SELECTED TOPICS CLASS FINAL REPORT

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In this report, I have tested the stochastic Navier-Stokes equations with different kind of noise, when i am doing this project, I have encountered several problem, how to quickly solve a very larger matrix efficiently, I have tested different matrix solver, pre-order softwares and blas , eg, SuperLU, Umfpack, Mumps, metis, pord, scrotch. standard blas , atlas, openblas etc. In a very short period of time to solve several thousands of million dimensional matrices is a very larger challenge, now i can not solve this problem for lack of a efficient HPC to use. I think if I have one hundreds fast CPU, i could get a much better results than what i have done. However, how to describe the different colored noise this still confused to me, I have look up many paper, I have not found a definitely definition for two dimensional space or higher. As we know that the measure of point-wise value is zeros under lebesgue measure. To define the correlation of random variable in two dimensional space from the point-wise view or from the Borel set view make me some confuse. In the past, I have used the KL-expansion to simulate the $\delta(t_1 - t_2)C(x, y)$, I am still not sure the algorithm i used is right or not. Several week ago, I suddenly have a idea to derivative the different noise from the H-valued Wiener process or cylinderical Wiener process, I wish this idea may be more accurate to represent the noise.

$$
\begin{array}{ll}
d\mathbf{u} - (\nu\Delta\mathbf{u}dt + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p)dt = dW & \text{in } \Omega\times(0,T), \\
\nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\times(0,T) \\
\mathbf{u} = 0 & \text{on } \partial\Omega\times(0,T) \\
\mathbf{u} = \mathbf{u}_0 & \Omega\times\{0\}
\end{array}
\tag{0.1}
$$

we assume that $W(t)$ is a Wiener process with covariance operator Q. This process may be considered in terms of its Fourier series, Suppose that Q is a bounded,linear, self-adjoint, positive definite operator on H, with eigenvalues $\gamma_l > 0$ and corresponding eigenfunctions e_l. Let $\beta_l, l = 1, 2, ...,$ be a sequence of real-valued independently and identically distributed Brownian motions. Then

$$
W(t) = \sum_{l=1}^{\infty} \gamma_l^{1/2} e_l \beta_l(t) \tag{0.2}
$$

is a Wiener process with covariance operator Q.

If Q is in trace class, then $W(t)$ is an H-valued process. If Q is not in trace class, for example $Q = I$, then W(t) does not belong to H, in which case $W(t)$ is called a cylindrical Wiener process.

Let $L_2^0 = HS(Q^{1/2}(H), H)$ denote the space of Hilbert-Schmidt operators from $Q^{1/2}(H)$ to H, i.e.,

$$
L_2^0 = \left\{ \psi \in L(H) : \sum_{l=1}^{\infty} \left|\left| \psi Q^{1/2} e_l \right|\right|^2 < \infty \right\} \tag{0.3}
$$

1. Computational Analysis. In this section we consider how to compute the approximate solution U^n of the solution u of (1.1). For simplicity, we assume that $\sigma(\cdot) = I$. Recall that the Wiener process $W(t)$ with covariance operator Q has the form (see Da Prato and Zabczyk [8, Chapter 4])

$$
W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} e_j \beta_j(t), \tag{1.1}
$$

where $\{\gamma_j, e_j\}_{j=1}^{\infty}$ is an eigensystem of Q, and $\{\beta_j(t)\}_{j=1}^{\infty}$ are independently and identically distributed real-valued Brownian motion. If $Tr(Q) < \infty$, then $W(t)$ is an H-valued process, In fact

$$\mathbb{E}\,||W(t)||^2 = \mathbb{E}\sum_{j=1}^{\infty}\gamma_j\beta_j(t)^2 = \sum_{j=1}^{\infty}\gamma_j(\mathbb{E}\beta_j(t)^2) = tTr(Q) < \infty. \tag{1.2}$$

If $Tr(Q) = \infty$, for example $Q = I$, then $W(t)$ is not H-valued.
Let U^n be the approximation in S_h of $u(t)$ at $t = t_n = nk$.
The skew symmetric form $B(\mathbf{u}, \mathbf{v})$ induces a trilinear form b defined as, $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{H}_0^1$.

