the tridiagonal matrix produced by the Lanczos algorithm, and (4.1) is identical to (2.30).

**Algorithm 4.1:** THE GMRES METHOD

Choose $\mathbf{u}^{(0)}$, compute $\mathbf{r}^{(0)} = \mathbf{f} - F\mathbf{u}^{(0)}$, $\beta_0 = \|\mathbf{r}^{(0)}\|$, $\mathbf{v}^{(1)} = \mathbf{r}^{(0)}/\beta_0$

**for** $k = 1, 2, \ldots$ **until** $\beta_k < \tau\beta_0$ **do**

  $\mathbf{w}_0^{(k+1)} = F\mathbf{v}^{(k)}$

  **for** $l = 1$ **to** $k$ **do**

    $h_{lk} = \langle \mathbf{w}_l^{(k+1)}, \mathbf{v}^{(l)} \rangle$

    $\mathbf{w}_{l+1}^{(k+1)} = \mathbf{w}_l^{(k+1)} - h_{lk}\mathbf{v}^{(l)}$

  **enddo**

  $h_{k+1,k} = \|\mathbf{w}_{k+1}^{(k+1)}\|$

  $\mathbf{v}^{(k+1)} = \mathbf{w}_{k+1}^{(k+1)}/h_{k+1,k}$

  Compute $\mathbf{y}^{(k)}$ such that $\beta_k = \left\|\beta_0\mathbf{e}_1 - \widehat{H}_k\mathbf{y}^{(k)}\right\|$ is minimized,

    where $\widehat{H}_k = [h_{ij}]_{1\le i\le k+1, 1\le j\le k}$

**enddo**

$\mathbf{u}^{(k)} = \mathbf{u}^{(0)} + V_k\mathbf{y}^{(k)}$

To derive a Krylov subspace iteration for solving systems of equations, we let $\mathbf{u}^{(k)} \in \mathbf{u}^{(0)} + \mathcal{K}_k(F, \mathbf{r}^{(0)})$. For the choice $\mathbf{v}^{(1)} = \mathbf{r}^{(0)}/\beta_0$, with $\beta_0 = \|\mathbf{r}^{(0)}\|$ as in Algorithm 4.1, this is equivalent to

$$\mathbf{u}^{(k)} = \mathbf{u}^{(0)} + V_k\mathbf{y}^{(k)} \tag{4.2}$$

for some $k$-dimensional vector $\mathbf{y}^{(k)}$. But the first line of (4.1) can be rewritten as $FV_k = V_{k+1}\widehat{H}_k$, and this implies that the residual satisfies

$$\mathbf{r}^{(k)} = \mathbf{r}^{(0)} - AV_k\mathbf{y}^{(k)} = V_{k+1}\left(\beta_0\mathbf{e}_1 - \widehat{H}_k\mathbf{y}^{(k)}\right), \tag{4.3}$$

where $\mathbf{e}_1 = (1, 0, \ldots, 0)^T$ is the unit vector of size $k$. The vectors $\{\mathbf{v}^{(j)}\}$ are pairwise mutually orthogonal, so that

$$\|\mathbf{r}^{(k)}\| = \beta_k = \left\|\beta_0\mathbf{e}_1 - \widehat{H}_k\mathbf{y}^{(k)}\right\|. \tag{4.4}$$

In particular, the residual of the iterate (4.2) with smallest Euclidean norm is determined by the choice of $\mathbf{y}^{(k)}$ that minimizes the expression on the right side of (4.4).

This upper-Hessenberg least squares problem can be solved by transforming $\widehat{H}_k$ into upper triangular form $\binom{R_k}{0}$, where $R_k$ is upper triangular, using $k+1$ plane rotations (which are also applied to $\beta_0\mathbf{e}_1$). Here, $\widehat{H}_k$ contains $\widehat{H}_{k-1}$ as a submatrix, so that in a practical implementation, $R_k$ can be updated from $R_{k-1}$. Moreover, by an analysis similar to that leading to (2.37), it can be shown that $\|\mathbf{r}^{(k)}\|$ is available at essentially no cost. Hence, a step of the GMRES algorithm