Linear Algebra Questions: Given a matrix A

- Can we solve $A * x = b$?
- How accurate is a numerical solution x going to be?
- If A is singular, or rectangular, can I still "solve" $A * x = b$?
- What is the determinant, rank, condition number, inverse of A?
- Can A be described in terms of special directions and values?
- What is the range of A? (vectors y such that $y = Ax$)
- What is the null space of A? (vectors x such that $A * x = 0$)
- Can we approximate A by a simpler matrix B?
- How do we measure closeness of two matrices A and B?

Data Questions: Given a set X of data vectors x

- What is the average element $\overline{x}$ in X?
- What is a good basis V for the subspace containing X?
- How do we represent vector x in basis V?
- How do we measure closeness of two vectors x_i and x_j?
- Does new data vector y "belong" to set X?

LU Factorization

$A = P' * L * U$, where

- P is a permutation matrix;
- L is a unit lower triangular matrix;
- U is an upper triangular matrix;

- Solve A*x=b by solving triangular systems;
- Determinant is product of diagonal elements of U * sign of P;
- Inverse can be computed by inverting P, L and U;
- Rank = number of nonzero diagonal elements of U;
- Condition can be "guessed" by range of diagonal elements of U;
- Underdetermined systems (M<N) solved using degrees of freedom;

QR Factorization

$A = Q * R$, where

- Q is an orthogonal matrix: $Q' = Q^{-1}$;
- R is an upper triangular matrix, with nonnegative diagonal entries;

- Solve A*x=b by solving triangular system $R * x = Q' * b$;
- Inverse is $R^{-1} * Q'$ (upper triangular system is easy to invert);
- Determinant is product of diagonal of R and determinant of Q;
- Rank = number of nonzero diagonal elements of R;
- If $R_{i,i}$ is zero (A is singular):
    - row i and subsequent rows must be completely zero;
    - can compute R's pseudoinverse R^+;
    - can compute A's pseudoinverse $A^+ = R^+ * Q'$;
    - "solution" $x = A^+ * b$ minimizes L2 norm of error;
- Solution of overdetermined system minimizes L2 norm of error;
- Solution of underdetermined system minimizes L2 norm of solution;

Eigen Factorization for square symmetric A

$A = X * \Lambda * X'$ where

- Λ is a diagonal matrix;
- X is orthogonal: $X' = X^{-1}$;

- Solve A*x=b by $x = X * \Lambda^{-1} * X' * b$;
- Inverse is $X * \Lambda^{-1} * X'$;
- Determinant is $\prod_{j=1}^{n} \lambda_j$;
- Rank $=$ number of nonzero diagonal elements of Λ;
- Condition $= \frac{\max(|\lambda_j|)}{\min(|\lambda_j|)}$;
- The columns of X are eigenvectors;
- The diagonal of Λ are eigenvalues;
- For eigenvector j, $A * x_j = \lambda_j * x_j$;
- Eigenvectors form a special basis in which A is diagonal;
- Eigenvalues measure expansion from x to $A * x$ along each direction;