STEP 9 : 2D Laplace equation

$$\boxed{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0}$$

Discretize:

$$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} = 0,$$

Transpose:

$$p^n_{ij} = \frac{\Delta y^2 (p^n_{i+1,j} + p^n_{i-1,j}) + \Delta x^2 (p^n_{i,j+1} + p^n_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}$$

IC : $p = 0$ everywhere

BC

$p = 0$ @ $x = 0$

$p = y$ @ $x = 2$

$\frac{\partial p}{\partial y} = 0$ @ $y = 0, 1$

Analytical solution

$$p(x,y) = \frac{x}{4} - 4 \sum_{n=1,\,odd}^{\infty} \frac{1}{(n\pi)^2 \sinh 2n\pi} \sinh n\pi x \cos n\pi y$$

$\square$.

STEP 10 2D Poisson Equation

$$\boxed{\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b}$$

Discretize:

$$\frac{p^n_{i+1,j} - 2p^n_{ij} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{ij} + p^n_{i,j-1}}{\Delta y^2} = b^n_{ij}$$

Step 10 (cont'd)

Transpose

$$p_{ij}^n = \frac{\Delta y^2 (p_{i+1,j}^n + p_{i-1,j}^n) + \Delta x^2 (p_{i,j+1}^n + p_{i,j-1}^n) - b_{ij}^n \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

IC — $p = 0$ everywhere

BC $p = 0$ @ $x = 0, 2$ / $y = 0, 1$

with:

$b_{ij} = 100$ @ $i = nx/4$ & $j = ny/4$

$b_{ij} = -100$ @ $i = (nx)3/4$ & $j = (ny)3/4$

$b_{ij} = 0$ everywhere else

$\boxtimes$.

STEP 11 — 2D Navier-Stokes (Cavity)

① $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

② $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

③ $$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \rho\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right]$$

Discretize:

① $$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + u_{i,j}^n \frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} + v_{i,j}^n \frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta y} =$$

$$-\frac{1}{\rho}\frac{p_{i+1,j}^n - p_{i-1,j}^n}{2\Delta x} + \nu\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right)$$

Step 11 (cont'd)

② $$\frac{v^{n+1}_{i,j} - v^n_{i,j}}{\Delta t} + u^n_{i,j}\frac{v^n_{i,j} - v^n_{i-1,j}}{\Delta x} + v^n_{i,j}\frac{v^n_{i,j} - v^n_{i,j-1}}{\Delta y} =$$

$$-\frac{1}{\rho}\frac{p^n_{i,j+1} - p^n_{i,j-1}}{2\Delta y} + \nu\left(\frac{v^n_{i+1,j} - 2v^n_{i,j} + v^n_{i-1,j}}{\Delta x^2} + \frac{v^n_{i,j+1} - 2v^n_{i,j} + v^n_{i,j-1}}{\Delta y^2}\right)$$

③ $$\frac{p^n_{i+1,j} - 2p^n_{i,j} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{i,j} + p^n_{i,j-1}}{\Delta y^2} =$$

$$\rho\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) - \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right)\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right)\right.$$

(minus) $$\left. - 2\left(\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\right)\left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}\right) - \left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right)\left(\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right)\right]$$

Transpose :

① $$u^{n+1}_{i,j} = u^n_{i,j} - u^n_{i,j}\frac{\Delta t}{\Delta x}\left(u^n_{i,j} - u^n_{i-1,j}\right)$$
$$- v^n_{i,j}\frac{\Delta t}{\Delta x}\left(u^n_{i,j} - u^n_{i,j-1}\right)$$
$$- \frac{\Delta t}{2\rho\Delta x}\left(p^n_{i+1,j} - p^n_{i-1,j}\right)$$
$$+ \nu\left\{\frac{\Delta t}{\Delta x^2}\left(u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}\right) + \frac{\Delta t}{\Delta y^2}\left(u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}\right)\right.$$

