ME 702 — Computational Fluid Dynamics

(L Barba)

Practical Module : "The 12 steps to computing the Navier-Stokes equations"

STEP 1 : 1D Linear Convection Equation

$$\boxed{\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0}$$

- Space-time discretization : $i \rightarrow$ index of grid in x
  $n \rightarrow$ index of grid in t

- Numerical scheme : Forward Difference (FD) in time
  Backward Difference (BD) in space

- Discrete equation :

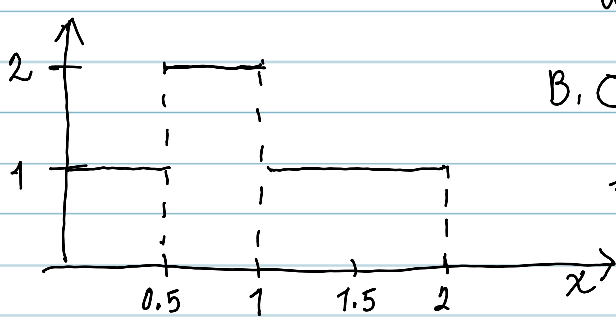
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

- Transpose, to obtain variable values at t_{n+1} from variable values at t_n

$$u_i^{n+1} = u_i^n - c\,\frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

Given an initial condition, this Difference Equation can be advanced in time!

- Consider as I.C. $u = 2$ @ $0.5 \le x \le 1$
  $u = 1$ @ everywhere else in $(0, 2)$



B.C. $u = 1$ @ $x = 0, 2$

"square wave"

STEP 2 : 1D Convection.

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

- Discretize as in STEP 1:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0.$$

- Transpose, as in STEP 1:

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right)$$

- I.C. $u = 2$ @ $0.5 \le x \le 1$
  $u = 1$ @ everywhere else in $(0,2)$

  B.C. $u = \phi$ @ $x = 0, 2$
  (same as in STEP 1)

$\boxtimes$.

For the next two steps, we need approximations of the Second Order derivative.
Geometrically, the second order derivative is the slope of the line tangent to the curve representing the first derivative.
So, we can use approximations for the first derivatives at 2 locations → all such approximations involve 3 points, at least

E.g. Central Difference, 2nd order:
combine FD & BD for the 1st derivative. Consider the Taylor expansion:

$$u_{i+1} = u_i + \Delta x \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i + \frac{\Delta x^3}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i + h.o.t$$

$$u_{i-1} = u_i - \Delta x \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i - \frac{\Delta x^3}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i + h.o.t$$

Add:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

STEP 3 : 1D Diffusion

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}}$$

- Scheme : FD in time & CD in space.
- Discretize: $$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$
- Transpose: $$u_i^{n+1} = u_i^n + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$
- Same I.C.'s & B.C.'s as STEPS 1 & 2.

$\square$.

STEP 4 : 1D Burger's equation

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}}$$

- Discretize:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Transpose

$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) + \frac{\nu \Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

I.C. $$u = -2\nu \frac{\partial \phi / \partial x}{\phi} + 4 \quad \text{with}$$

$$\phi = \exp\left(-x^2/4\nu\right) + \exp\left(-(x-2\pi)^2/4\nu\right)$$

B.C. $u(0) = u(2\pi)$

This problem has the analytical solution:

$* = (t+1)$ (missing below)

$$u = -2\nu \frac{\partial \phi / \partial x}{\phi} + 4 \quad \text{with}$$

$$\phi = \exp\left(-(x-4t)^2/4\nu^{*}\right) + \exp\left(-(x-4t-2\pi)^2/4\nu^{*}\right)$$

$\square$.

STEP 5 : 2D Linear Convection

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0}$$

Discretize: $i \to$ index in x & $j \to$ index in y.

$$\frac{u_{ij}^{n+1} - u_{i,j}^{n}}{\Delta t} + c\,\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + c\,\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} = 0$$

(nested loops will be needed for i & j)

Transpose: $$u_{i,j}^{n+1} = u_{i,j}^{n} - \frac{c\Delta t}{\Delta x}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) - \frac{c\Delta t}{\Delta x}\left(u_{i,j}^{n} - u_{i,j-1}^{n}\right)$$

I.C. $u = 2$ @ $0.5 \le x \le 1$ & $0.5 \le y \le 1$

$u = 1$ @ everywhere else in $(0,2) \times (0,2)$

B.C. $u = 1$ @ $x = 0, 2$ & $y = 0, 2$

$\square$.

