



A collection of 2D elliptic problems for testing adaptive grid refinement algorithms[☆]



William F. Mitchell

National Institute of Standards and Technology, 100 Bureau Dr., Stop 8910, Gaithersburg, MD 20899-8910, United States

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ABSTRACT

Adaptive grid refinement is a critical component of the improvements that have recently been made in algorithms for the numerical solution of partial differential equations (PDEs). The development of new algorithms and computer codes for the solution of PDEs usually involves the use of proof-of-concept test problems. 2D elliptic problems are often used as the first test bed for new algorithms and codes. This paper contains a set of twelve parametrized 2D elliptic test problems for adaptive grid refinement algorithms and codes. The problems exhibit a variety of types of singularities, near singularities, and other difficulties.

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1. Introduction

The numerical solution of partial differential equations (PDEs) is the most compute-intensive part of a wide range of scientific and engineering applications. Consequently the development and application of faster and more accurate methods for solving partial differential equations has received much attention in the past fifty years. Self-adaptive methods to determine a quasi-optimal grid are a critical component of the improvements. Although adaptive grid refinement techniques are now in widespread use in applications, they remain an active field of research, particularly in the context of *hp*-adaptive techniques.

The development of new algorithms and computer codes for the solution of PDEs usually involves the use of proof-of-concept test problems. Such test problems have a variety of uses such as demonstrating that a new algorithm is effective, verifying that a new code is correct in the sense of achieving the theoretical order of convergence, and comparing the performance of different algorithms and codes. Nearly every paper on algorithms for solving PDEs contains a numerical results section with one or more test problems.

In this paper, the focus is on 2D elliptic problems, as they are often used as the first test bed for new algorithms and codes for solving PDEs. Other classes of problems (3D, hyperbolic, goal-oriented, etc.) can be found on the NIST Adaptive Mesh Refinement Benchmark Problems web page [1]. This paper contains a set of twelve 2D elliptic test problems for adaptive grid refinement algorithms and codes. Most of the problems are taken from the numerical results section of papers in the adaptive grid refinement literature. The problems exhibit a variety of types of singularities (point and line singularities on the boundary and in the interior), near singularities (sharp peaks, boundary layers, and wave fronts), and other difficulties. Most of the problems are parametrized to allow “easy” and “hard” variations on the problem. This collection of problems can be used for testing correctness of a computer code or optimality of an algorithm by observing the convergence rate of a norm (energy, H^1 , L^2 , etc.) of the error with respect to the number of degrees of freedom, to test the efficiency index of error estimates, to benchmark the computation time of different computer codes and algorithms, etc.

We primarily consider elliptic partial differential equations of the form

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E-mail address: william.mitchell@nist.gov
URL: <http://math.nist.gov/~WMitchell>

$$-\frac{\partial}{\partial x} \left(p(x, y) \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(q(x, y) \frac{\partial u}{\partial y} \right) + r(x, y)u = f(x, y) \quad \text{in } \Omega \quad (1)$$

$$u = g_D(x, y) \quad \text{on } \partial\Omega_D \quad (2)$$

$$p(x, y) \frac{\partial u}{\partial x} \frac{\partial y}{\partial s} - q(x, y) \frac{\partial u}{\partial y} \frac{\partial x}{\partial s} + c(x, y)u = g_N(x, y) \quad \text{on } \partial\Omega_N \quad (3)$$

with coefficient functions p , q , r and c , and right hand sides f , g_D and g_N , where Ω is a bounded, connected, polygonal, open region in \mathbb{R}^2 with boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$. Differentiation with respect to s is with respect to a counter-clockwise parametrization of the boundary $(x(s), y(s))$ with $\|(dx/ds, dy/ds)\| = 1$. Eq. (2) represents Dirichlet boundary conditions, and Eq. (3) represents natural boundary conditions if $c = 0$ and mixed boundary conditions otherwise. We assume the data in Eqs. (1)–(3) satisfy the usual ellipticity and regularity assumptions. Some of the test problems extend this to a coupled system of two equations, and the inclusion of first order derivative and mixed derivative terms.

Many of the test problems use the special case of Poisson's Equation

$$-\nabla^2 u := -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

and Laplace's Equation, which is Poisson's equation with $f = 0$. All but one problem are defined by the method of manufactured solution. When the right side of the equation is simply given as “ f ” and the exact solution is known, f is determined by applying the operator to the exact solution. Similarly, if the boundary conditions are said to be Dirichlet and the exact solution is known, g_D is given by the exact solution.

We use the terms singular and singularity rather loosely. We consider a function to be singular (or to have a singularity) if it or a derivative of any order is singular. More precisely, we consider it to be singular if there exists a finite positive m such that the function does not lie in H^m , where H^m is the usual Sobolev space of functions whose derivatives of order m are square integrable, and the usual extensions to noninteger m . We refer to the largest m such that the function is in H^m as the Sobolev regularity of the function.

2. The set of test problems

2.1. Analytic solution

This is a well behaved problem with a smooth solution that has no trouble spots. It can be used for seeing how an adaptive algorithm behaves in a context where adaptivity is not really needed.

Equation: Poisson.

Domain: unit square.

Boundary conditions: Dirichlet.

Solution: $2^{4a}x^a(1-x)^ay^a(1-y)^a$.

Parameters: the integer a determines the degree of the polynomial solution. It should be chosen to be large enough that the highest order finite elements to be used will not give the exact solution.

The solution with $a = 10$ is shown in Fig. 1, both as a color-mapped image and as a surface in perspective. The other figures that show solutions also present these two views.

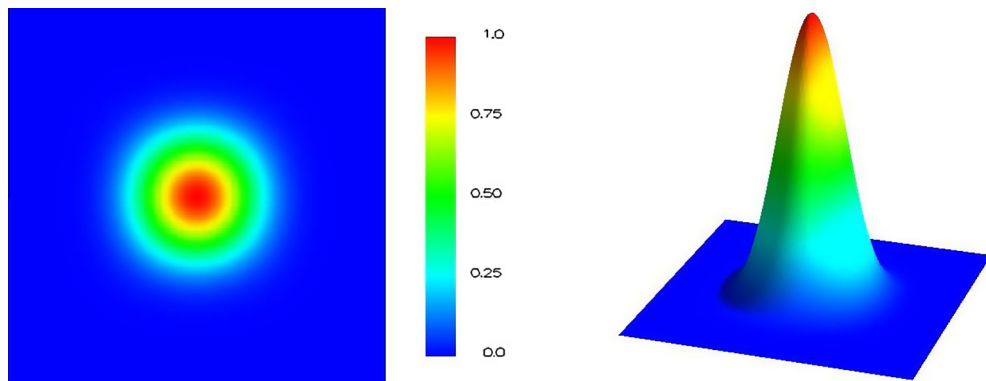


Fig. 1. The solution of the analytic problem with $a = 10$.

2.2. Reentrant corner

For elliptic partial differential equations, reentrant corners in the domain are a source of singularities in the solution. In particular, for a corner with an angle ω as shown in Fig. 2, the solution behaves like r^α where r is the distance from the corner and $\alpha = \pi/\omega$. The solution is in $H^{1+\alpha-\epsilon} \forall \epsilon > 0$ [2].

Equation: Laplace.

Domain: $(-1, 1) \times (-1, 1)$ with a section removed from the clockwise side of the positive x axis, as shown in Fig. 2.

Boundary conditions: Dirichlet.