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (B(\mathbf{u}, \mathbf{v}), \mathbf{w}) = \frac{1}{2}\left\{(\mathbf{u}\cdot\nabla\mathbf{v}, \mathbf{w}) - (\mathbf{u}, \mathbf{w}, \mathbf{v})\right\} \tag{1.3}$$

For convenience, the following notations will be used throughout this paper

$$\mathbf{u}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n), p^{n+\frac{1}{2}} = \frac{1}{2}(p^{n+1} + p^n), \tag{1.4}$$

ALGORITHM 1.1. *Suppose that $\mathbf{u}_h^i \in V_h$ for $i = 0, 1, \ldots, n_0$ and $p_h^i \in Q_h$. For each n, find $\mathbf{u}_h^{n+1}, p_h^{n+1} \in V_h \times Q_h$ satisfying*

$$(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h) + \nu(\nabla\mathbf{u}^{n+i}, \nabla\mathbf{v}_h) + (B(\mathbf{u}^{n+i}, \mathbf{u}^{n+i}), \mathbf{v}_h) - (p^{n+i}, \nabla\cdot\nabla\mathbf{v}_h) = \frac{1}{\Delta t}(\int_{t_n}^{t_{n+1}} dW, \mathbf{v}_h)$$
$$(div\mathbf{u}^{n+1}, q_h) = 0$$

Since W(t) is $\dot{H}^{\beta-1}$-valued with $\beta \in [0,1]$, $P_hW(t)$ is well defined. We therefore can write

$$\int_{t_{n-1}}^{t_n} dW(s) = (W(t_n) - W(t_{n-1})) = \sum_{j=1}^{\infty}\gamma_j^{1/2}e_j(\beta_j(t_n) - \beta_j(t_{n-1})). \tag{1.5}$$

Here

$$\frac{1}{\sqrt{k}}\,(\beta_j(t_n) - \beta_j(t_{n-1}) = \mathcal{N}(0,1), \tag{1.6}$$

where $\mathcal{N}(0,1)$ is the real-valued Gaussian random variable. Thus the right hand side of (4.2) can be computed by truncating the following series of J terms;

$$\frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}} dW(s), \chi\right) = \frac{1}{\Delta t}\left(\sum_{j=1}^{\infty}\gamma_j^{1/2}e_j\left(\beta_j(t_{n+1}) - \beta_j(t_n)\right), \chi\right) \tag{1.7}$$
$$= \frac{1}{\Delta t}\sum_{j=1}^{\infty}\gamma_j^{1/2}\left(\beta_j(t_{n+1}) - \beta_j(t_n)\right)(e_j, \chi) \tag{1.8}$$
$$= \frac{1}{\sqrt{\Delta t}}\sum_{j=1}^{\infty}\gamma_j^{1/2}\xi_i(e_j, \chi) \tag{1.9}$$
$$\tag{1.10}$$

LEMMA 1.2. *[2] Let U_h^J and u_h be defined by () and (4.5), respectively, Assume that $\{S_h\}$ is defined on a quasi-uniform family of triangulations, and let N_h be the dimension of S_h. Assume that $\left|\left|A^{(\beta-1)/2}\right|\right|_{L_2^0} < \infty$ for some $\beta \in [0,1]$, If $J \geq N_h$, then we have, for $t > 0$,*

$$\left|\left|u_h^J - u_h\right|\right|_{L_2(\Omega,H)} \leq Ch^\beta\left|\left|A^{(\beta-1)/2}\right|\right|_{L_2^0} < \infty \tag{1.11}$$

2. Numerical Experiments.

In this section, we shall report some numerical experiments results for the Stochastic Navier Stokes equation with color noise of continuous covariance kernel function . From the above numerical results of KL expansion, we can find that sometimes the Kl expansion are not unique for given kernel function, In my experiment, the Gram Schmidt orthogonalization has been applied to generate a set of numerical orthonormal eigenfunction values.

In our experiment, we want to see how the error estimates depend on space mesh size. to do this, we choose fixed very small $\Delta t = (\frac{1}{100})^3$ and a sequence of $h_i = 2^{-i}, i = 1, ..., 6$. we consider $M = 100$ simulations. For each simulation , we generate different normal random numbers according the truncation number of KL expansion. and compute $\mathbf{U}^n \approx \mathbf{u(t_n)}$ at time $t_n = 100\Delta t$.

we then compute the following L_2 norm of the error at t_n for each simulation $\Omega_j, j = 1, 2, ..., M$,

$$\epsilon(\omega_j) = ||U^n(\omega_j) - u(t_n, w_j)||^2 \tag{2.1}$$

where the true solution at $u(t_n, \omega_j)$ is approximated by a solution computed by small space step $h = 2^{-7}$ and fixed time step Δt.

