ME 702 – Computational Fluid Dynamics

(L Barba)

Lecture 0 — Review: Differential form of the Fluid Eqns.

1) Conservation of Mass :

For a system : $\dfrac{D M_{sys}}{Dt} = 0$.

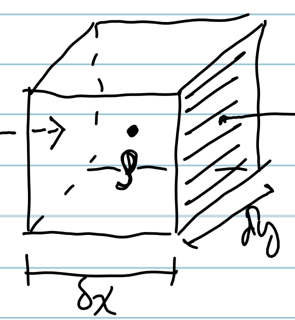
For a C.V. : $$\underbrace{\frac{\partial}{\partial t}\int_{cv} \rho \, d\mathcal{V}}_{\text{① Rate of change of mass in C.V}} + \underbrace{\int_{cs} \rho \vec{V}\cdot\hat{n}\, dA}_{\text{② Net rate of flow of mass across C.S.}} = 0$$

Differential Form — consider a small element $\delta x \delta y \delta z$

① $\dfrac{\partial}{\partial t}\displaystyle\int_{cv} \rho \, d\mathcal{V} = \dfrac{\partial \rho}{\partial t} \delta x \delta y \delta z$ $\quad \{ \rho =$ uniform in $\delta \mathcal{V}$

② Rate of mass flow : in the x-direction , ρu : x-comp. of the mass flow rate

$\left[\rho u - \dfrac{\partial}{\partial x}(\rho u)\dfrac{\delta x}{2}\right]\delta y \delta z$ $\longrightarrow$ $\left[\rho u + \dfrac{\partial}{\partial x}(\rho u)\dfrac{\delta x}{2}\right]\delta y \delta z$



Net rate of mass outflow in x = $\dfrac{\partial}{\partial x}(\rho u)\, \delta x \delta y \delta z$

Similarly : y-direction = $\dfrac{\partial}{\partial y}(\rho v)\, \delta x \delta y \delta z$

z-direction = $\dfrac{\partial}{\partial z}(\rho w)\, \delta x \delta y \delta z$ $\quad \}+$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

VECTOR NOTATION $\rightarrow$ $\boxed{\dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0}$

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Lecture ∅ - Review: Differential form of the Fluid Equations

2) Conservation of Momentum :

System : $$\vec{F} = \frac{D}{Dt}\int_{sys} \vec{V}\, dm$$

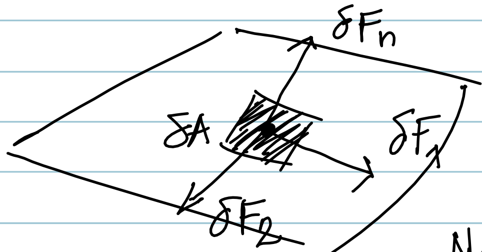
for a C.V : $$\sum \vec{F}_{cv} = \frac{\partial}{\partial t}\int_{cv} \vec{V} \rho\, d\mathcal{V} + \int_{cs} \vec{V} \rho\, \vec{V}\cdot\hat{n}\, dA$$

- Infinitesimal fluid mass δm : $\delta \vec{F} = \frac{D}{Dt}(\vec{V}\delta m) = \delta m \frac{D\vec{V}}{Dt}$

$$\delta \vec{F} = \delta m \cdot \vec{a}$$

- Types of forces — Body forces → weight $\delta \vec{F}_b = \delta m \cdot \vec{g}$

— Surface forces → normal & tangential



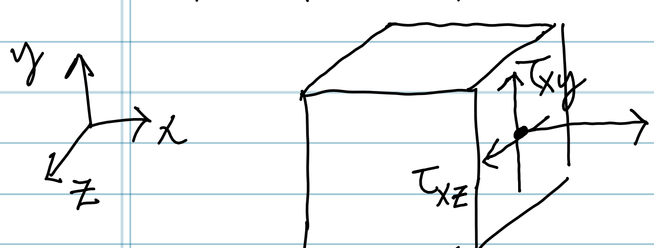
δF_n : normal to δA
δF_1, δF_2 ; tangent to δA

Normal Stress : $\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$

Shearing Stresses : $\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$; $\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$

With reference to a coordinate system :

Plane parallel to y-z

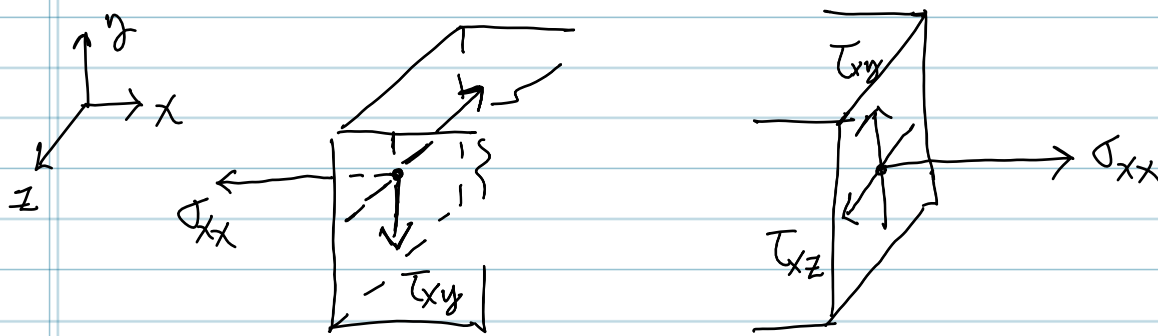


σ_{xx} : Normal stress
τ_{xy} & τ_{xz} Shearing Stresses

Subscript Notation : 1st refers direction of Normal Vector
2nd refers direction of stress vector.