Solution: $r^\alpha \sin(\alpha\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

Parameters: ω determines the angle of the reentrant corner, and consequently α and the strength of the singularity. Varying ω can be used to study the effect of the strength of the singularity on adaptive algorithms. Using $\omega = 3\pi/2$ gives the infamous “L domain” problem used heavily in the adaptive refinement community. With $\omega = 2\pi$ the domain is a square with a slit. A solution with ω slightly larger than π is nearly linear.

The solutions for $\omega = \pi + 0.01$, $5\pi/4$, $3\pi/2$, $7\pi/4$, and 2π are shown in Figs. 3–7.

2.3. Linear elasticity

Several papers [3–6] use a problem from linear elasticity as an example. This is a coupled system of two equations with a mixed derivative in the coupling term. In [3] the equations are given as

$$\begin{cases} -E \frac{1-v^2}{1-2v} \frac{\partial^2 u}{\partial x^2} - E \frac{1-v^2}{2-2v} \frac{\partial^2 u}{\partial y^2} - E \frac{1-v^2}{(1-2v)(2-2v)} \frac{\partial^2 v}{\partial x \partial y} = F_x \\ -E \frac{1-v^2}{2-2v} \frac{\partial^2 v}{\partial x^2} - E \frac{1-v^2}{1-2v} \frac{\partial^2 v}{\partial y^2} - E \frac{1-v^2}{(1-2v)(2-2v)} \frac{\partial^2 u}{\partial x \partial y} = F_y \end{cases} \quad (4)$$

where u and v are the x and y displacements, E is Young's Modulus, and v is Poisson's ratio.

Two solutions are given in [3] in polar coordinates; a mode 1 solution

$$\begin{cases} u = \frac{1}{2G} r^\lambda [(\kappa - Q(\lambda + 1)) \cos(\lambda\theta) - \lambda \cos((\lambda - 2)\theta)] \\ v = \frac{1}{2G} r^\lambda [(\kappa + Q(\lambda + 1)) \sin(\lambda\theta) + \lambda \sin((\lambda - 2)\theta)] \end{cases} \quad (5)$$

and a mode 2 solution

$$\begin{cases} u = \frac{1}{2G} r^\lambda [(\kappa - Q(\lambda + 1)) \sin(\lambda\theta) - \lambda \sin((\lambda - 2)\theta)] \\ v = -\frac{1}{2G} r^\lambda [(\kappa + Q(\lambda + 1)) \cos(\lambda\theta) + \lambda \cos((\lambda - 2)\theta)] \end{cases} \quad (6)$$

where $\kappa = 3 - 4\nu$, $G = E/(2(1 + \nu))$, and λ and Q are constants. The solutions have a point singularity at the origin and are in $H^{1+\lambda-\epsilon} \forall \epsilon > 0$ [4].

The domain is taken to be a square with a slit (a cracked plate) as in [6]. Some of the other references use an L-shaped domain.

Equation: coupled system of two equations given by Eq. (4).

Domain: $(-1, 1) \times (-1, 1)$ with a slit from $(0, 0)$ to $(1, 0)$.

Boundary conditions: Dirichlet.

Solution: Two solutions as given in Equations 5 and 6.

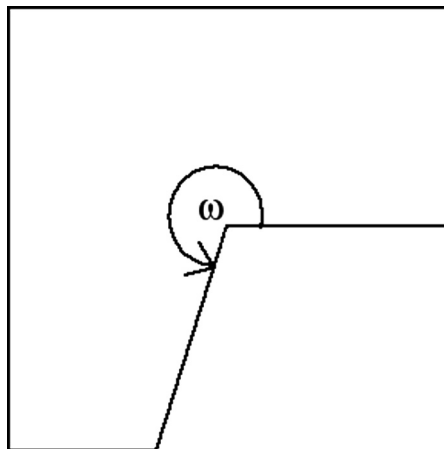


Fig. 2. Domain for the reentrant corner problem.

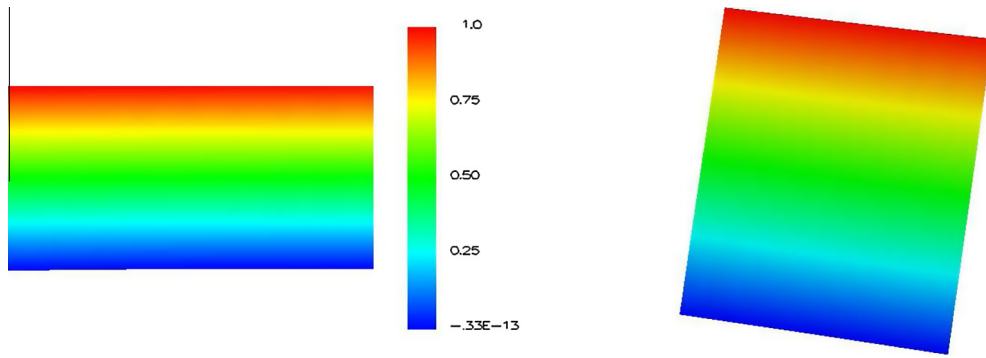


Fig. 3. The solution of the reentrant corner problem with $\omega = \pi + .01$.

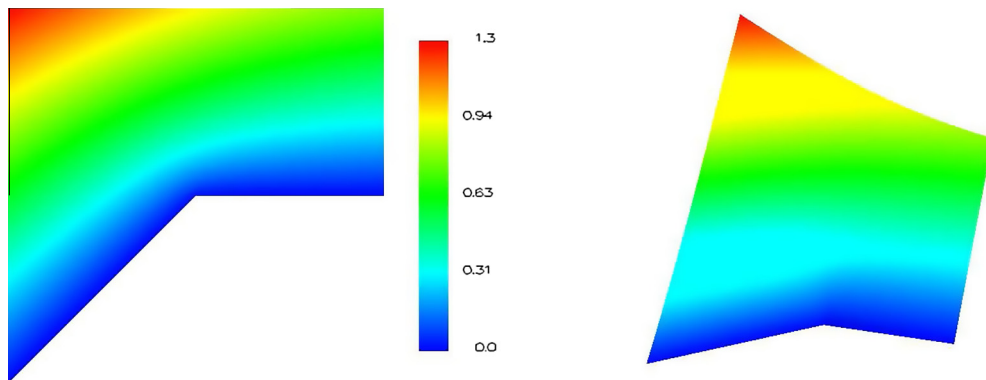


Fig. 4. The solution of the reentrant corner problem with $\omega = 5\pi/4$.

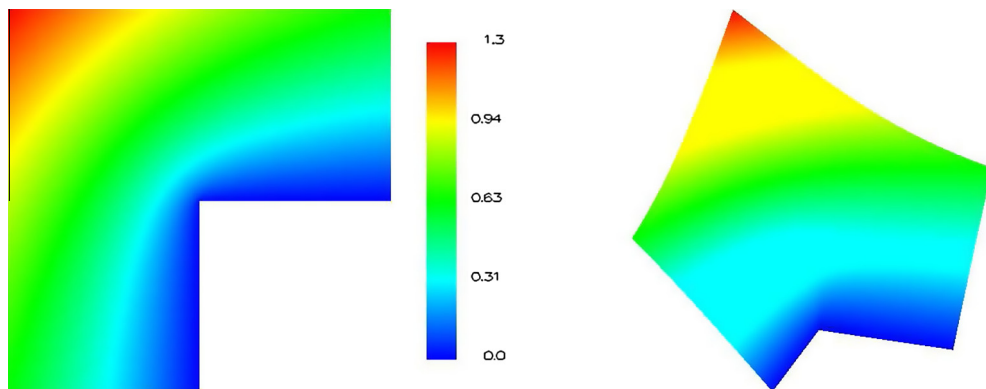


Fig. 5. The solution of the reentrant corner problem with $\omega = 3\pi/2$.

