# 4

# SOLUTION OF DISCRETE CONVECTION–DIFFUSION PROBLEMS

As shown in Chapter 3, the coefficient matrix arising from discretization of the convection–diffusion equation is nonsymmetric. To develop iterative solution algorithms for these problems, as well as those arising in other settings such as the Navier–Stokes equations, the algorithms discussed in Chapter 2 must be adapted to handle nonsymmetric systems of linear equations. In this chapter, we outline the strategies and issues associated with Krylov subspace iteration for general nonsymmetric systems, together with specific details for convection–diffusion systems associated with preconditioning and multigrid methods.

## 4.1 Krylov subspace methods

We are considering iterative methods for solving a system $F\mathbf{u} = \mathbf{f}$, where, for the moment (i.e. in this section), $F$ represents an arbitrary nonsymmetric matrix of order $n$. Recall that for symmetric positive-definite systems, the conjugate gradient method has two properties that make it an effective iterative solution algorithm. It is *optimal*, in the sense that at the $k$th step, the energy norm of the error is minimized with respect to the $k$-dimensional Krylov space $\mathcal{K}_k(F, \mathbf{r}^{(0)})$. (Equivalently, the error is orthogonal to $\mathcal{K}_k(F, \mathbf{r}^{(0)})$ with respect to the energy inner product.) In addition, it is *inexpensive*: the number of arithmetic operations required at each step of the iteration is independent of the iteration count $k$. This also means that the storage requirements are fixed. Unfortunately, there are no generalizations of CG directly applicable to arbitrary nonsymmetric systems that have both of these properties. A Krylov subspace method for nonsymmetric systems of equations can display at most one of them: it can retain optimality but allow the cost per iteration to increase as the number of iterations grows, or it can require a fixed amount of computational work at each step but sacrifice optimality.

Before discussing what can be done, we note that one way to apply Krylov subspace methods to a nonsymmetric system is to simply create a symmetric positive definite one such as that defined by the normal equations $F^T F\mathbf{u} = F^T\mathbf{f}$. The problem could then be solved by applying the conjugate gradient method to the new system. This approach clearly inherits some of the favorable features of CG. However, the Krylov subspace generated is $\mathcal{K}_k(F^T F, F^T\mathbf{r}^{(0)})$ and therefore convergence will depend on properties of $F^T F$. For example, recall Theorem 2.4, which specifies a bound that depends on the condition number of the coefficient