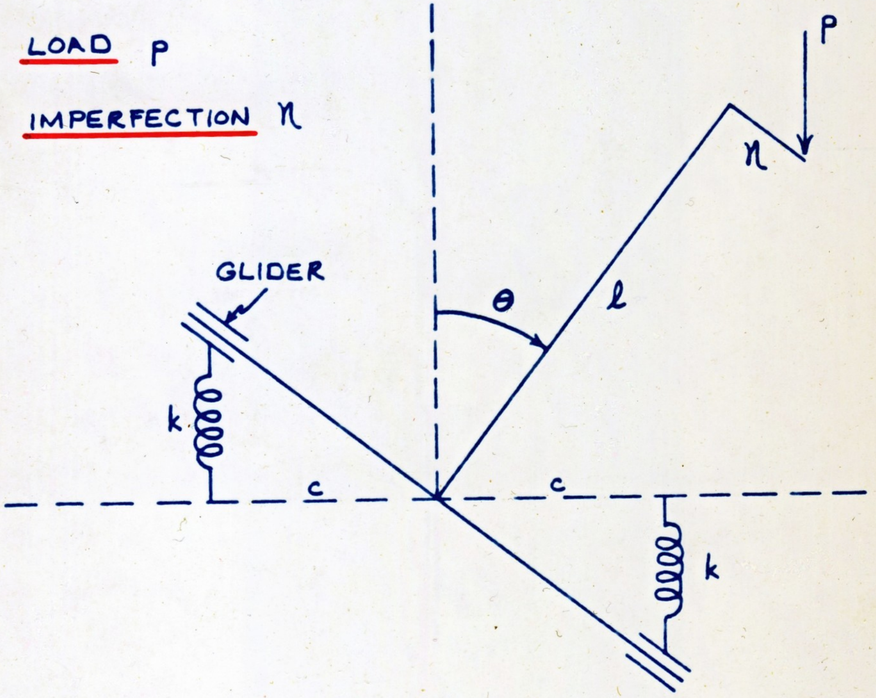
FLEXIBLE COLUMN - BIFURCATION DIAGRAM



BUCKLING OF STRUT WITH IMPERFECTION

LOAD P

IMPERFECTION η



BUCKLING OF STRUT WITH IMPERFECTION

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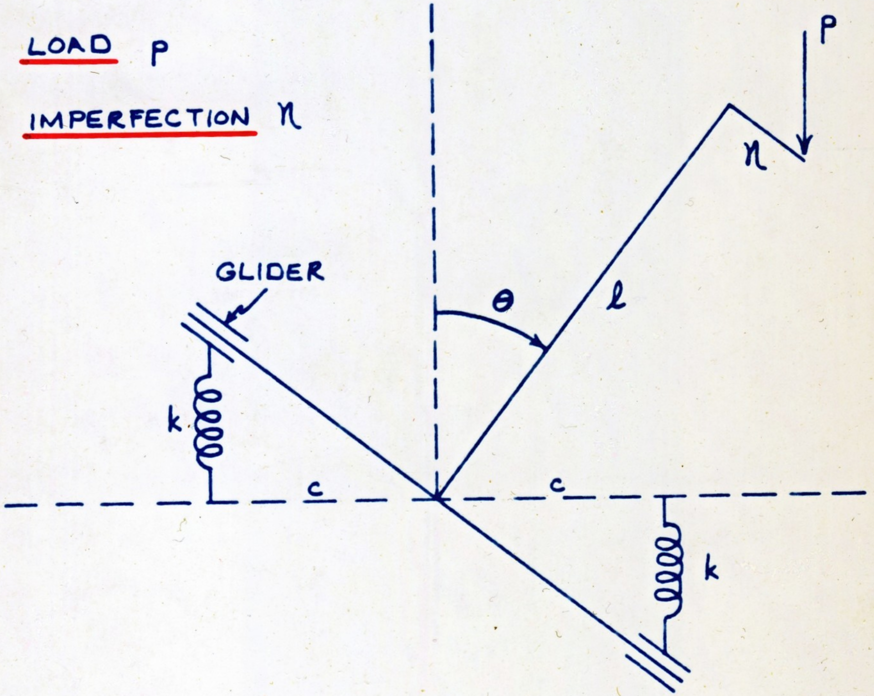