Parameters: All the above references use $\nu = 0.3$ and $E = 1$. The values for λ and Q differ in the two solutions, and are given in Table 1. See [3] for the derivation of these constants. With these solutions and parameters, $F_x = F_y = 0$. The solutions are shown in Figs. 8–11.

2.4. Peak

This problem has an exponential peak in the interior of the domain. It is based on Problem 10 in [7].
 Equation: Poisson.
 Domain: Unit square.
 Boundary conditions: Dirichlet.

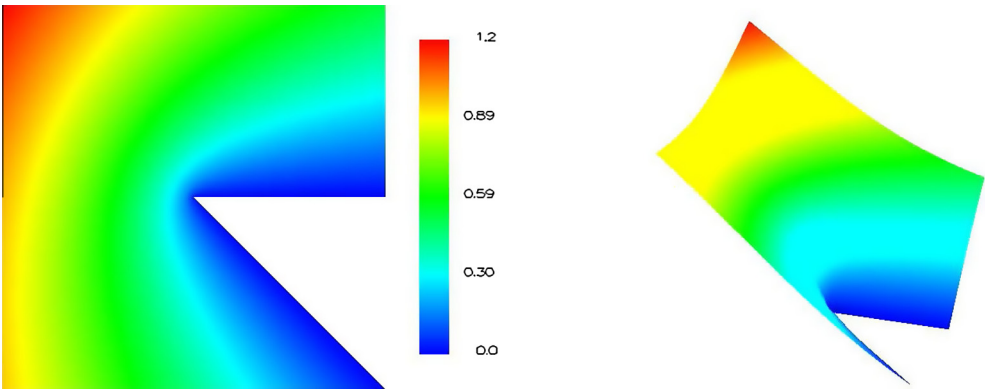


Fig. 6. The solution of the reentrant corner problem with $\omega = 7\pi/4$.

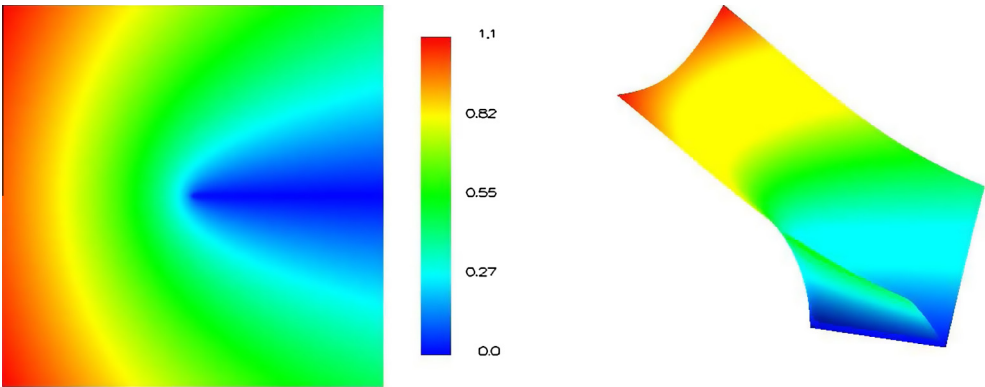


Fig. 7. The solution of the reentrant corner problem with $\omega = 2\pi$.

Table 1
Parameters for the linear elasticity problem.

mode	λ	Q
1	0.5444837367825	0.5430755788367
2	0.9085291898461	−0.2189232362488

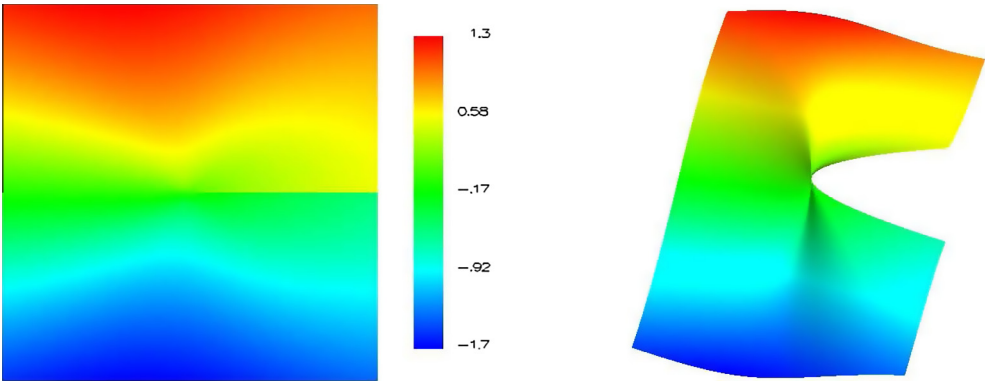


Fig. 8. The u component of the mode 1 solution of the linear elasticity problem.

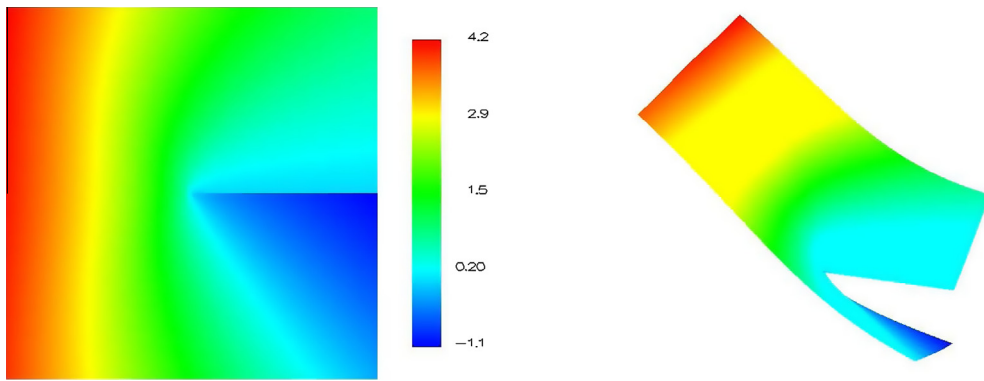


Fig. 9. The v component of the mode 1 solution of the linear elasticity problem.

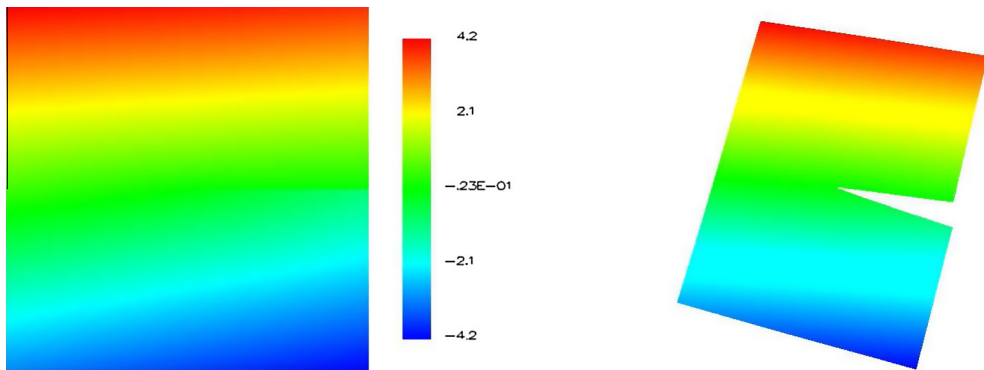


Fig. 10. The u component of the mode 2 solution of the linear elasticity problem.

