Computational Fluid Dynamics, Vol. I
by Klaus A. Hoffmann and Steve T. Chiang, EES Books, 2000 (4th edition)

Chapter 1

Classification of Partial Differential Equations

1.1 Introductory Remarks

Since the solution procedure of a partial differential equation (PDE) depends on the type of the equation, it is important to study various classifications of PDEs. Imposition of initial and/or boundary conditions also depends on the type of PDE. Most of the governing equations of fluid mechanics and heat transfer are expressed as second-order PDEs and therefore classification of such equations is considered in this chapter. In addition, a system of first-order PDEs and a system of second-order PDEs are considered as well.

1.2 Linear and Nonlinear PDEs

Partial differential equations can be classified as linear or nonlinear. In a linear PDE, the dependent variable and its derivatives enter the equation linearly, i.e., there is no product of the dependent variable or its derivatives. Individual solutions of this type of PDE can be superimposed, e.g., two solutions to the governing equation can be added together to give a third solution to the original equation. An example of a linear PDE is the one-dimensional wave equation

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x}$$

where a is the speed of sound which is assumed constant.

On the other hand, a nonlinear PDE contains a product of the dependent variable and/or a product of its derivatives. Two solutions to a nonlinear equation cannot be added to produce a third solution that also satisfies the original equation. An example of a nonlinear PDE is the inviscid Burgers equation:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

If a PDE is linear in its highest order derivatives, it is called a quasi-linear PDE.

1.3 Second-Order PDEs

To classify the second-order PDE, consider the following equation

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} + D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial y} + F\phi + G = 0 \tag{1-1}$$

where, in general, the coefficients A, B, C, D, E, F, and G are functions of the independent variables x and y and of the dependent variable ϕ. Assume that $\phi = \phi(x, y)$ is a solution of the differential equation. This solution describes a surface in space, on which space curves may be drawn. These curves patch various solutions of the differential equation and are known as the characteristic curves. Some fundamental concepts of characteristics are provided in Appendix A.

By definition, the second-order derivatives along the characteristic curves are indeterminate and, indeed, they may be discontinous across the characteristics. However, no discontinuity of the first derivatives is allowed, i.e., they are continuous functions of x and y. Thus, the differentials of ϕ_x and ϕ_y, which represent changes from location (x, y) to $(x + dx,\ y + dy)$ across the characteristics, may be expressed as

$$d\phi_x = \frac{\partial \phi_x}{\partial x}dx + \frac{\partial \phi_x}{\partial y}dy = \frac{\partial^2 \phi}{\partial x^2}dx + \frac{\partial^2 \phi}{\partial x \partial y}dy \tag{1-2}$$

and

$$d\phi_y = \frac{\partial \phi_y}{\partial x}dx + \frac{\partial \phi_y}{\partial y}dy = \frac{\partial^2 \phi}{\partial x \partial y}dx + \frac{\partial^2 \phi}{\partial y^2}dy \tag{1-3}$$

The original equation, i.e., Equation (1-1), may be expressed as follows

$$A\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + C\frac{\partial^2 \phi}{\partial y^2} = H \tag{1-4}$$

where

$$H = -\left(D\frac{\partial \phi}{\partial x} + E\frac{\partial \phi}{\partial y} + F\phi + G\right)$$

Now Equation (1-4), along with Equations (1-2) and (1-3), can be solved for the second-order derivatives of ϕ. For example, using Cramer's rule,

$$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\begin{vmatrix} A & H & C \\ dx & d\phi_x & 0 \\ 0 & d\phi_y & dy \end{vmatrix}}{\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}} \tag{1-5}$$

Since it is possible to have discontinuities in the second-order derivatives of the dependent variable across the characteristics, these derivatives are indeterminate. Thus, setting the denominator equal to zero,

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \tag{1-6}$$

yields the equation

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{1-7}$$

Solving this quadratic equation yields the equations of the characteristics in physical space:

$$\left(\frac{dy}{dx}\right)_{\alpha,\beta} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \tag{1-8}$$

Setting the numerator of (1-5) equal to zero provides a set of characteristic curves in the ϕ_x, ϕ_y plane. These are known as hodograph characteristics. Depending on the value of $(B^2 - 4AC)$, characteristic curves can be real or imaginary. For problems in which real characteristics exist, a disturbance can propagate only over a finite region, as shown in Figure 1-1. The downstream region affected by a disturbance at point A is called the zone of influence (indicated by horizontal shading). A signal at point A will be felt only if it originated from a finite region called the zone of dependence of point A (vertical shading).

