ME 702 – COMPUTATIONAL FLUID DYNAMICS

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(lecture 8 (2011) handout)

PREVIOUSLY — We discussed STABILITY & CONVERGENCE of numerical schemes

USING : 1D linear convection, we looked at
* Truncation error
* Modified differential equation (Mod DE)

e.g. Using CD in x & Forward D in t
the Mod DE is:

$$\bar{u}_t + c\bar{u}_x = -\frac{\Delta t}{2} c^2 \bar{u}_{xx} + O(\Delta t^2, \Delta x^2)$$

↳ a convection–diffusion equation with negative diffusion coefficient ⇒ scheme is unstable

but : 1st order upwind
the Mod DE is:

$$\bar{u}_t + c\bar{u}_x = \underbrace{c\frac{\Delta x}{2}\left(1 - \frac{c\Delta t}{\Delta x}\right)}_{\text{Numerical diffusion}} \bar{u}_{xx}$$

↳ Stability requires $c > 0$ & $\boxed{\sigma = \frac{c\Delta t}{\Delta x} < 1}$ (CFL condition)

— We then discussed Von Neumann stability analysis

IN SUMMARY : Schemes can have
- conditional stability
- unconditional stability
- unconditional instability

NOTE — Explicit schemes are (at best) conditionally stable.
Implicit schemes are (in general) unconditionally stab

* CONDITIONAL STABILITY PUTS A LIMIT ON THE TIME STEP *

⇒ to maintain stability, one cannot progress too rapidly in time. This can be a severe requirement, especially for convection-dominated cases.

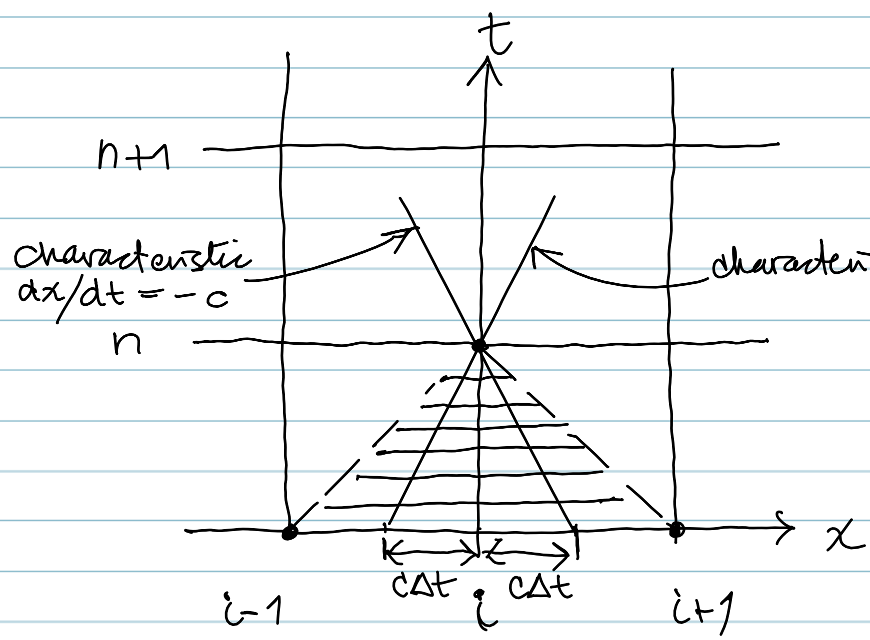
- Implicit methods: no time step restrictions, but more computational work per step
- For diffusion equations: explicit time step restriction

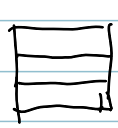
$$\frac{\nu \Delta t}{\Delta x^2} \le \frac{1}{2}$$

, is generally not so severe.

Physical interpretation of CFL condition $\quad c\frac{\Delta t}{\Delta x} < 1$

* Expresses that the distance covered by the solution during interval Δt should be smaller than Δx.



 indicates domain of dependence of the numerical scheme

$c > 0$: points $i-1$ and i

$c < 0$: points i and $i+1$

— CFL condition $\sigma < 1$ ensures that the domain of dependence of the Differential Eqn. should be entirely contained in the <u>Numerical</u> domain of dependence of the discretized equation

↳ u_i^{n+1} must be able to include all the physical information influencing the system at x_i

— This interpretation: Useful in 2D & 3D when it may be difficult to express analytically the stability condition

Also illustrates why a B.D. scheme, based on points i and $i-1$ would be unstable when $c<0$: the solution would be convected from right to left and the value at x_i should depend on x_{i+1}

↳ we need a Forward Diff. scheme

(and using x_{i-1} is contrary to the physics)

NEW SCHEMES FOR CONVECTION EQUATION

① <u>Leapfrog scheme</u> : CD in x and t.

$$\boxed{\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0}$$

or:

$$u_i^{n+1} = u_i^{n-1} - \sigma\left(u_{i+1}^n - u_{i-1}^n\right)$$

Von Neumann:
$$V^{n+1} = V^{n-1} - \sigma V^n\left(e^{I\phi} - e^{-I\phi}\right)$$

$$G = \frac{V^{n+1}}{V^n} = \frac{V^n}{V^{n-1}}$$

↳ $G - 1/G = -\sigma\left(e^{I\phi} - e^{-I\phi}\right)$

* quadratic eqn. for G:
$$G = I\sigma\sin\phi \pm \sqrt{1 - \sigma^2 \sin^2\phi}$$

- $\sigma > 1$, term under sqrt can become negative leading to G purely imaginary and $\|G\| > 1$
  $\Rightarrow$ unstable
- $|\sigma| \leq 1$, term under sqrt always real and

$$G \cdot G^* = Re(G)^2 + Im(G)^2 = (1 - \sigma^2 \sin^2\phi) + \sigma^2 \sin^2\phi$$
$$= 1$$

$\Rightarrow$ Neutrally stable

(2) Lax-Friedrichs scheme (1954)

* As a way of stabilizing FTCS (forward time - central scheme)
  ↳ Replace u_i by the average $\frac{1}{2}(u_{i-1} + u_{i+1})$
  - introduces an error of $O(\Delta x)$, reducing the scheme to first order in space.

$$\boxed{u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right)}$$

Von Neumann $\rightarrow$ $G = \cos\phi - I\sigma \sin\phi$
an ellipse in the complex G-plane $\Rightarrow$ $|\sigma| \leq 1$ for stability

(3) Lax-Wendroff scheme (LW)

* derived from Taylor expansion

$$\boxed{u_i^{n+1} = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\sigma^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}$$

Von Neumann $\rightarrow$ $G = 1 - I\sigma\sin\phi - \sigma^2(1 - \cos\phi)$
an ellipse in the complex G-plane $\Rightarrow$ $|\sigma| \leq 1$ for stability