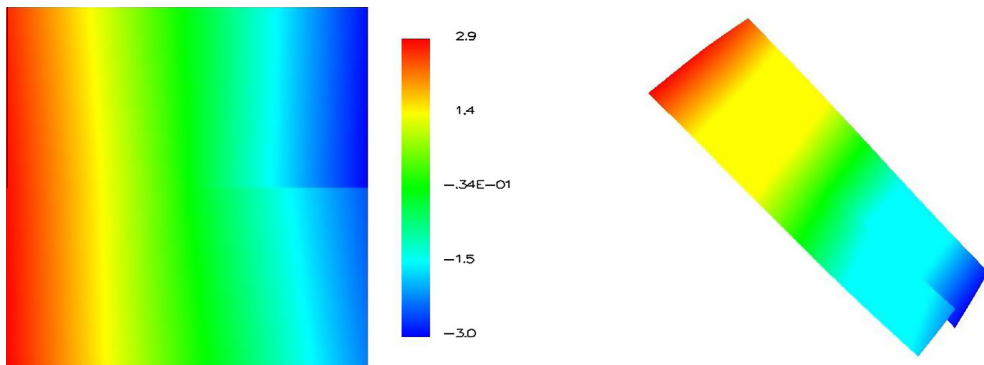


Fig. 11. The v component of the mode 2 solution of the linear elasticity problem.

Solution: $e^{-2((x-x_c)^2+(y-y_c)^2)}$.

Parameters: (x_c, y_c) is the location of the peak, and α determines the strength of the peak.

The solutions of two instances of this problem are shown in Figs. 12 and 13. The first has a mild peak at the vertex $(0.5, 0.5)$ with $\alpha = 1000$. The second has a sharp peak at $(0.51, 0.117)$ with $\alpha = 100000$.

2.5. Battery

This problem comes from [6], where it is attributed to Ivo Babuška and Sandia National Laboratories. It features piecewise constant coefficient functions p and q and right hand side f , and mixed boundary conditions. The solution has multiple point singularities in the interior of the domain. The equation models heat conduction in a battery with nonhomogeneous materials. The domain is the rectangle shown in Fig. 14. The numbered regions indicate the areas of different material constants,

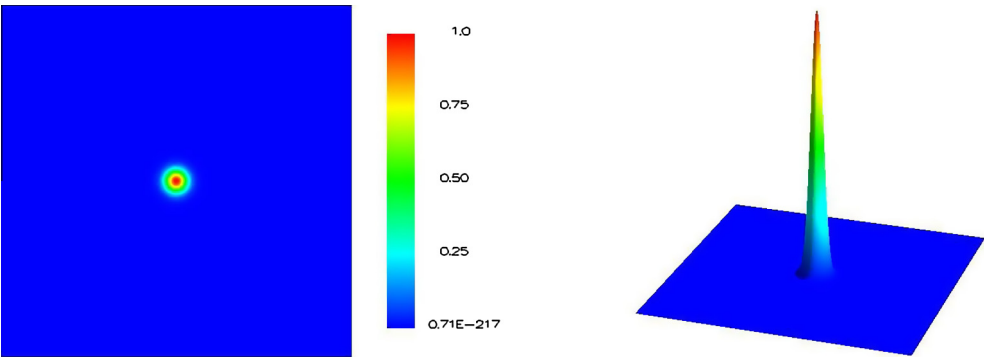


Fig. 12. The solution of the peak problem with $\alpha = 1000$, $(x_c, y_c) = (0.5, 0.5)$.

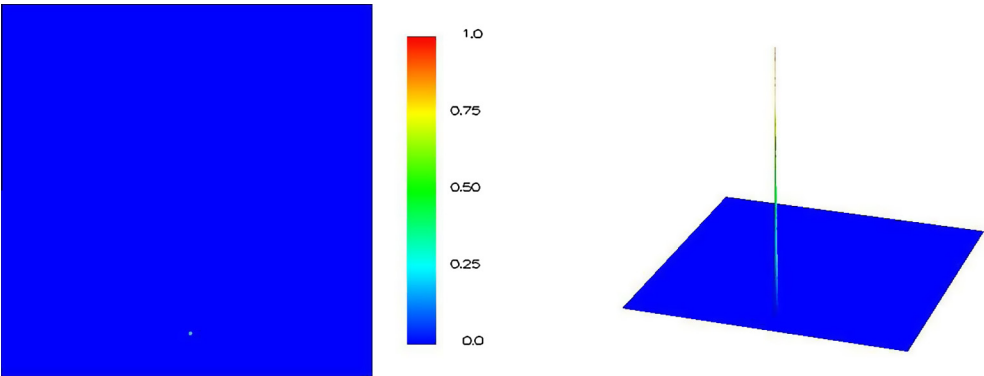


Fig. 13. The solution of the peak problem with $\alpha = 100000$, $(x_c, y_c) = (0.51, 0.117)$.

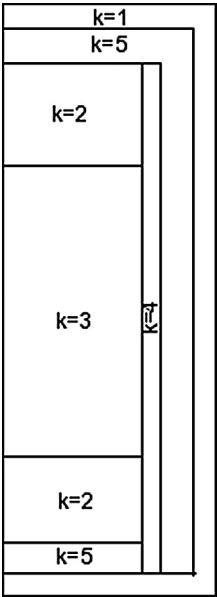


Fig. 14. Domain for the battery problem.

with the constants given in Table 2. The location of the line segments that separate the regions are given in Table 3. The coefficients of the mixed boundary conditions are given in Table 4. The solution has singularities at the points where three or

Table 2

Piecewise constant coefficient functions for the battery problem. k designates the region from Fig. 14.

k	p	q	f
1	25.0	25.0	0.0
2	7.0	0.8	1.0
3	5.0	0.0001	1.0
4	0.2	0.2	0.0
5	0.05	0.05	0.0

Table 3

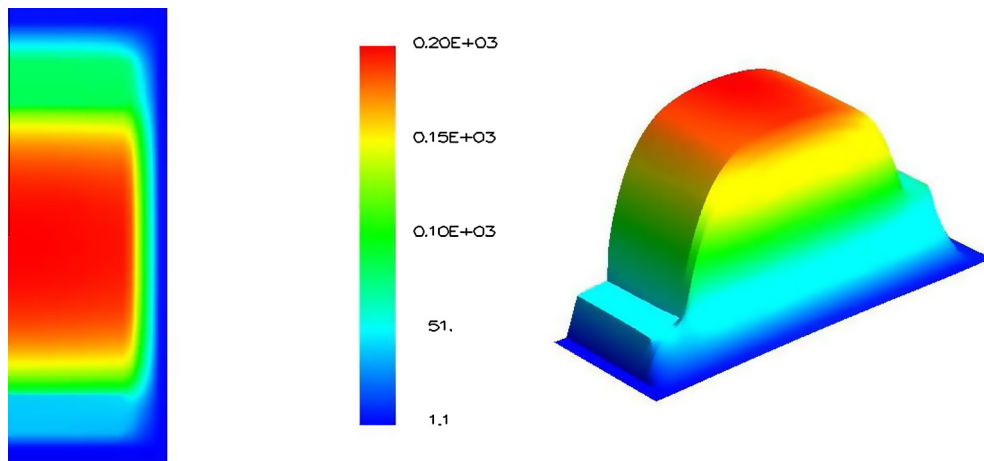
The locations of the line segments that separate the regions of the battery problem. For example, the line $y = 24.0$ is the top of the region in Fig. 14.

x	y
0.0	0.0
6.1	0.8
6.5	1.6
8.0	3.6
8.4	18.8
	21.2
	23.2
	24.0

Table 4

Boundary condition coefficients for the four sides of the battery problem domain in Fig. 14.

side	c	g_N
left	0.0	0.0
top	1.0	3.0
right	2.0	2.0
bottom	3.0	1.0

**Fig. 15.** The solution of the battery problem.

more materials meet. For any $\epsilon > 0$ there exists coefficients such that the solution is in $H^{1+\epsilon}$. By observing the rate of convergence with uniform h -refinement and comparing it with the theoretical *a priori* error bound, we estimate that ϵ is about $1/2$ for the given set of coefficients. The solution is shown in Fig. 15.