P.E. OF SPRINGS $= 2 \cdot \frac{1}{2} k (c \tan\theta)^2$

BUCKLING OF STRUT WITH IMPERFECTION

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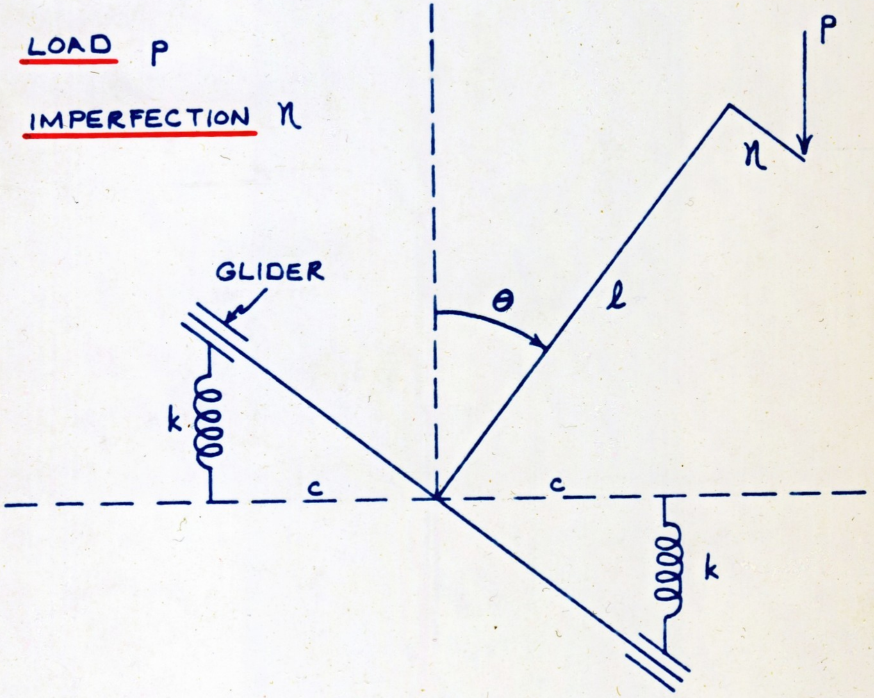


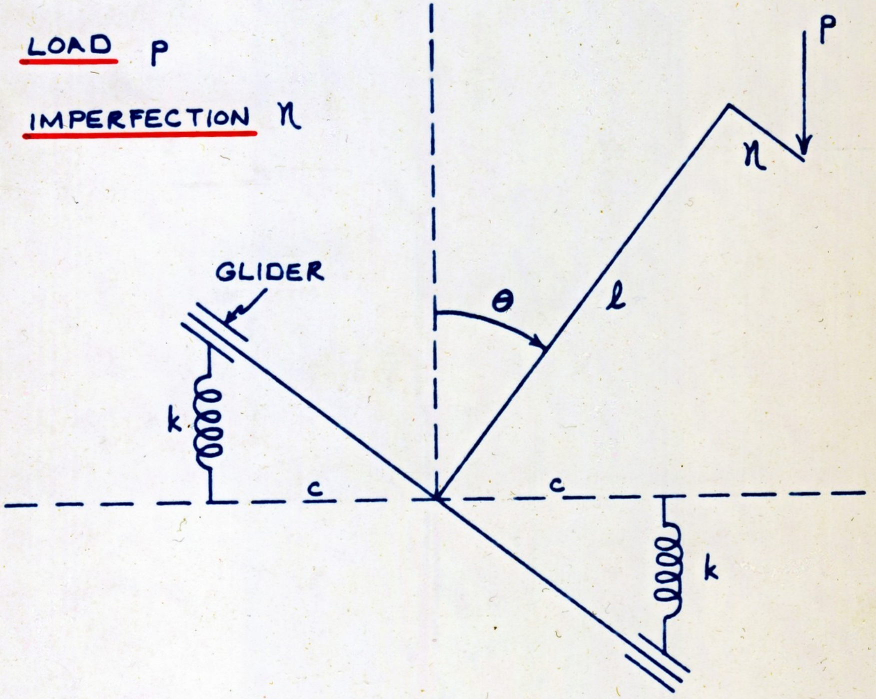
P.E. OF SPRINGS $= 2 \cdot \frac{1}{2} k \left(c \tan\theta\right)^2$

P.E. OF LOAD $= -p\left(\ell - \ell\cos\theta + \eta\sin\theta\right)$

BUCKLING OF STRUT WITH IMPERFECTION

LOAD p

IMPERFECTION η



P.E. OF SPRINGS $= 2\cdot\frac{1}{2}k(c\tan\theta)^2$

P.E. OF LOAD $= -p(\ell-\ell\cos\theta+\eta\sin\theta)$

TOTAL P.E. $V = kc^2\tan^2\theta - p(\ell-\ell\cos\theta+\eta\sin\theta)$

$$\approx \frac{3}{8}\theta^4 - \frac{1}{2}(p-1)\theta^2 - p\eta\theta \qquad \left[\begin{array}{l}\ell = 1\\ kc^2 = \frac{1}{2}\end{array}\right.$$

POTENTIAL ENERGY OF STRUT

TOTAL P.E. $V = kc^2\tan^2\theta - p(\ell - \ell\cos\theta + \eta\sin\theta)$

$$= \frac{1}{2}\tan^2\theta - p + p\cos\theta - p\eta\sin\theta \qquad \left[\begin{array}{l} \ell = 1 \\ kc^2 = \frac{1}{2} \end{array}\right.$$

POTENTIAL ENERGY OF STRUT

TOTAL P.E. $V = kc^2\tan^2\theta - p(\ell - \ell\cos\theta + \eta\sin\theta)$

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ASSUME: η, θ SMALL, p CLOSE TO 1 SO THAT