we then average $\epsilon(\omega_j)$ with respect to Ω_j to obtain the following approximation of $||U^n(\omega_j) - u(t_n)||^2$:

$$S(h_i) = \left(\frac{1}{M} \sum_{j=1}^{M} \epsilon(\omega_j) \right)^{1/2} = \left(\frac{1}{M} \sum_{j=1}^{M} ||U^n(\omega_j) - u(t_n, w_j)||^2 \right)^{1/2} \tag{2.2}$$

$$\approx ||U^n(\omega_j) - u(t_n, w_j)||_{L_2}, \tag{2.3}$$

where

$$\approx ||U^n(\omega_j) - u(t_n, w_j)||_{L_2} = \left(\int_{\Omega} ||U^n(\omega_j) - u(t_n, w)||^2 \, dP(\Omega) \right)^{1/2} \tag{2.4}$$

the convergence order α is obtain by

$$\frac{S(h_i)}{S(H_{i+1})} \approx \left(\frac{h_i}{h_{i+1}} \right)^{\alpha} = 2^{\alpha} \tag{2.5}$$

Let us introduce the KL expansion numerical expansion approximation.

2.1. example 1.

we consider the following two-dimensional stochastic Navier Stokes driven by color noise.

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f(x,t) + \xi(x,t,\omega) &\quad \text{in } (0,T)\times\mathcal{D}\times\Omega, \\ \nabla\cdot\mathbf{u} = 0 &\quad \text{in } (0,T)\times\mathcal{D}\times\Omega, \\ \mathbf{u} = \mathbf{g} &\quad \text{on } (0,T)\times\partial\mathcal{D}\times\Omega, \\ \mathbf{u} = \mathbf{u}_0 &\quad \text{on } \mathcal{D}\times\{0\}\times\Omega. \end{aligned} \tag{2.6}$$

Where $\xi(x,t,\omega)$ denotes the color noise with mean zeros and gaussian variance function $C_X(x,y) = \sigma^2 e^{-\frac{|x-y|^2}{L_c}}, x, y \in D$ and $\nu = 1, L_c = 10$, and

$$\mathbf{g}(t,x) = (e^{-t}\cos(2\pi y)\sin(2\pi x), -e^{-t}\cos(2\pi x)\sin(2\pi y)) \tag{2.7}$$

$$\begin{aligned} f(x,t) = &(2x + \pi e^{-2t}\sin(4\pi x) - e^{-t}\cos(2\pi y)\sin(2\pi x) + 8\pi^2\nu e^{-t}\cos(2\pi y)\sin(2\pi x), \\ &2y + \pi e^{-2t}\sin(4\pi y) + e^{-t}\cos(2\pi x)\sin(2\pi y) - 8\pi^2\nu e^{-t}\cos(2\pi x)\sin(2\pi y)) \end{aligned}$$

$$\mathbf{u}_0 = (\cos(2\pi y)\sin(2\pi x), \cos(2\pi x)\sin(2\pi y)) \tag{2.8}$$

h	$\|\|u(T)-u_h(T)\|\|$	order	$\|\|v(T)-v_h(T)\|\|$	order	$\|\|p(T)-p_h(T)\|\|$	order
1/2	2.524270e-02	-	2.524270e-02	-	1.263379e+00	-
1/4	1.069344e-02	1.239141	1.068352e-02	1.240480	7.144075e-01	0.822468
1/8	1.337853e-03	2.998734	1.336509e-03	2.998846	3.077256e-01	1.215103
1/16	1.656229e-04	3.013946	1.651406e-04	3.016702	9.342183e-02	1.719813
1/32	2.081968e-05	2.991882	2.076527e-05	2.991451	2.723982e-02	1.778043
1/64	2.609404e-06	2.996156	2.591168e-06	3.002498	6.451420e-03	2.078028

TABLE 2.1
the computational results for 100 simulations

2.2. exmaple 2. In this example, we have tested the color noise with exponential covariance function $C_X(x,y) = e^{-\frac{|x-y|}{L_c}}, x, y \in D$, all others parameter are same as the example 1 except the KL truncation number,

From the above simulation of KL expansion for color noise with exponent correlation function, we know that the eigenvalues decay much slowly compared to gaussian correlation function. so we remain 30 terms of KL expansion in this test. we have the following test results.

h	$\|\|u(T)-u_h(T)\|\|$	order	$\|\|v(T)-v_h(T)\|\|$	order	$\|\|p(T)-p_h(T)\|\|$	order
1/2	2.524308e-02	-	2.524308e-02	-	8.020224e-01	-
1/4	1.069348e-02	1.239157	1.068345e-02	1.240510	4.292909e-01	0.901687
1/8	1.337855e-03	2.998737	1.336515e-03	2.998829	1.601010e-01	1.422973
1/16	1.656494e-04	3.013717	1.651818e-04	3.016350	3.629905e-02	2.140979
1/32	2.082091e-05	2.992028	2.077364e-05	2.991229	1.020693e-02	1.830382
1/64	2.604166e-06	2.999140	2.590207e-06	3.003615	2.698158e-03	1.919503