Equation: $-\frac{\partial}{\partial x} \left(p \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left(q \frac{\partial u}{\partial y} \right) = f$ with p , q and f given in Table 2.

Domain: $(0, 8.4) \times (0, 24)$.

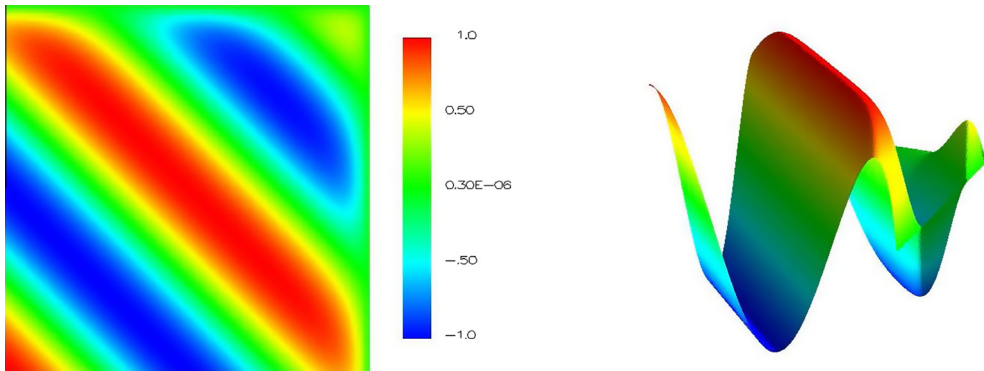


Fig. 16. The solution of the boundary layer problem with $\epsilon = 10^{-1}$.

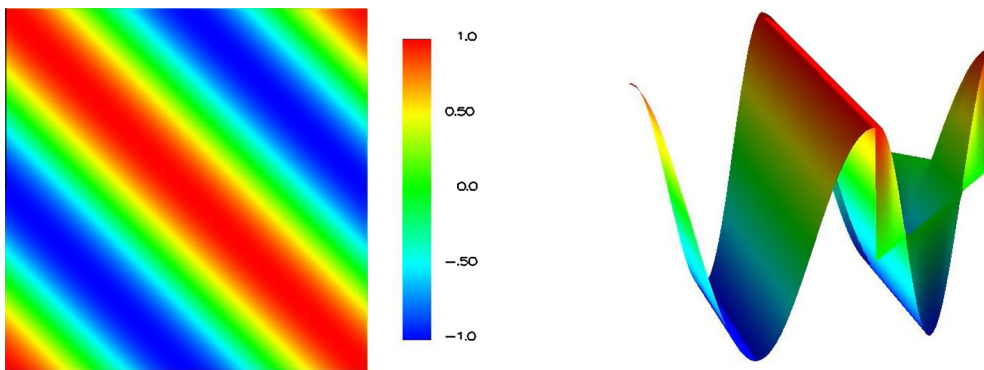


Fig. 17. The solution of the boundary layer problem with $\epsilon = 10^{-3}$.

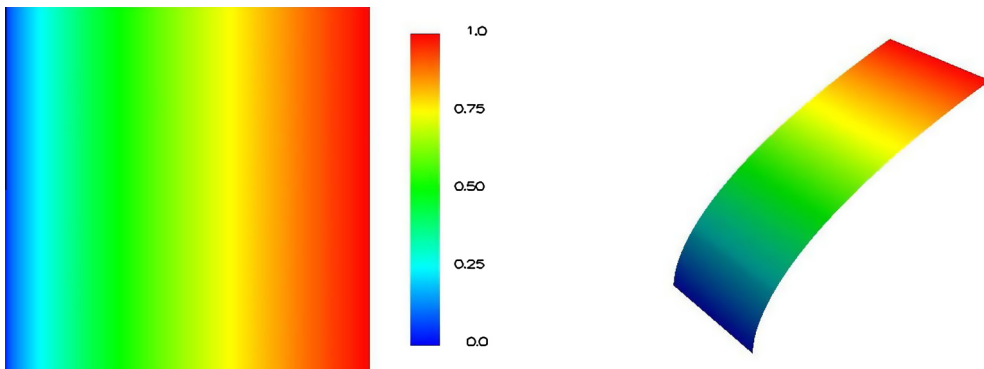


Fig. 18. The solution of the boundary line singularity problem with $\alpha = 0.6$.

Boundary conditions: Mixed, with c and g_N given in Table 4.

Solution: Unknown.

Parameters: None.

2.6. Boundary layer

This problem comes from [8]. It has an $O(\epsilon)$ boundary layer along the right and top sides of the domain. It is a convection–diffusion equation with first order terms.

Equation: $-\epsilon \nabla^2 u + 2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f$.

Domain: $(-1, 1) \times (-1, 1)$.

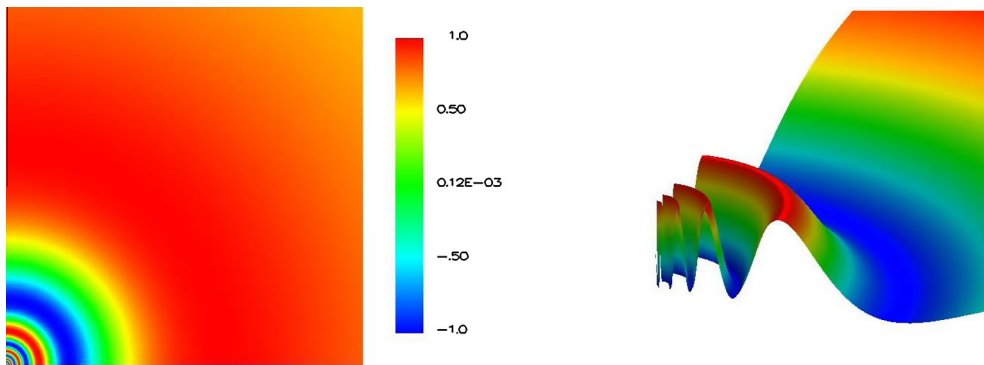


Fig. 19. The solution of the oscillatory problem with $\alpha = \frac{1}{10\pi}$.