- TERMS ONLY UP TO θ^4 ARE SIGNIFICANT
- $p\eta\theta^3$ IS SMALL COMPARED TO $p\eta\theta$
- $(p-1)\theta^4$ IS SMALL COMPARED TO $(p-1)\theta^2$

POTENTIAL ENERGY OF STRUT

TOTAL P.E. $V = kc^2 \tan^2\theta - p(\ell - \ell\cos\theta + \eta\sin\theta)$

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- $p\eta\theta^3$ IS SMALL COMPARED TO $p\eta\theta$
- $(p-1)\theta^4$ IS SMALL COMPARED TO $(p-1)\theta^2$

$$V = \tfrac{1}{2}\tan^2\theta - (1-\cos\theta) - (p-1)(1-\cos\theta) - p\eta\sin\theta$$

$$= \tfrac{1}{2}\left(\theta + \frac{\theta^3}{3} + \cdots\right)^2 - \left(1 - 1 + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right)$$

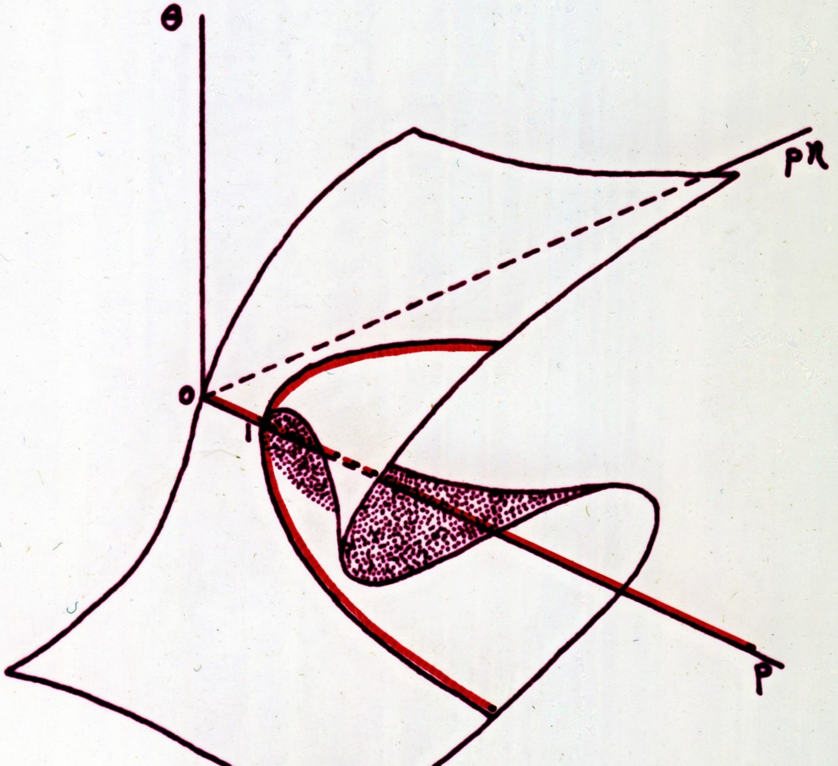
$$- (p-1)\left(1 - 1 + \frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \cdots\right) - p\eta\left(\theta - \frac{\theta^3}{3!} + \cdots\right)$$

$$\approx \tfrac{1}{2}\theta^2 + \tfrac{1}{3}\theta^4 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4 - \tfrac{1}{2}(p-1)\theta^2 - p\eta\theta$$

$$= \underline{\tfrac{3}{8}\theta^4 - \tfrac{1}{2}(p-1)\theta^2 - p\eta\theta}$$

EQUILIBRIUM SURFACE OF STRUT

$$\frac{\partial V}{\partial \theta} = 0 : \quad \underline{\tfrac{3}{2}\theta^3 - (p-1)\theta - p\varkappa = 0}$$



SOLUTION MANIFOLD

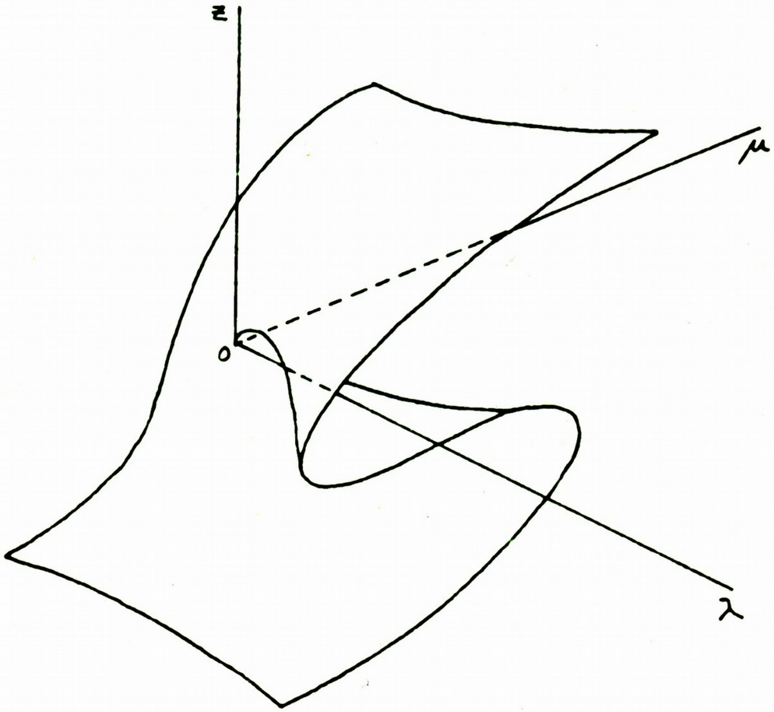
$\rho - 1 \to \lambda, \quad \rho h \to \mu, \quad \theta \to z$

