Theorem 2.4 for CG. This is partly due to the factor $\kappa(V)$, but it also depends on the need for bounds from approximation theory for $\max_\lambda |p_k(\lambda)|$.

The results of Theorem 4.1 can be used to gain insight into the convergence behavior of GMRES by taking the $k$th root of (either of) the bounds. In particular,

$$\left(\frac{\|\mathbf{r}^{(k)}\|}{\|\mathbf{r}^{(0)}\|}\right)^{1/k} \leq \kappa(V)^{1/k}\left(\min_{p_k \in \Pi_k,\, p_k(0)=1} \max_{\lambda_j} |p_k(\lambda_j)|\right)^{1/k}.$$

$\kappa(V)$ does not depend on $k$, and it therefore follows that $\kappa(V)^{1/k} \to 1$ as $k$ increases. This suggests consideration of the limit

$$\rho := \lim_{k\to\infty}\left(\min_{p_k \in \Pi_k,\, p_k(0)=1} \max_{\lambda_j} |p_k(\lambda_j)|\right)^{1/k}. \tag{4.8}$$

Since GMRES constructs the exact solution in a finite number of steps, this does not lead to a simple statement about the error at any given step of the computation. However, it does give insight into the asymptotic behavior for large enough $k$: as the iteration proceeds, it can be expected that the norm of the residual will be reduced by a factor roughly equal to $\rho$ at each step. We refer to $\rho$ of (4.8) as the *asymptotic convergence factor* of the GMRES iteration. It is an interesting fact that asymptotic estimates for large $k$ are often descriptive of observed convergence behavior for $k \ll n$. It is rarely the case that $n$ iterations are necessary for an accurate solution to be obtained.

For later analysis, we mention that for a simple (stationary) iteration (as in (2.26)) with iteration matrix $T = M^{-1}R$, the norms of successive error vectors will asymptotically (for large numbers of iterations) reduce by a factor which is simply the eigenvalue of $T$ of maximum modulus. We will therefore denote by $\rho(T)$ the eigenvalue of $T$ of maximum modulus since this reflects the ultimate rate of convergence for a simple iteration as does $\rho$ defined above for GMRES iteration.

Returning to GMRES, a bound on $\rho$ can be obtained using the fact that any polynomial $\chi_k \in \Pi_k$ with $\chi_k(0) = 1$ satisfies, for any set $\mathcal{E}$ that contains the eigenvalues of $F$,

$$\min_{p_k \in \Pi_k,\, p_k(0)=1} \max_{\lambda\in\mathcal{E}} |p_k(\lambda)| \leq \max_{\lambda\in\mathcal{E}} |\chi_k(\lambda)|.$$

This was used in Theorem 2.4 to construct a bound on the error for the CG method, where $\chi_k$ was taken to be a scaled and translated Chebyshev polynomial. The same approach can be used to derive a bound on the asymptotic convergence factor for the GMRES method when the enclosing set $\mathcal{E}$ is an ellipse in the complex plane.

**Theorem 4.2.** *Suppose $F$ is diagonalizable and its eigenvalues all lie in an ellipse $\mathcal{E}$ with center $c$, foci $c \pm d$ and semi-major axis $a$, and $\mathcal{E}$ does not contain the origin. Then the asymptotic convergence factor for* GMRES *iteration is*