3. Colored in Space and white in space. We consider two Hilbert spaces H and U , and a symmetric nonnegative operator $Q \in L(U)$. we will consider first the case when $TrQ < +\infty$, Then there exist a complete orthonormal system $\{e_k\}$ in U, and a bounded sequence of nonnegative real number λ_k such that

$$Qe_k = \lambda_k e_k, k = 1, 2, ... \tag{3.1}$$

Now we set $Q = e^{||x-y||^2} \in L(U)$ in the distribution sense

$$Qe_k := \int e^{||x-y||^2} e_k = \lambda_k e_k \tag{3.2}$$

It is well know the trace $TrQ < +\infty$

4. white in Space and white in space. The concept of cylindrical wiener process is closely related to that of space-time white noise. Loosely speaking, the latter is the time derivative of a cylindrical Wiener process, Assume that $(E, \mathcal{E})$ is a measurable space and that λ finite measure on E, write $\mathcal{E}_{fin} := \{A \in \mathcal{E} : \lambda(A) < \infty\}$

Let W be a cylindrical Wiener process on L^2O, Let $\{e_n\}$ be an orthonormal basis of $L^2(o)$, and let $W_n(t) = W(t, e_n)$. Then

$$\chi(A) = \sum_n \int_0^\infty \int_O \chi_A(\xi,t) e_n(\xi) d\xi dW_n(t), A \in \mathcal{E}_{fin} \tag{4.1}$$

defines a space time white noise on O,ref [1].

Proof. Since the W_n are independent and χ_A and e_n are non-random, the stochastic integrals

$$\chi(A) = \sum_n \int_0^\infty \int_O \chi_A(\xi,t) e_n(\xi) d\xi dW_n(t), A \in \mathcal{E}_{fin} \tag{4.2}$$

are independent Gaussian random variables in R, thus for $A, B \in \mathcal{E}_{fin}$

$$E\chi(A)\chi(B) = \sum_n \int_0^\infty \int_o \chi_A(t,\xi) e_n(\xi) d\xi \tag{4.3}$$

$$= \sum_n \int_0^\infty \int_o \chi_{A\cap B} e_n(\xi)^2 d\xi \tag{4.4}$$

□

5. white in Space and white in space. The time integral of the Wiener process

$$W^{-1}(t) := \int_0^t W(s) ds \tag{5.1}$$

is called integrated Brownian motion, it arised in many applications and can be shown to have the distribution $N(0, t^3/3)$

The second one is easy

$$E(exp(uw(t))) = E(exp(u\sqrt(t)N(0,1))) \tag{5.2}$$

6. Remark. The above computational results looks very well, but i think i can not be describe by $\delta(t_1 - t_2)C(x,y)$, I need to do some derivation to better understand the relationship between KL expansion and the Classic Q-value-H-wiener process. I wish I can finish this project this as soon as possible.

This is the new test results for this week, when I use the least square method to approximate the convergence rate, i got the results (1.05,1.06,1.5) for (u,v,p), if i use the classic $ln(e_i/e_{i+1})/ln(2)$ to compute the order , it looks very bad.

```
 allL2Error
5.83324591220848922E-003  6.09836464373105132E-003   79.037100073119802
4.02175720282692186E-003  4.16742654087437752E-003   26.142389741221812
2.35403643168293625E-003  2.40199550741585551E-003   9.3632191444671609
1.17822204976905328E-003  1.21309910223968117E-003   3.2803548561869320
4.72318649184909009E-004  4.84148729153322682E-004   1.2208554166192229
1.44953465495548242E-004  1.48502868824882951E-004  0.38338233007955769
 0.0000000000000000        0.0000000000000000        0.0000000000000000
 0.0000000000000000        0.0000000000000000        0.0000000000000000
 0.0000000000000000        0.0000000000000000        0.0000000000000000
output the convergence rate
0.53647290769487599       0.54926564790260590       1.5961389868733020
0.77268933937407802       0.79492331639136415       1.4813144980178590
0.99852519047505650       0.98553603869955653       1.5131527358998031
 1.3187790520282210        1.3251752013599518        1.4259595334395805
 1.7041706752105930        1.7049595057927047        1.6710366045725038
```

```
                +Infinity                +Infinity                +Infinity
                      NaN                      NaN                      NaN
                      NaN                      NaN                      NaN
total time =   332813.72962200001
```

REFERENCES

[1] S. Peszat and J. Zabczyk, *Stochastic partial differential equations with Lévy noise*, Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007, An evolution equation approach. MR 2356959 (2009b:60200)

[2] Yubin Yan, *Galerkin finite element methods for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal. **43** (2005), no. 4, 1363–1384 (electronic). MR 2182132 (2007a:65013)