$$F(z, \lambda, \mu) = \frac{3}{2} z^3 - \lambda z - \mu = 0 \qquad F : \mathbb{R}^1 \times \mathbb{R}^2 \to \mathbb{R}^1$$

M IS A 2-DIMENSIONAL MANIFOLD IN 3-SPACE.

PARAMETERS λ, μ

STATE z



SET-UP & PROBLEM

$$F : \underset{\substack{\uparrow \\ \text{STATE} \\ \text{SPACE}}}{Z} \times \underset{\substack{\uparrow \\ \text{PARAMETER} \\ \text{SPACE}}}{\Lambda} \longrightarrow Y$$

SOLUTION MANIFOLD (EQUILIBRIUM SURFACE):

$$F(z, \lambda) = 0$$

SET-UP & PROBLEM

$$F : \underset{\substack{\nearrow \\ \text{STATE} \\ \text{SPACE}}}{Z} \times \underset{\substack{\nwarrow \\ \text{PARAMETER} \\ \text{SPACE}}}{\Lambda} \longrightarrow Y$$

SOLUTION MANIFOLD (EQUILIBRIUM SURFACE):

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PROBLEM: DETERMINE THE STRUCTURE & INTERESTING FEATURES OF A SOLUTION MANIFOLD

SET-UP & PROBLEM

$$F : \underset{\substack{\nearrow \\ \text{STATE} \\ \text{SPACE}}}{Z} \times \underset{\substack{\nwarrow \\ \text{PARAMETER} \\ \text{SPACE}}}{\Lambda} \longrightarrow Y$$

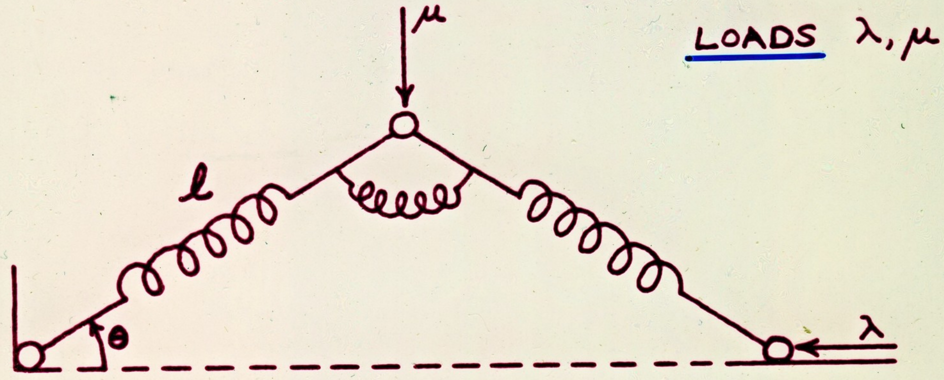
SOLUTION MANIFOLD (EQUILIBRIUM SURFACE):

$$F(z, \lambda) = 0$$

PROBLEM: DETERMINE THE STRUCTURE & INTERESTING FEATURES OF A SOLUTION MANIFOLD

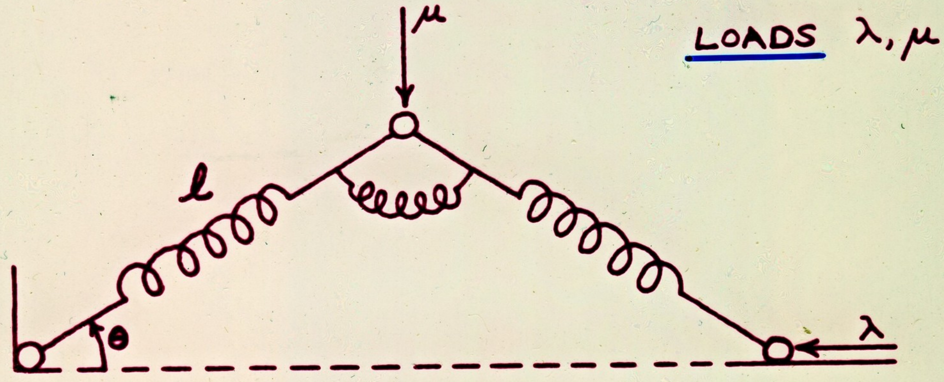
COMMONLY USED METHOD: USE NUMERICAL PROCEDURES TO TRACE 1-DIMENSIONAL PATHS (SUBMANIFOLDS) ON MANIFOLD & THEN TRY TO CONSTRUCT FULL MANIFOLD