The second-order PDE previously expressed as Equation (1-1) is classified according to the sign of the expression $(B^2 - 4AC)$. It will be

(a) elliptic if $B^2 - 4AC < 0$

(b) parabolic if $B^2 - 4AC = 0$ or

(c) hyperbolic if $B^2 - 4AC > 0$

Note that the classification depends only on the coefficients of the highest order derivatives.

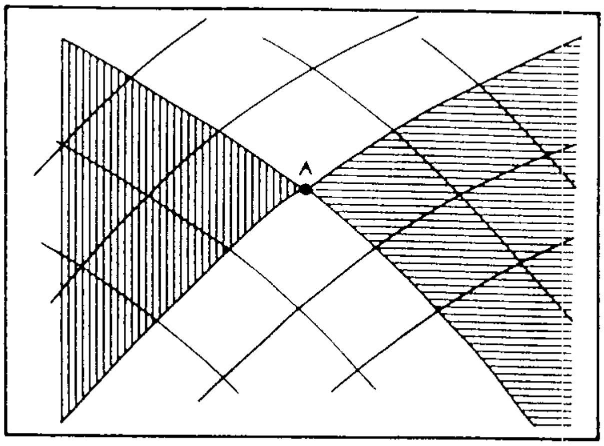

Figure 1-1. Zone of influence (horizontal shading) and zone of dependence (vertical shading) of point A.

1.4 Elliptic Equations

A partial differential equation is elliptic in a region if $(B^2 - 4AC) < 0$ at all points of the region. An elliptic PDE has no real characteristic curves. A disturbance is propagated instantly in all directions within the region. Examples of elliptic equations are Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1-9}$$

and Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x,y) \tag{1-10}$$

The domain of solution for an elliptic PDE is a closed region, R, shown in Figure 1-2. On the closed boundary of R, either the value of the dependent variable, its normal gradient, or a linear combination of the two is prescribed. Providing the boundary conditions uniquely yields the solution within the domain.

1.5 Parabolic Equations

A partial differential equation is classified as parabolic if $(B^2 - 4AC) = 0$ at all points of the region. The solution domain for a parabolic PDE is an open region, as shown in Figure 1-3. For a parabolic partial differential equation there exists one characteristic line. Unsteady heat conduction in one dimension

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1-11}$$

and diffusion of viscosity, expressed as

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1-12}$$

are examples of parabolic PDEs. An initial distribution of the dependent variable and two sets of boundary conditions are required for a complete description of the problem. The boundary conditions are prescribed as the value of the dependent variable or its normal derivative or a linear combination of the two. The solution of the parabolic equation marches downstream within the domain from the initial plane of data satisfying the specified boundary conditions. The parabolic partial differential equation is the counterpart to an initial value problem in an ordinary differential equation (ODE).

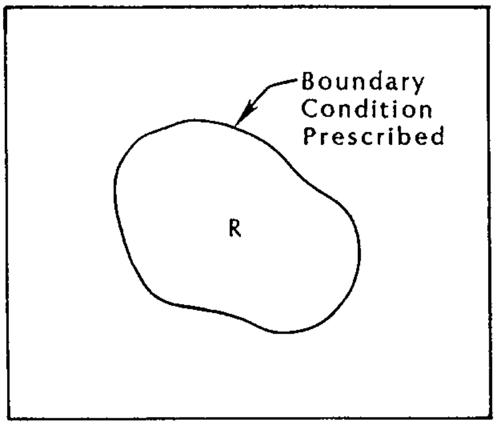

Figure 1-2. The domain of solution for an elliptic PDE.

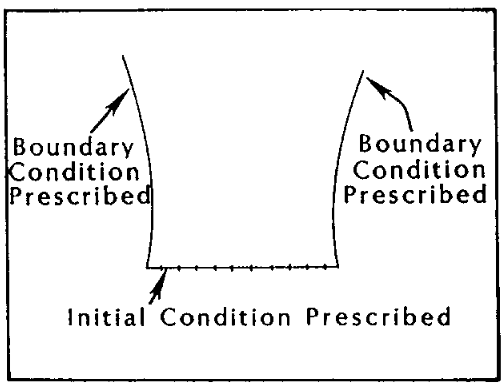

Figure 1-3. The domain of solution for a parabolic PDE.