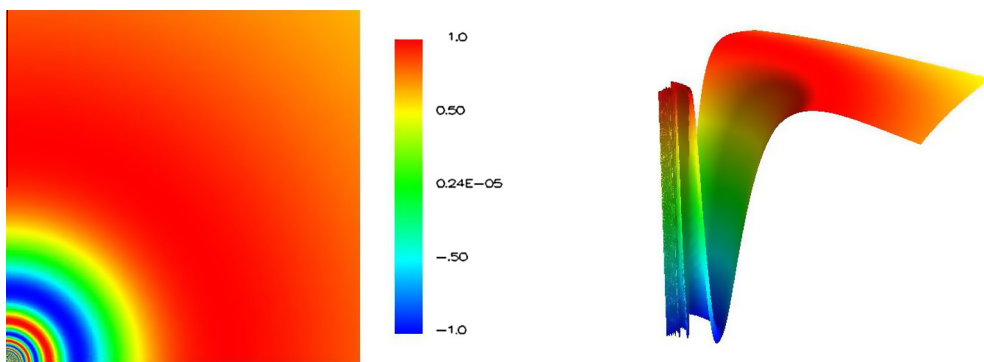


Fig. 20. The solution of the oscillatory problem with $\alpha = \frac{1}{50\pi}$.

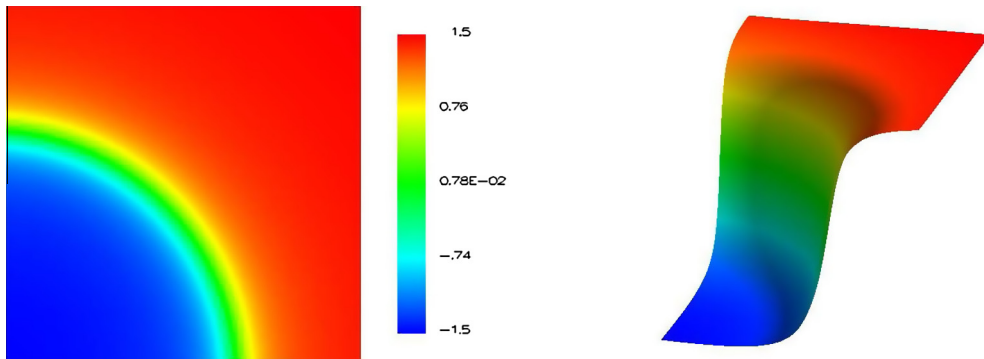


Fig. 21. The solution of the mild wave front problem.

Boundary conditions: Dirichlet.

Solution: $(1 - e^{-(1-x)/\epsilon})(1 - e^{-(1-y)/\epsilon}) \cos(\pi(x+y))$.

Parameters: ϵ determines the strength of the boundary layer.

Figs. 16 and 17 show solutions with a mild boundary layer from $\epsilon = 10^{-1}$, and a steep boundary layer from $\epsilon = 10^{-3}$.

2.7. Boundary line singularity

Many papers [6,9–11] use a 1D example with a singularity of the form x^α at the left endpoint of the domain. The solution is in $H^{\alpha+1/2-\epsilon} \forall \epsilon > 0$ [9]. This can be extended to 2D by simply making the solution be constant in y . On the unit square, the result is a solution that is singular along the left boundary.

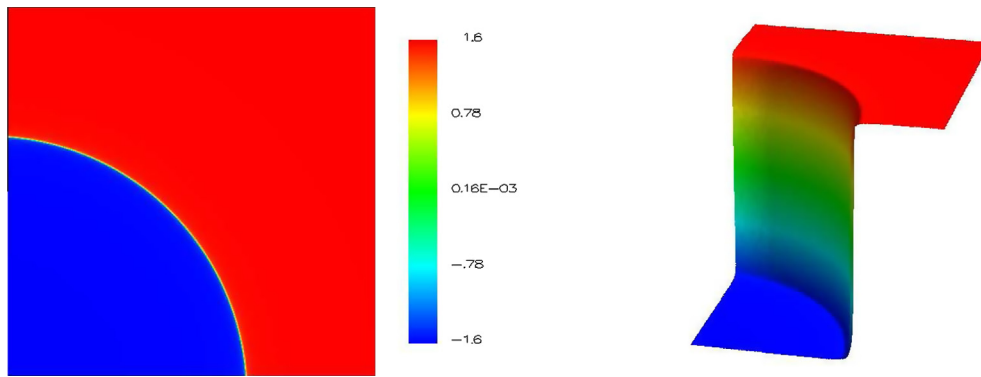


Fig. 22. The solution of the steep wave front problem.

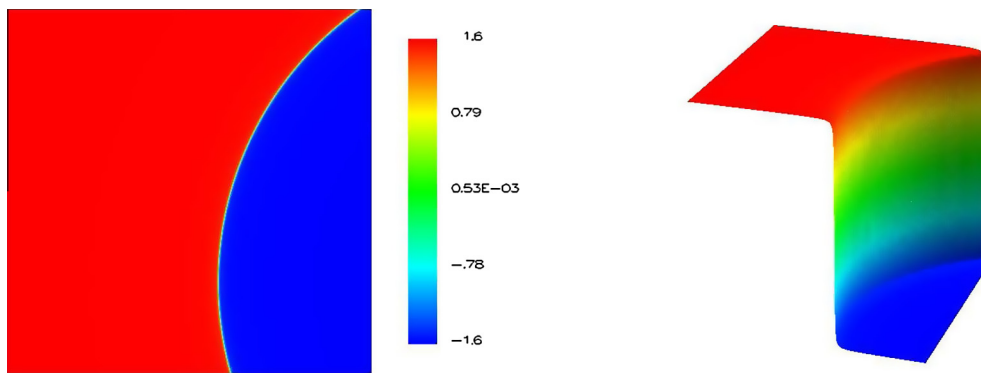


Fig. 23. The solution of the asymmetric wave front problem.

Equation: Poisson.

Domain: Unit square.

Boundary conditions: Dirichlet.

Solution: x^α .

Parameters: $\alpha \geq 1/2$ determines the strength of the singularity. All of the cited references use $\alpha = 0.6$.

The solution with $\alpha = 0.6$ is shown in Fig. 18.

2.8. Oscillatory

This problem is inspired by the wave function that satisfies a Schrödinger equation model of two interacting atoms [12]. It is highly oscillatory near the origin, with the wavelength decreasing closer to the origin. The number of oscillations, N , is determined by the parameter $\alpha = \frac{1}{N\pi}$. We use a Helmholtz equation for this problem.

Equation: $-\nabla^2 u - \frac{1}{(\alpha+r)^4} u = f$, where $r = \sqrt{x^2 + y^2}$.

Domain: Unit square.

Boundary conditions: Dirichlet.

Solution: $\sin\left(\frac{1}{\alpha+r}\right)$.

Parameters: α determines the number of oscillations.

The solutions of a relatively easy problem with $\alpha = \frac{1}{10\pi}$ and a more difficult problem with $\alpha = \frac{1}{50\pi}$ are shown in Figs. 19 and 20. In the surface plot of Fig. 19 we have zoomed in on the origin to show the detail.

2.9. Wave front

A commonly used example for testing adaptive refinement algorithms is Poisson's equation with a solution that has a steep wave front in the interior of the domain [2,6,9,11]. Usually the wave front is given by an arc tangent, but sometimes a hyperbolic tangent is used. In this problem we use a circular wave front as in [6,11]. Parameters determine the steepness and location of the wave front. With the arc tangent wave front, there is a mild singularity at the center of the circle. By

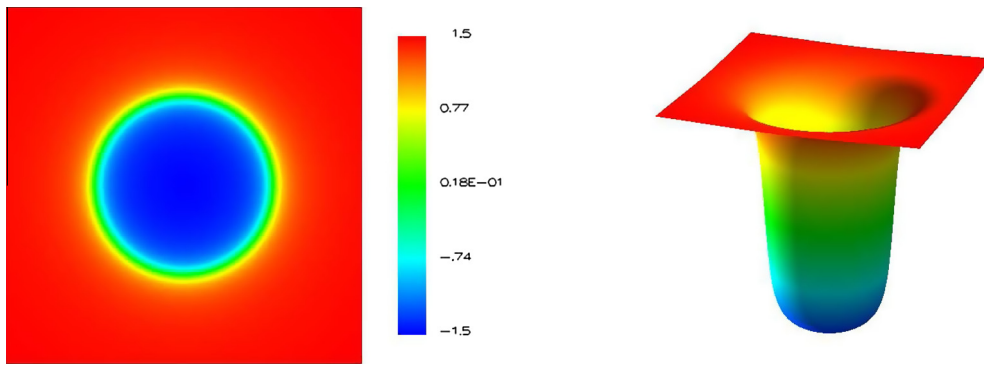


Fig. 24. The solution of the wave front “well” problem.