BUCKLING OF SPRING



LOADS λ, μ

BUCKLING OF SPRING

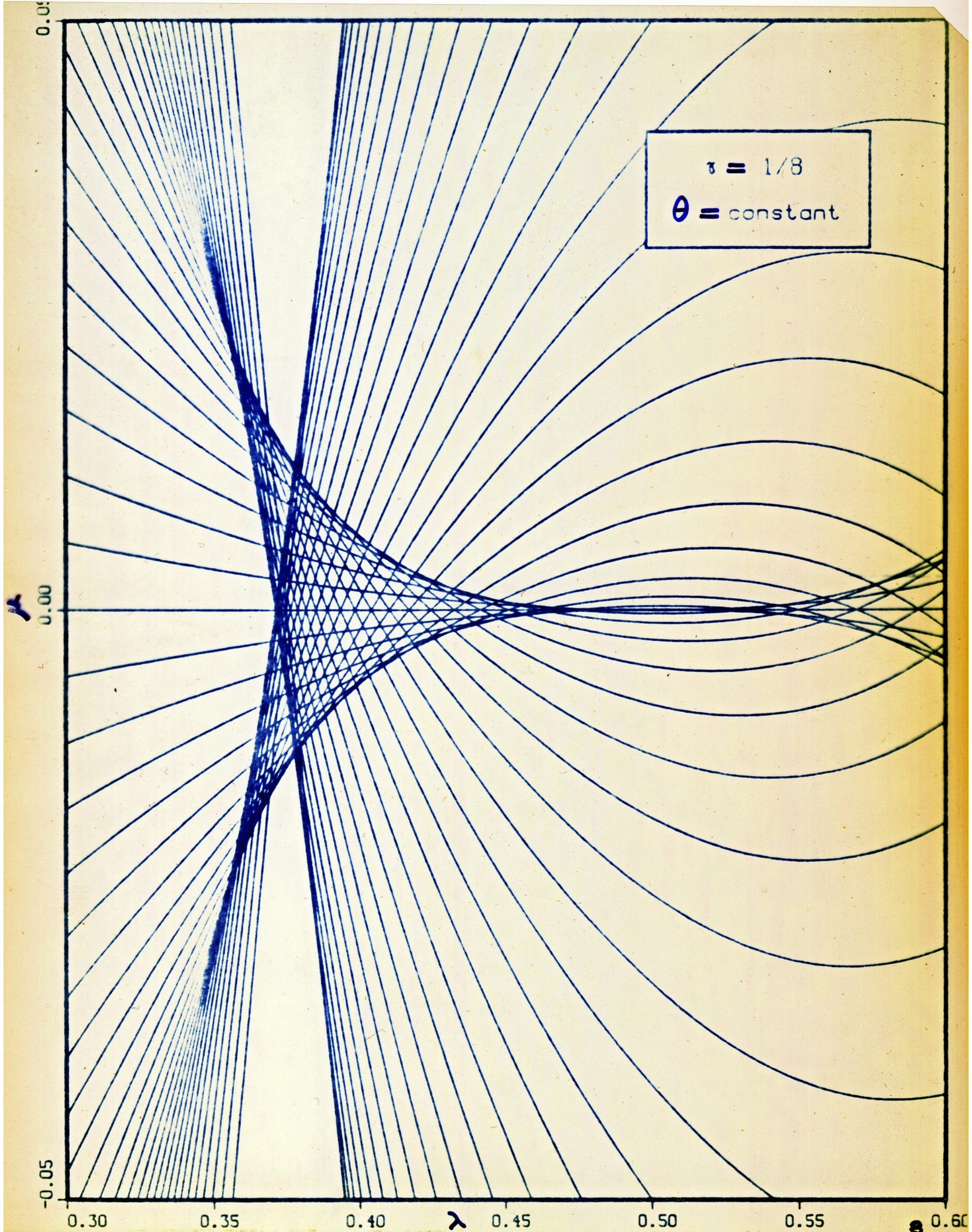


LOADS λ, μ

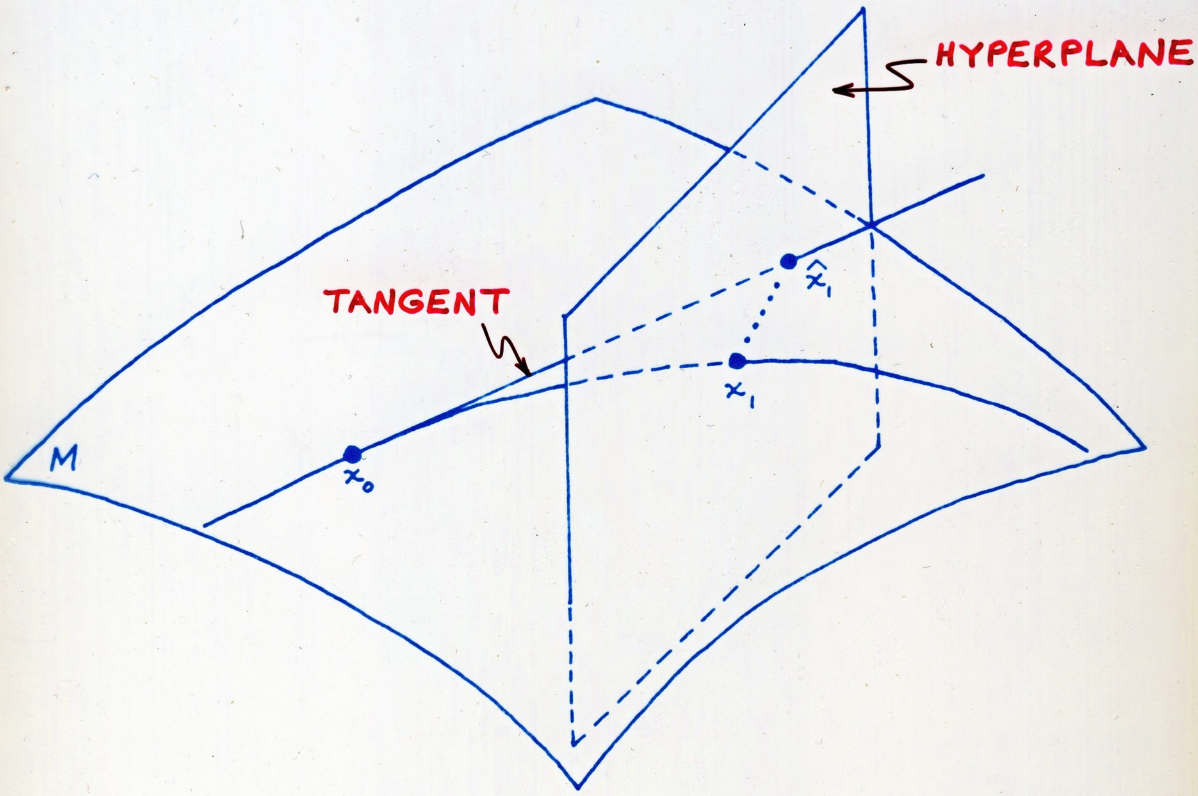
$$F(\ell,\theta,\lambda,\mu) = \begin{bmatrix} -2(1-\ell) + 2\lambda\cos\theta + \mu\sin\theta \\ \frac{1}{2}\theta - 2\lambda\ell\sin\theta + \mu\ell\cos\theta \end{bmatrix} = 0$$