② $$v^{n+1}_{i,j} = v^n_{i,j} - u^n_{i,j}\frac{\Delta t}{\Delta x}\left(v^n_{i,j} - v^n_{i-1,j}\right) - v^n_{i,j}\frac{\Delta t}{\Delta x}\left(v^n_{i,j} - v^n_{i,j-1}\right)$$
$$- \frac{\Delta t}{2\rho\Delta y}\left(p^n_{i,j+1} - p^n_{i,j-1}\right)$$
$$+ \nu\left\{\frac{\Delta t}{\Delta x^2}\left(v^n_{i+1,j} - 2v^n_{i,j} + v^n_{i-1,j}\right) + \frac{\Delta t}{\Delta y^2}\left(v^n_{i,j+1} - 2v^n_{i,j} + v^n_{i,j-1}\right)\right\}$$

③ $$p^n_{i,j} = \frac{(p^n_{i+1,j} + p^n_{i-1,j})\Delta y^2 + (p^n_{i,j+1} + p^n_{i,j-1})\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$$

$$- \frac{\rho \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \left\{ \frac{1}{\Delta t} \left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \right) \ldots \right.$$

$$- \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \ldots$$

$$\cancel{+}\ 2\, \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} \frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} \ldots$$

(minus)

$$\left. - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y} \right\}.$$

I.C. $u, v, p = 0$ everywhere

B.C. $u = 1$ @ $y = 2$ $\qquad$ $p = 0$ @ $y = 2$

$u, v = 0$ elsewhere $\qquad$ $\partial p / \partial x = 0$ @ $x = 0, 2$

$\partial p / \partial y = 0$ @ $y = 0$

$\square$.

STEP 12 — 2D Navier Stokes ("channel")

① $$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + F$$

(simulates a pressure grad. driving the flow).

② $$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).$$

③ $$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \rho \left\{ \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial u}{\partial x} \frac{\partial u}{\partial x} \ldots \right.$$

$$\left. - 2 \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} \frac{\partial v}{\partial y} \right\}$$

Discretizations are the same as in Step 11, except for the new source term in equation ①.

① will be:

$$u^{n+1}_{i,j} = u^n_{i,j} - u^n_{i,j}\frac{\Delta t}{\Delta x}\left(u^n_{i,j} - u^n_{i-1,j}\right) - v^n_{i,j}\left(u^n_{i,j} - u^n_{i,j-1}\right)$$

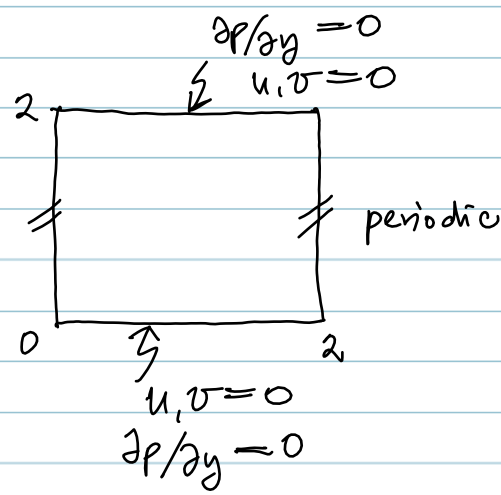
$$- \frac{\Delta t}{2\rho\Delta x}\left(p^n_{i+1,j} - p^n_{i-1,j}\right)$$

$$+ \nu\left\{\frac{\Delta t}{\Delta x^2}\left(u^n_{i+1,j} - 2u^n_{i,j} + u^n_{i-1,j}\right) + \frac{\Delta t}{\Delta y^2}\left(u^n_{i,j+1} - 2u^n_{i,j} + u^n_{i,j-1}\right)\right\}$$

$$+ \underbrace{F\Delta t}_{\text{extra source term.}} .$$

I.C. $u, v, p = 0$ everywhere

B.C. u, v, p periodic @ $x = 0, 2$
$u, v = 0$ @ $y = 0, 2$
$\partial p/\partial y = 0$ @ $y = 0, 2$
$F = 1$ everywhere



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