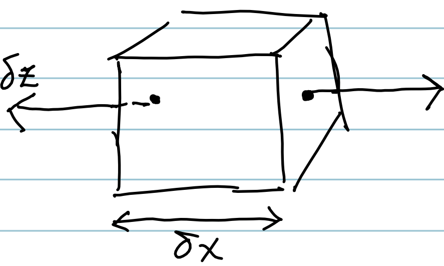
2) Conservation of Momentum (cont'd)

- sign convention for stresses :

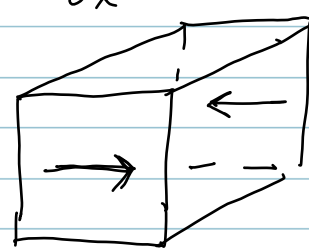


- Surface forces in terms of stresses :

$$\left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right)\delta y\,\delta z$$



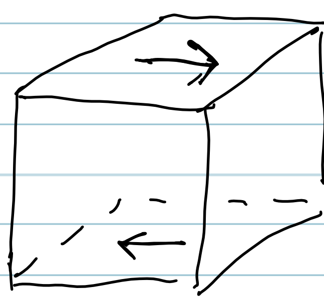
$$\left(\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\cdot\frac{\delta x}{2}\right)\delta y\,\delta z$$



$$\left(\tau_{zx} - \frac{\partial}{\partial z}\tau_{zx}\frac{\delta z}{2}\right)\delta x\,\delta y$$

$$\left(\tau_{zx} + \frac{\partial}{\partial z}\tau_{zx}\frac{\delta z}{2}\right)\delta x\,\delta y$$

$$\left(\tau_{yx} + \frac{\partial}{\partial y}\tau_{yx}\frac{\delta y}{2}\right)\delta x\,\delta z$$



$$\left(\tau_{yx} - \frac{\partial}{\partial y}\tau_{yx}\frac{\delta y}{2}\right)\delta x\,\delta z$$

All stresses shown in positive direction.

2) Conservation of Momentum (cont'd)

$$\delta F_{sx} = \left(\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} \right)\delta x \delta y \delta z$$

- Similarly, other directions

$$\delta F_{sy} = \left(\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + \frac{\partial}{\partial z}\tau_{zy} \right)\delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\sigma_{zz} \right)\delta x \delta y \delta z$$

Equations of motion : $\delta \vec{F} = \delta m \cdot \vec{a}$

$\delta m = \rho\, \delta x \delta y \delta z$

$$\left\{ \begin{aligned}
\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) \\
\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) \\
\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right)
\end{aligned} \right.$$

→ 3 equations + continuity = 4 eqns.
unknowns : u, v, w + all the stresses !

- **Inviscid Flow** → No shearing stresses

$\left. \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p \right\}$ pressure

Euler's Eqns : $\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right)$

VECTOR NOTATION :

$$\boxed{\rho \vec{g} - \nabla p = \rho\left(\frac{\partial}{\partial t}\vec{V} + (\vec{V}\cdot\nabla)\vec{V} \right)}$$

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Lecture 0 – Review : Differential Form of the Fluid Equations

3) The Navier-Stokes equations :

Newtonian fluids $\rightarrow$ Linear relationship between stresses & rates of deformation

Normal stresses : $\sigma_{xx} = -p + 2\mu \, \partial u / \partial x$

$$\sigma_{yy} = -p + 2\mu \, \partial v / \partial y \quad ; \quad \sigma_{zz} = -p + 2\mu \, \partial w / \partial z$$

$$\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) = -p \quad \Big\} \quad \nabla \cdot \vec{V} = 0 \text{ incompressible}$$

Shearing stresses : $\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \quad ; \quad \tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

Stress terms , x-direction momentum eqn:

$$\underbrace{\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial \tau_{zx}}{\partial z}}$$

$$-\frac{\partial}{\partial x}p + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x} \right) + \mu \left(\frac{\partial^2 w}{\partial z \partial x} + \frac{\partial^2 u}{\partial z^2} \right)$$

rewrite;

$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x} \right)$$

$$\mu \cdot \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)$$

incompressible $\Big\{ \quad = 0$

3) The Navier Stokes equations (cont'd).

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

use vector notation:

$$\boxed{\rho\left(\frac{\partial}{\partial t}\vec{V} + (\vec{V}\cdot\nabla)\vec{V}\right) = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V}}$$

Navier Stokes

$\rho = \text{const.}$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

3 equations + continuity = 4 eqns.

Unknowns: $u, v, w, p, \rho \Rightarrow$ 5 unknowns

$\rightarrow$ Need an equation of state

* Nonlinear, Second order PDE $\Rightarrow$ Very few known solution!

$\downarrow$

$(\vec{V}\cdot\nabla)\vec{V}$

- Only general approach is computational