$$F: \underset{\substack{\uparrow \\ \text{STATES} \\ \ell,\theta}}{\mathbb{R}^2} \times \underset{\substack{\uparrow \\ \text{PARAMETERS} \\ \lambda,\mu}}{\mathbb{R}^2} \to \mathbb{R}^2$$

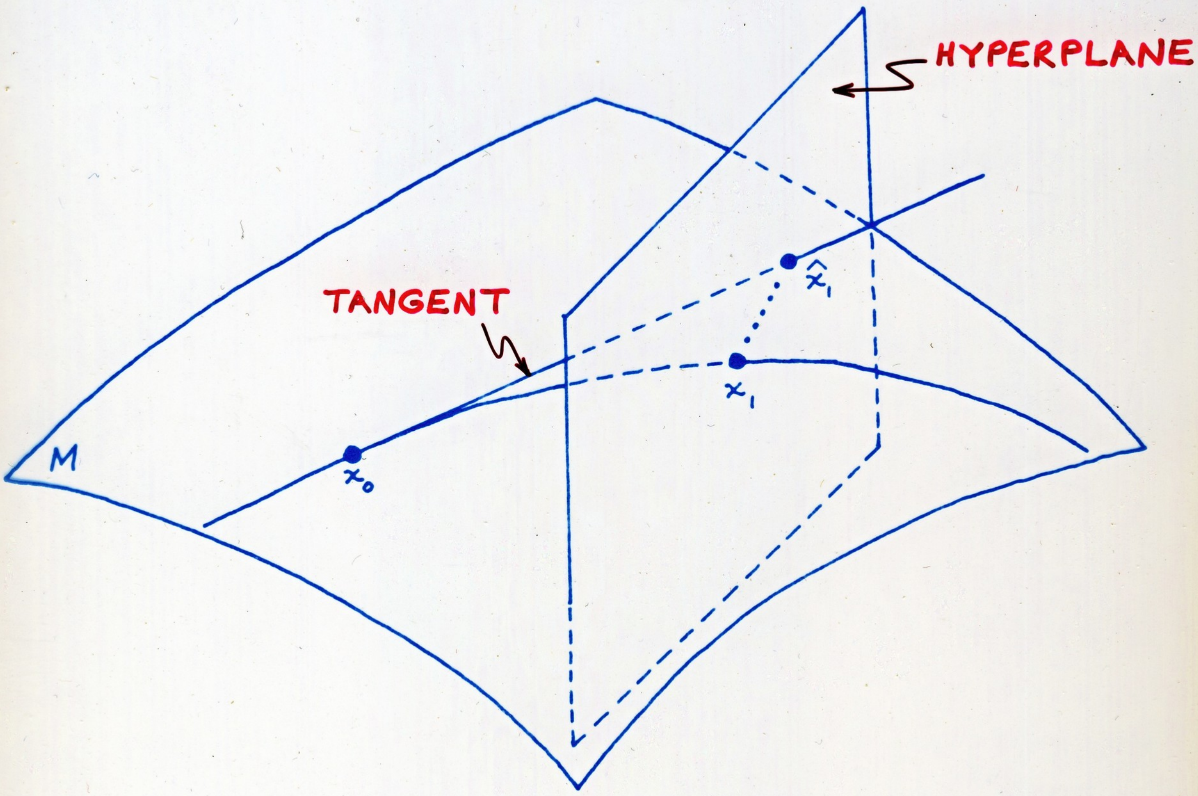
M IS A 2-DIMENSIONAL MANIFOLD IN 4-SPACE.

