Lanczos algorithm, and (4.1) is identical

ETHOD

$Fu^{(0)}$, $\beta_0 = \|\mathbf{r}^{(0)}\|$, $\mathbf{v}^{(1)} = \mathbf{r}^{(0)}/\beta_0$

lo

$\left\|\beta_0 \mathbf{e}_1 - \widehat{H}_k \mathbf{y}^{(k)}\right\|$ is minimized,

$j \le k$

on for solving systems of equations, we
hoice $\mathbf{v}^{(1)} = \mathbf{r}^{(0)}/\beta_0$, with $\beta_0 = \|\mathbf{r}^{(0)}\|$ as

$$) + V_k \mathbf{y}^{(k)} \tag{4.2}$$

t the first line of (4.1) can be rewritten
t the residual satisfies

$$= V_{k+1}\left(\beta_0 \mathbf{e}_1 - \widehat{H}_k \mathbf{y}^{(k)}\right), \tag{4.3}$$

vector of size k. The vectors $\{\mathbf{v}^{(j)}\}$ are

$$\left|\beta_0 \mathbf{e}_1 - \widehat{H}_k \mathbf{y}^{(k)}\right\|. \tag{4.4}$$

te (4.2) with smallest Euclidean norm is
minimizes the expression on the right side

s problem can be solved by transforming
where R_k is upper triangular, using $k+1$
1 to $\beta_0 \mathbf{e}_1$). Here, $\widehat{H}_k$ contains $\widehat{H}_{k-1}$ as a
nentation, R_k can be updated from R_{k-1}.
at leading to (2.37), it can be shown that
t. Hence, a step of the GMRES algorithm

consists of constructing a new Arnoldi vector $\mathbf{v}^{(k+1)}$, determining the residual norm of the iterate $\mathbf{r}^{(k)}$ that would be obtained from $\mathcal{K}_k(F, \mathbf{r}^{(0)})$, and then either constructing $\mathbf{u}^{(k)}$ if the stopping criterion is satisfied, or proceeding to the next step otherwise.

By construction, the iterate $\mathbf{u}^{(k)}$ generated by the GMRES method is the member of the translated Krylov space

$$\mathbf{u}^{(0)} + \mathcal{K}_k(F, \mathbf{r}^{(0)})$$

for which the Euclidean norm of the residual vector is minimal. That is,

$$\|\mathbf{r}^{(k)}\| = \min_{p_k \in \Pi_k,\, p_k(0)=1} \|p_k(F)\mathbf{r}^{(0)}\|. \tag{4.5}$$

As in the analysis of the CG method, the Cayley–Hamilton theorem implies that the exact solution is obtained in at most n steps. Bounds on the norm of the residuals associated with the GMRES iterates are derived from the optimality condition.

Theorem 4.1. *Let $\mathbf{u}^{(k)}$ denote the iterate generated after k steps of* GMRES *iteration, with residual $\mathbf{r}^{(k)}$. If F is diagonalizable, that is, $F = V \Lambda V^{-1}$ where Λ is the diagonal matrix of eigenvalues of F, and V is the matrix whose columns are the eigenvectors, then*

$$\frac{\|\mathbf{r}^{(k)}\|}{\|\mathbf{r}^{(0)}\|} \le \kappa(V) \min_{p_k \in \Pi_k,\, p_k(0)=1} \max_{\lambda_j} |p_k(\lambda_j)|, \tag{4.6}$$

where $\kappa(V) = \|V\|\,\|V^{-1}\|$ is the condition number of V. If, in addition, $\mathcal{E}$ is any set that contains the eigenvalues of F, then

$$\frac{\|\mathbf{r}^{(k)}\|}{\|\mathbf{r}^{(0)}\|} \le \kappa(V) \min_{p_k \in \Pi_k,\, p_k(0)=1} \max_{\lambda \in \mathcal{E}} |p_k(\lambda)|. \tag{4.7}$$

Proof Assertion (4.6) is derived from the observations that, for any polynomial p_k,

$$\begin{aligned} \|p_k(F)\mathbf{r}^{(0)}\| &= \|V p_k(\Lambda) V^{-1}\mathbf{r}^{(0)}\| \\ &\le \|V\|\,\|V^{-1}\|\,\|p_k(\Lambda)\|\,\|\mathbf{r}^{(0)}\| \\ &\le \|V\|\,\|V^{-1}\| \max_{\lambda_j} |p_k(\lambda_j)|\,\|\mathbf{r}^{(0)}\|. \end{aligned}$$

The bound (4.7) is an immediate consequence of (4.6). □

These "minimax" bounds generalize the analogous results (2.11) and (2.12) for the conjugate gradient method. There are, however, two significant differences. First, there is the presence of the condition number $\kappa(V)$ of the matrix of eigenvectors. It is difficult to bound this quantity, but its presence is unavoidable for polynomial bounds entailing the eigenvalues of F. Second, it is more difficult to derive an error bound for the GMRES iterates in a form that is as clean as