1.6 Hyperbolic Equations

A partial differential equation is called hyperbolic if $(B^2 - 4AC) > 0$ at all points of the region. A hyperbolic PDE has two real characteristics. An example of a hyperbolic equation is the second-order wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2} \tag{1-13}$$

A complete description of the flow governed by a second-order hyperbolic PDE requires two sets of initial conditions and two sets of boundary conditions. The initial conditions at $t = 0$ may be expressed as

$$\phi(x, 0) = f(x)$$

and

$$\phi_t(x, 0) = g(x)$$

where the functions f and g are specified for a particular problem.

For a first-order hyperbolic equation, such as

$$\frac{\partial \phi}{\partial t} = -a \frac{\partial \phi}{\partial x}$$

only one initial condition needs to be specified. Note that the initial condition cannot be specified along a characteristic line.

A classical method of solving a hyperbolic PDE with two independent variables is the method of characteristics (MOC). Along the characteristic lines, the PDE reduces to an ODE, which can be easily integrated to obtain the desired solution. Details of MOC and the appropriate solution schemes will not be discussed here. However, some essential elements of characteristics are provided in Appendix A. Additional materials on MOC may be found in References [1-1] or [1-2].

To illustrate classification of a second-order PDE, an example is proposed as follows:

Example 1.1: Classify the steady two-dimensional velocity potential equation.

$$\left(1 - M^2\right) \phi_{xx} + \phi_{yy} = 0$$

Solution: According to notations used in Equation (1-1),

$$A = (1 - M^2), \quad B = 0, \quad \text{and} \quad C = 1$$

Thus, $(B^2 - 4AC) = -4(1 - M^2)$. If $M < 1$ (subsonic flow), then $(B^2 - 4AC) < 0$ and the equation is elliptic. For $M = 1$ (sonic flow), $(B^2 - 4AC) = 0$ and the equation is parabolic. For $M > 1$ (supersonic flow), $(B^2 - 4AC) > 0$ and the equation is hyperbolic.

Now consider the physical interpretation of various classifications. Assume that a body moving with a velocity u in an inviscid fluid is creating disturbances which propagate with the speed of sound, a. If the velocity u is smaller than a, that is, if the flow is subsonic, then the disturbance is felt everywhere in the flowfield (Figure 1-4a). Note that this is what happens for an elliptic PDE.

As the speed of the body u increases and approaches the speed of sound, a front is developed, with a region ahead of it which does not feel the presence of the disturbance (Figure 1-4b). This region is known as the zone of silence. Thus the disturbance is felt only behind the front. This region is known as the zone of action. When the speed u is further increased, to the extent that it exceeds the speed of sound, a conical front (in three-dimensional analysis) is formed (Figure 1-4c). The effect of the disturbance is felt only within this cone.

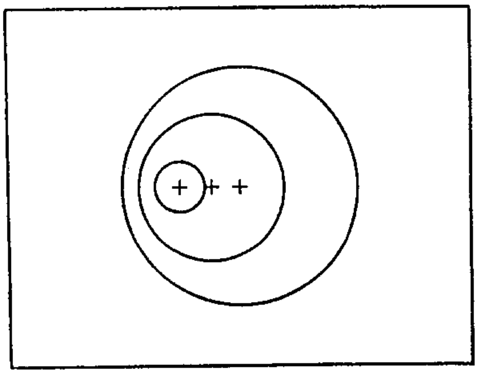

Figure 1-4a. Propagation of disturbance in subsonic flow.

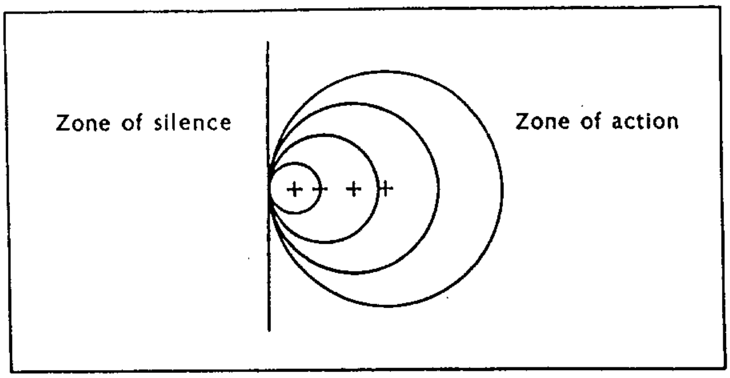

Figure 1-4b. Propagation of disturbance in sonic flow.