τ = 1/8
θ = constant
0.0
0.00
−0.05
0.30
0.35
0.40
λ
0.45
0.50
0.55
0.6

IDEA OF "PITCON"

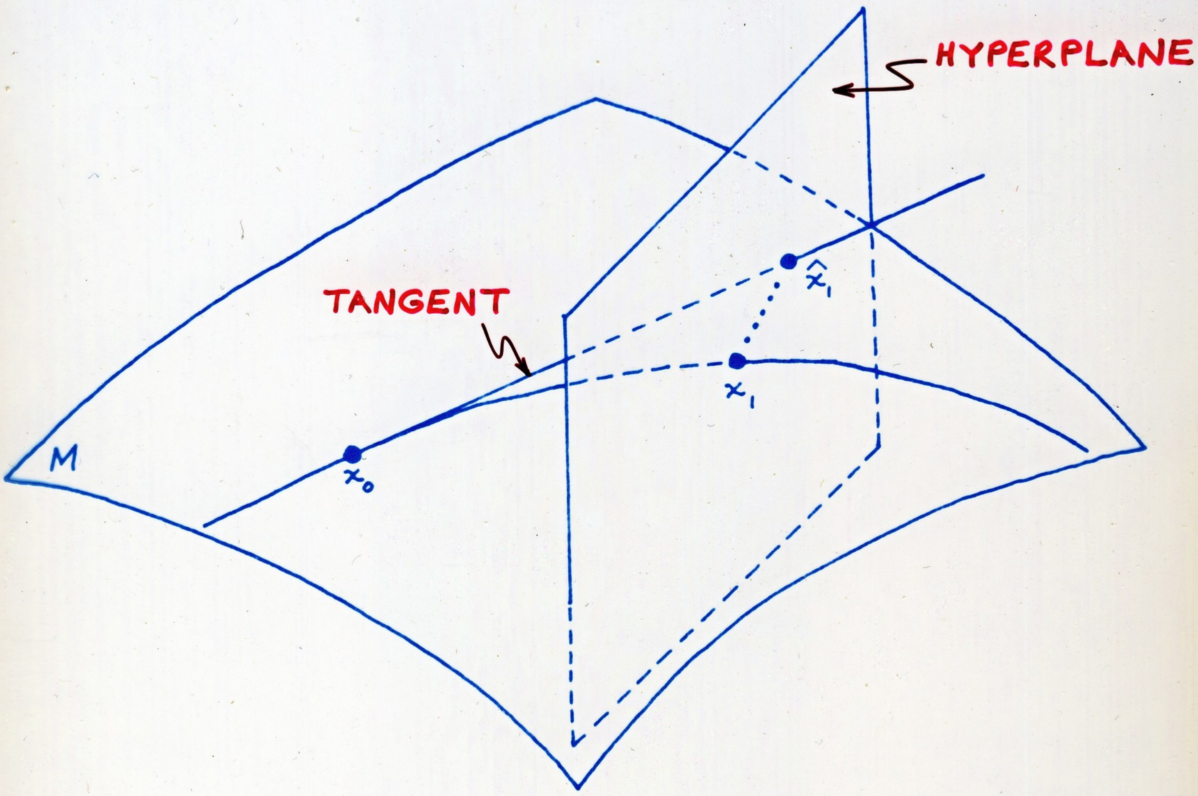


IDEA OF "PITCON"