Table 5

Parameters for the wave front problem.

name	α	x_c	y_c	r_0
mild	20	−.05	−.05	0.7
steep	1000	−.05	−.05	0.7
asymmetric	1000	1.5	0.25	0.92
well	50	0.5	0.5	0.25

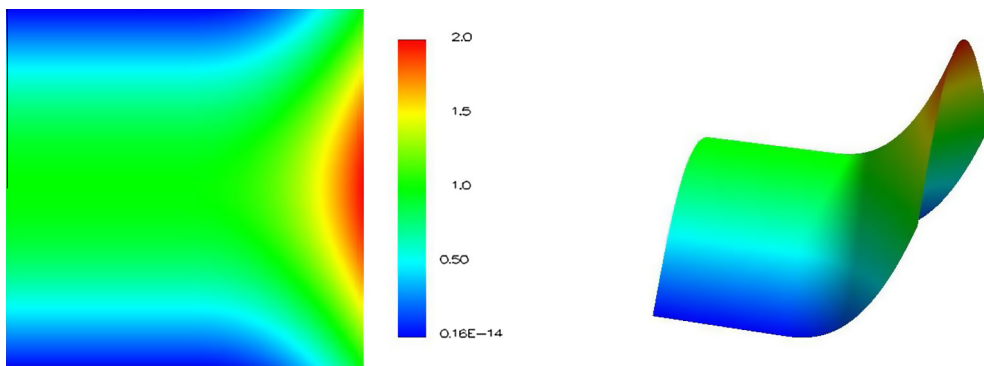


Fig. 25. The solution of the interior line singularity problem with $\alpha = 2.5$ and $\beta = 0$.

observing the convergence rate with uniform h -refinement, we estimate that the solution is in H^m with $m \approx 2$ if the center of the circle is in the closure of the domain, and smooth otherwise.

Equation: Poisson.

Domain: Unit square.

Boundary conditions: Dirichlet.

Solution: $\tan^{-1}(\alpha(r - r_0))$ where $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$.

Parameters: (x_c, y_c) is the center of the circular wave front, r_0 is the distance from the wave front to the center of the circle, and α gives the steepness of the wave front.

Four example solutions are shown in Figs. 21–24, with the parameters given in Table 5. In the first three we choose the center of the circle to be outside the domain so that we are examining the performance on the wave front, not the singularity. These solutions are characterized as a mild wave front, a steep wave front, and a steep wave front that is not symmetric about the origin. In the fourth example, the entire circle is inside the domain, resulting in a solution that is a well with a mild singularity at the bottom.

2.10. Interior line singularity

Houston et al. [9] extend the 1D x^2 problem in Section 2.7 to 2D by extending the 1D domain to $(-1, 1)$, defining the solution to be 0 for $x < 0$, extending the domain to 2D with $y \in (-1, 1)$, and adding $\cos(\pi y/2)$. We extend this further to allow a

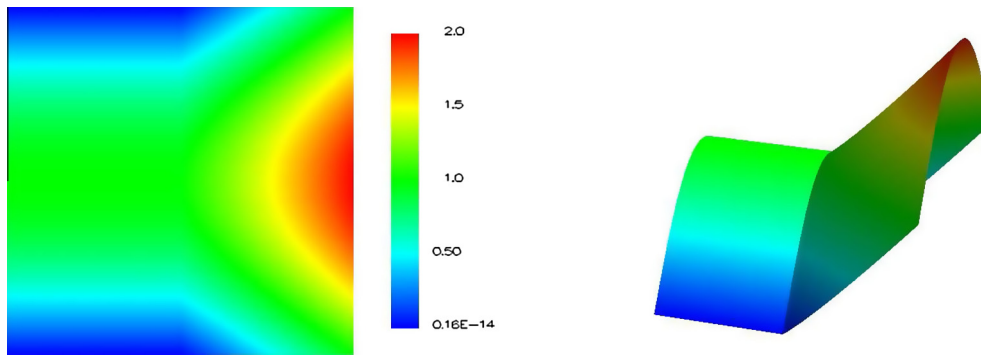


Fig. 26. The solution of the interior line singularity problem with $\alpha = 1.1$ and $\beta = 0$.

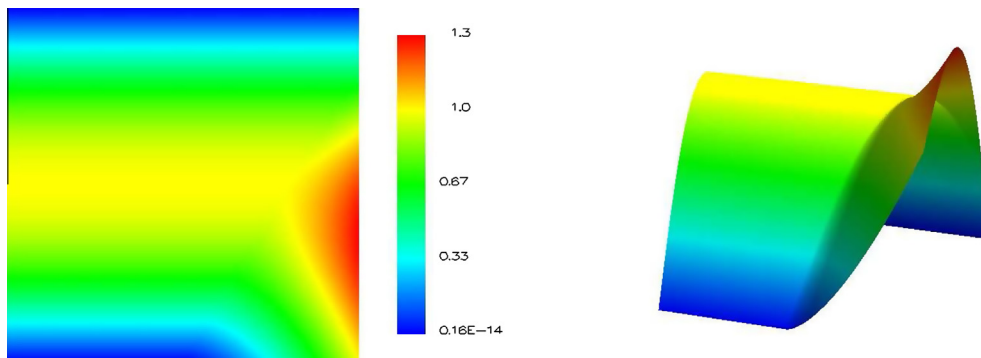


Fig. 27. The solution of the interior line singularity problem with $\alpha = 1.5$ and $\beta = 0.6$.

sloped line so that the singularity does not necessarily coincide with element edges. There is no PDE in [9] since they are only evaluating their estimate of the regularity. We use a Poisson equation with Dirichlet boundary conditions. The solution is in $H^{\alpha+1/2-\epsilon} \forall \epsilon > 0$ [9].

Equation: Poisson.

Domain: $(-1, 1) \times (-1, 1)$.

Boundary conditions: Dirichlet. Solution:
$$\begin{cases} \cos(\pi y/2) & x \leq \beta(y+1) \\ \cos(\pi y/2) + (x - \beta(y+1))^\alpha & x > \beta(y+1) \end{cases}$$

Parameters: α determines the strength of the singularity and β determines the slope of the singularity line.

Solutions containing a mild singularity with $\alpha = 2.5, \beta = 0$, a stronger singularity with $\alpha = 1.1, \beta = 0$, and a slanted line singularity with $\alpha = 1.5, \beta = 0.6$ are shown in Figs. 25–27.

