MATH 728D: Machine Learning Solutions to Homework #1: Linear Algebra

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January 30, 2019

1 Householder Transformations

1. Show that Q is symmetric;

$$\begin{aligned} Q =& I - \frac{2}{v'v} v v' \\ Q' =& (I - \frac{2}{v'v} v v')' \\ =& I' - \frac{2}{v'v} (v v')' \\ =& I - \frac{2}{v'v} v v' \\ =& Q; \end{aligned}$$

2. Show that Q is orthogonal;

$$\begin{aligned} Q * Q' =& Q * Q \\ =& (I - \frac{2}{v'v} v v') * (I - \frac{2}{v'v} v v') \\ =& I - \frac{2}{v'v} v v' - \frac{2}{v'v} v v' + \frac{4}{(v v')^2} v v' v v' \\ =& I - \frac{4}{v'v} v v' + \frac{4}{(v'v)^2} v (v'v) v' \\ =& I - \frac{4}{v'v} v v' + \frac{4}{v'v} v v' \\ =& I \end{aligned}$$

3. Show that $Q\,v = -v$;

$$\begin{aligned} Q * v =& (I - \frac{2}{v'v} v v') * v \\ =& v - \frac{2}{v'v} v v' v \\ =& v - \frac{2}{v'v} v (v'v) \\ =& v - 2v \\ =& -v \end{aligned}$$

2 The QR Factorization

1. Based on this fact, what is the Q factor for A?

$$
\begin{aligned}
Q_1 * Q_2 * ... * Q_k * A =& R \\
Q'_k * ... * Q'_2 * Q'_1 * Q_1 * Q_2 * ... * Q_k * A =& Q'_k * ... * Q'_2 * Q'_1 * R \\
A =& Q'_k * ... * Q'_2 * Q'_1 * R \\
=& Q * R
\end{aligned}
$$

2. How do you know that Q is an orthogonal matrix?
If Q_1 is orthogonal, then so is Q'_1;
If Q'_1 and Q'_2 are orthogonal, then so is $Q'_2 * Q'_1$;
Repeating this argument, $Q = Q'_k * ... * Q'_2 * Q'_1$ is orthogonal.
3. The value k counts the number of Householder transformations we must apply. The j-th transformation reduces column j to upper triangular form. What is the typical value of k when A is an $n \times n$ matrix?
We have to "upper triangularize" $n-1$ columns of A, so $k = n-1$;
What happens if A is an $m \times n$ matrix?
If $m < n$, $k = m-1$;
if $n < m$, $k = n$.

3 The LR Factorization

1. Describe how this factorization can be used to solve a linear system $A * x = b$;

$$
\begin{aligned}
A * x =& b \\
P' * L * R * x =& b \\
L * R * x =& P * b \text{ (easy to multiply by permutation)} \\
\text{Solve } L * (R * x) =& P * b \text{ (easy to solve unit lower triangular system)} \\
\text{Solve } R * x =& L^{-1} * P * b \text{ (easy to solve upper triangular system)} \\
x =& R^{-1} * L^{-1} * P * b
\end{aligned}
$$

2. The inverse of a triangular matrix is easy to compute. Assuming this is so, what is a formula for the inverse of a matrix A for which we have computed the LR factorization?

$$A^{-1} = R^{-1} * L^{-1} * P$$

4 The SVD Factorization

1. Show that the columns of U are eigenvectors of AA'

$$
\begin{aligned}
A * A' * U =& U * S * V' * (V * S' * U') * U \\
=& U * S * (V' * V) * S' * (U' * U) \\
=& U * S^2
\end{aligned}
$$

Here S^2 is an $m \times m$ diagonal matrix.
2. Show that the values σ_i^2 are (some of the) eigenvalues of AA' and $A'A$
A similar argument shows that $A' * A * V = V * S^2$, where S^2 is an $n \times n$ diagonal matrix;

3. Suppose that A is symmetric and positive definite, so that it has a full set of eigenvectors X and nonnegative eigenvalues Λ. Write a formula for the singular value decomposition of A;

$$\begin{aligned} X' * A * X =& \Lambda; \\ A =& X * \Lambda * X' \\ A =& U (= X) * \; S (= \Lambda) * \; V' (= X') \end{aligned}$$

4. Suppose that A is a square, invertible matrix. Write a formula for the inverse of A in terms of the singular value decomposition;

$$\begin{aligned} A =& U * S * V' \\ A^{-1} =& V * S^{-1} * U' \text{ (Assertion)} \\ A * A^{-1} =& U * S * V' * V * S^{-1} * U' \\ =& U * S * S^{-1} * U' \\ =& U * U' \\ =& I \end{aligned}$$

and similarly for $A^{-1} * A$. S^{-1} is meaningful because S is square, and because each σ_i is nonzero.