STEP 1: COMPUTE THE TANGENT DIRECTION AT x_0

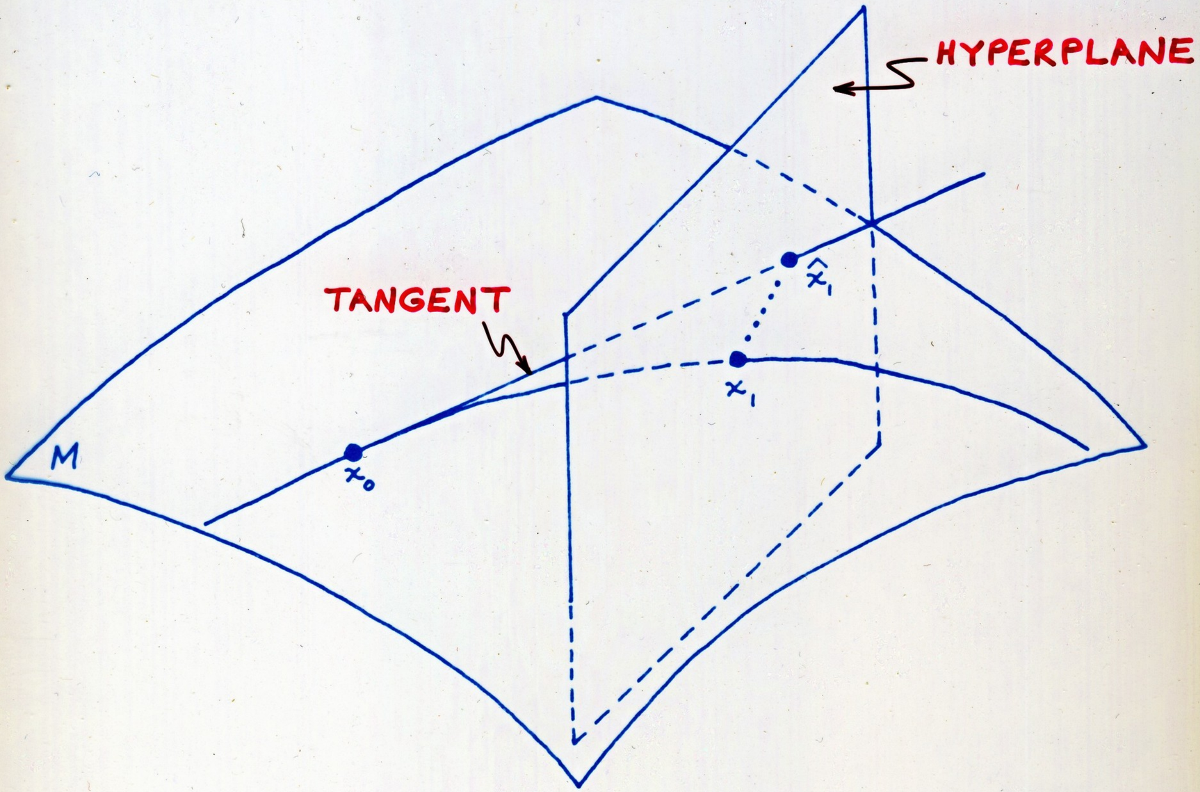
IDEA OF "PITCON"



STEP 1: COMPUTE THE TANGENT DIRECTION AT x_0

STEP 2: COMPUTE A PREDICTED POINT $\hat{x}_1$ IN THE TANGENT DIRECTION

IDEA OF "PITCON"



STEP 1: COMPUTE THE TANGENT DIRECTION AT x_0

STEP 2: COMPUTE A PREDICTED POINT $\hat{x}_1$ IN THE TANGENT DIRECTION

STEP 3: STARTING AT $\hat{x}_1$, APPLY AN ITERATIVE CORRECTOR PROCESS (SUCH AS NEWTON'S METHOD) IN A HYPERPLANE NOT CONTAINING THE TANGENT DIRECTION

AIRCRAFT VARIABLES

$x = (z, \lambda) \in \mathbb{R}^8$

x_1 = ROLL RATE

x_2 = PITCH RATE

x_3 = YAW RATE

x_4 = ANGLE OF ATTACK

x_5 = SIDE-SLIP ANGLE

STATE VARIABLES

z

x_6 = ELEVATOR DEFLECTION

x_7 = AILERON DEFLECTION

x_8 = RUDDER DEFLECTION

CONTROL PARAMETERS

λ

AIRCRAFT EQUATION

$$F(x) = Ax + \Phi(x) = 0, \quad x \in \mathbb{R}^8$$

$$A = \begin{bmatrix} -3.933 & 0.107 & 0.126 & 0 & -9.99 & 0 & -45.83 & -7.64 \\ 0 & -0.987 & 0 & -22.95 & 0 & -28.37 & 0 & 0 \\ 0.002 & 0 & -0.235 & 0 & 5.67 & 0 & -0.921 & -6.51 \\ 0 & 1.0 & 0 & -1.0 & 0 & -0.168 & 0 & 0 \\ 0 & 0 & -1.0 & 0 & -0.196 & 0 & -0.0071 & 0 \end{bmatrix}$$

$$\Phi(x) = \begin{bmatrix} -0.727\, x_2 x_3 + 8.39\, x_3 x_4 - 684.4\, x_4 x_5 + 63.5\, x_4 x_7 \\ -0.949\, x_1 x_3 + 0.173\, x_1 x_5 \\ -0.716\, x_1 x_2 - 1.578\, x_1 x_4 + 1.132\, x_4 x_7 \\ -x_1 x_5 \\ x_1 x_4 \end{bmatrix}$$

AIRCRAFT STABILITY

x_1 = ROLL RATE

x_6 = ELEVATOR DEFLECTION

= FIXED

x_7 = AILERON DEFLECTION

= VARIABLE

x_8 = RUDDER DEFLECTION

= 0