STEP 6 : 2D Convection

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= 0 \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= 0 \end{aligned}}$$

Discretize:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\,\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\,\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} = 0$$

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\,\frac{v_{i,j}^{n} - v_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\,\frac{v_{i,j}^{n} - v_{i,j-1}^{n}}{\Delta y} = 0$$

Transpose:

$$u_{i,j}^{n+1} = u_{i,j}^{n} - u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(u_{i,j}^{n} - u_{i-1,j}^{n}\right) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(u_{i,j}^{n} - u_{i,j-1}^{n}\right)$$

$$v_{i,j}^{n+1} = v_{i,j}^{n} - u_{i,j}^{n}\frac{\Delta t}{\Delta x}\left(v_{i,j}^{n} - v_{i-1,j}^{n}\right) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}\left(v_{i,j}^{n} - v_{i,j-1}^{n}\right.$$

STEP 6 (cont'd)

IC $\begin{cases} u=2,\ v=2 & @\ 0.5 \le x \le 1 \ \& \ 0.5 \le y \le 1 \\ u=1,\ v=1 & @ \text{ everywhere else in } (0,2)\times(0,2) \end{cases}$

BC $u = 1$ (one), $v = 1$ (one) @ $x = 0, 2$ & $y = 0, 2$ ☒.

STEP 7 : 2D Diffusion

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}}$$

Discretize

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \nu \frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,y}^{n}}{\Delta x^2} + \nu \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i}^{n}}{\Delta y^2}$$

Transpose

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \frac{\nu \Delta t}{\Delta x^2}\left(u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}\right) + \frac{\nu \Delta t}{\Delta y^2}\left(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}\right)$$

IC $\begin{cases} u=2 & @\ 0.5 \le x \le 1 \ \& \ 0.5 \le y \le 1 \\ u=1 & @ \text{ everywhere else in } (0,2)\times(0,2) \end{cases}$

B.C $u = 1$ @ $x = 0, 2$ & $y = 0, 2$ ☒.

STEP 8 : 2D Burgers equation

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{aligned}}$$

STEP 8 (cont'd)

Discretize

1) $$\frac{u^{n+1}_{i,j} - u^{n}_{i,j}}{\Delta t} + u^{n}_{i,j}\frac{u^{n}_{i,j} - u^{n}_{i-1,j}}{\Delta x} + v^{n}_{i,j}\frac{u^{n}_{i,j} - u^{n}_{i,j-1}}{\Delta y}$$

$$= \nu\left(\frac{u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j}}{\Delta x^2} + \frac{u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j-1}}{\Delta y^2}\right)$$

2) $$\frac{v^{n+1}_{i,j} - v^{n}_{i,j}}{\Delta t} + u^{n}_{i,j}\frac{v^{n}_{i,j} - v^{n}_{i-1,j}}{\Delta x} + v^{n}_{i,j}\frac{v^{n}_{i,j} - v^{n}_{i,j-1}}{\Delta y}$$

$$= \nu\left(\frac{v^{n}_{i+1,j} - 2v^{n}_{i,j} + v^{n}_{i-1,j}}{\Delta x^2} + \frac{v^{n}_{i,j+1} - 2v^{n}_{i,j} + v^{n}_{i,j-1}}{\Delta y^2}\right)$$

Transpose

1) $$u^{n+1}_{i,j} = u^{n}_{i,j} - \frac{\Delta t}{\Delta x}u^{n}_{i,j}\left(u^{n}_{i,j} - u^{n}_{i-1,j}\right) - \frac{\Delta t}{\Delta y}v^{n}_{i,j}\left(u^{n}_{i,j} - u^{n}_{i,j-1}\right.$$

$$+ \frac{\nu\Delta t}{\Delta x^2}\left(u^{n}_{i+1,j} - 2u^{n}_{i,j} + u^{n}_{i-1,j}\right)$$

$$+ \frac{\nu\Delta t}{\Delta y^2}\left(u^{n}_{i,j+1} - 2u^{n}_{i,j} + u^{n}_{i,j+1}\right)$$

2) $$v^{n+1}_{i,j} = v^{n}_{i,j} - \frac{\Delta t}{\Delta x}u^{n}_{i,j}\left(v^{n}_{i,j} - v^{n}_{i-1,j}\right) - \frac{\Delta t}{\Delta y}v^{n}_{i,j}\left(v^{n}_{i,j} - v^{n}_{i,j-1}\right)$$

$$+ \frac{\nu\Delta t}{\Delta x^2}\left(v^{n}_{i+1,j} - 2v^{n}_{i,j} + v^{n}_{i-1,j}\right)$$

$$+ \frac{\nu\Delta t}{\Delta y^2}\left(v^{n}_{i,j+1} - 2v^{n}_{i,j} + v^{n}_{i,j-1}\right)$$

- IC's & B.C's → use the same as STEP 7.