2.11. Intersecting interfaces

This problem comes from a paper by Kellogg [13] in which he studies Poisson problems with intersecting interfaces. Two interfaces, given by the lines $y = 0$ and $y = \tan(\phi)x$, divide the plane into four regions for a given $\phi \in (0, \pi/2]$. The PDE coefficients p and q are a piecewise constant function taking the value p_i in the i th region counterclockwise from the positive x -axis. Let $\psi = \pi/2 - \phi$. The solution is given in polar coordinates by

$$u = r^{a_1} \mu(\theta) \quad (7)$$

where

$$\mu(\theta) = \begin{cases} \cos((\psi - \beta_1)a_1) \cos((\theta - \phi + \alpha_1)a_1) & 0 \leq \theta \leq \phi \\ \cos(\alpha_1 a_1) \cos((\theta - \pi + \beta_1)a_1) & \phi \leq \theta \leq \pi \\ \cos(\beta_1 a_1) \cos(\theta - \pi - \alpha_1)a_1) & \pi \leq \theta \leq \pi + \phi \\ \cos((\phi - \alpha_1)a_1 \cos(\theta - \phi - \pi - \beta_1)a_1) & \pi + \phi \leq \theta \leq 2\pi \end{cases} \quad (8)$$

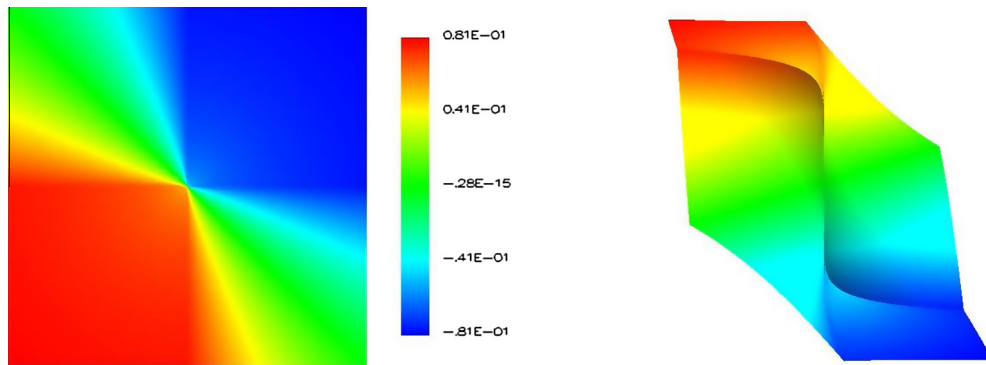


Fig. 28. The solution of the intersecting interfaces problem.

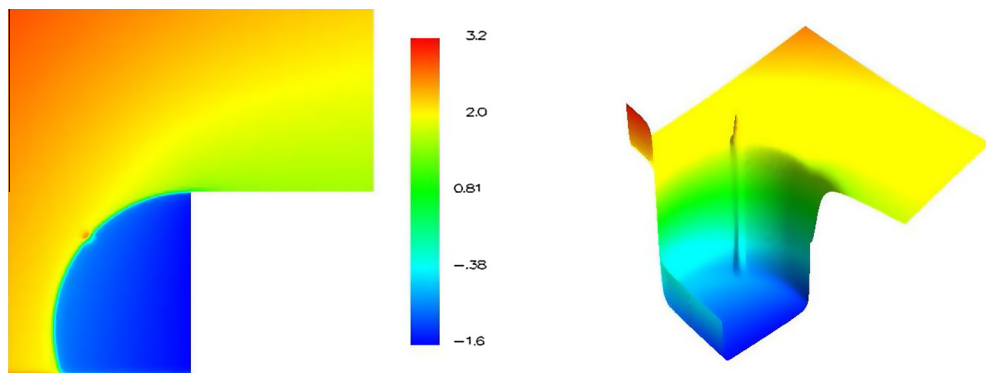


Fig. 29. The solution of the problem with multiple difficulties.

and where the numbers a_1 , α_1 and β_1 satisfy the relations

$$\begin{cases} p_1/p_2 = -\tan((\psi - \beta_1)a_1) \cot(\alpha_1 a_1) \\ p_2/p_3 = -\tan(\alpha_1 a_1) \cot(\beta_1 a_1) \\ p_3/p_4 = -\tan(\beta_1 a_1) \cot((\phi - \alpha_1)a_1) \\ p_4/p_3 = -\tan((\phi - \alpha_1)a_1) \cot((\psi - \beta_1)a_1) \\ 0 < a_1 < \pi/\psi \\ \max\{0, 2\phi a_1 - \pi\} < 2a_1\alpha_1 < \min\{2\phi a_1, \pi\} \\ \max\{0, \pi - 2\psi a_1\} < -2a_1\beta_1 < \min\{\pi, 2\pi - 2\psi a_1\} \end{cases} \quad (9)$$

The solution has a discontinuous derivative along the interfaces, and an infinite derivative at the origin. The solution is in $H^{1+a_1-\epsilon} \forall \epsilon > 0$, so a_1 can be chosen to make the Sobolev regularity arbitrarily close to 1.

Equation: $-\nabla \cdot p \nabla u = 0$ where p is piecewise constant.

Domain: $(-1, 1) \times (-1, 1)$.

Boundary conditions: Dirichlet.

Solution: given by Eqs. (7) and (8)

Parameters: As in [14], we take $\theta = \pi/2$, $p_1 = p_3 = R$, $p_2 = p_4 = 1$, and $a_1 = 0.1$. The conditions in Eq. (9) can then be solved to obtain $R = 161.4476387975881$, $\alpha_1 = \pi/4$ and $\beta_1 = -14.92256510455152$.

The solution for the given parameters is shown in Fig. 28.

2.12. Multiple difficulties

In [11] one of the test cases involves both a singularity due to a reentrant corner and a sharp gradient in the form of an arctangent wave front. In this problem we combine four or five difficulties of different strengths into the same problem by combining some of the features of the other test problems. It contains a point singularity due to a reentrant corner, a circular wave front (which might include a singularity at the center of the circle), a sharp peak, and a boundary layer.

Equation: Poisson.

Domain: L-shaped domain $(-1, 1) \times (-1, 1) \setminus (0, 1) \times (-1, 0)$.

Boundary conditions: Dirichlet.

Solution: $r^{(\pi/\omega)} \sin(\theta\pi/\omega) + \tan^{-1}(\alpha_w(\sqrt{(x-x_w)^2 + (y-y_w)^2} - r_0)) + e^{-\alpha_p((x-x_p)^2 + (y-y_p)^2)} + e^{-(1+y)/\epsilon}$

Parameters: This problem has the same parameters as the problems that it combines. The boundary layer was placed on $y = -1$. It could instead be placed on $y = 1$, $x = -1$ or $x = 1$, or any combination of them.

Fig. 29 shows a solution where the wave front intersects the boundary layer and corner singularity, and the peak is centered on the wave front. The following parameters were used. For the reentrant corner, $\omega = 3\pi/2$. The wave front is defined by $(x_w, y_w) = (0, -3/4)$, $r_0 = 3/4$ and $\alpha_w = 200$. The peak is centered at $(x_p, y_p) = (\sqrt{5}/4, -1/4)$ with strength $\alpha_p = 1000$. The boundary layer is given by $\epsilon = 1/100$.

3. Future work

The collection of problems given in this paper were selected to represent a variety of solution features for which adaptive grid refinement is appropriate in the solution of elliptic partial differential equations in 2D. Clearly this is a limited class of problems. Standard benchmark problems are also needed for 1D, 3D, time dependent problems, problems that are motivated by real-world applications without manufactured solutions, goal-oriented adaptivity, etc. A web resource [1] is being developed, from which researchers can obtain standard adaptive grid refinement benchmark problems from a rich collection of problems representing many classes of partial differential equations.

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