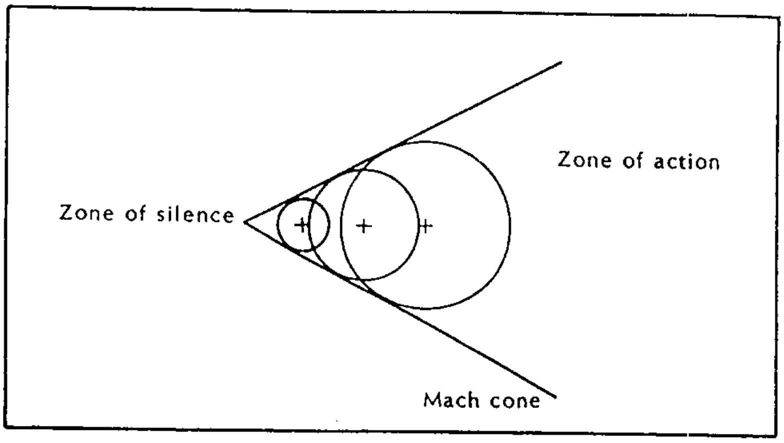

Figure 1-4c. Propagation of disturbance in supersonic flow.

This conical front is known as the Mach cone in three-dimensional space or as Mach lines in two-dimensional space. Mach lines patch two different solutions of the PDE and thus represent the characteristics of the PDE.

1.7 Model Equations

Several partial differential equations will be used as model equations in the following chapters. These equations will be used to illustrate the application of various finite differencing techniques and stability analyses. By observing and analyzing the behavior of the numerical methods when applied to simple model equations, an understanding should be developed which will be useful in studying more complex problems. The selected equations are primarily derived from principles of fluid mechanics and heat transfer. However, this selection should not limit our discussion to problems in fluid mechanics. Many PDEs in science and engineering may be represented by the selected model equations investigated here.

The selected model PDEs which will be used in the next chapters are as follows:

1. Laplace's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1-14}$$

2. Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = f(x, y) \tag{1-15}$$

3. The equation for unsteady heat conduction:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \tag{1-16}$$

4. The y-component of the Navier-Stokes equation reduced to Stokes' first problem:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1-17}$$

5. The wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \left(\frac{\partial^2 u}{\partial x^2} \right) \tag{1-18}$$

6. The Burgers equation:

$$\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} \tag{1-19}$$

These equations are expressed in one- or two-space dimensions in the Cartesian coordinate system. Some of the model equations in two-space dimensions will be reduced to one-space dimension in the upcoming discussions.

In most cases, the selected model equation subject to imposed initial and boundary conditions has an analytical solution. In such instances, the analytical solution is used as a basis for comparison with various numerical solutions. These comparisons are very useful in determining the accuracy of the various numerical algorithms employed.

1.8 System of First-Order PDEs

The equations of fluid motion are composed of conservation of mass, conservation of momentum, and conservation of energy. The governing equations may be expressed by partial differential equations, thus forming a system of second-order PDEs. For certain classes of problems, the governing equations are reduced to a system of first-order PDEs. For example, the equations of fluid motion for inviscid flowfields, known as the Euler equations, belong to this category. Furthermore, in some applications a higher-order PDE may be reduced to a system of first-order PDEs by introducing new viariables. In this section, the conditions under which a system of first-order PDEs is classified will be explored. Consider a set of first-order PDEs expressed in the following form

$$\frac{\partial \Phi}{\partial t} + [A]\frac{\partial \Phi}{\partial x} + [B]\frac{\partial \Phi}{\partial y} + \Psi = 0 \tag{1-20}$$

where Φ represents a vector (or column matrix) containing the unknown variables. The elements of the coefficient matrices $[A]$ and $[B]$ are functions of x, y, and t; and the vector Ψ is a function of Φ, x, and y. For example, a set of two first-order PDEs could be represented by the following equations:

$$\frac{\partial u}{\partial t} + a_1\frac{\partial u}{\partial x} + a_2\frac{\partial v}{\partial x} + a_3\frac{\partial u}{\partial y} + a_4\frac{\partial v}{\partial y} + \Psi_1 = 0 \tag{1-21}$$