Partial Solution to Last Issue's Homework Assignment

MORE MODELS OF INFECTION: IT'S EPIDEMIC

By Dianne P. O'Leary

Problem 1. Model 1 consists of the differential equation

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - I(t)k,$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t),$$

$$\frac{dR(t)}{dt} = I(t)/k.$$

We start the model by assuming some proportion of infected individuals—for example, $I(0) = 0.005$, $S(0) = 1 - I(0)$, and $R(0) = 0$. Run Model 1 for $k = 4$ and $\tau = 0.8$ until either $I(t)$ or $S(t)$ drops below 10^{-5}. Plot $I(t)$, $S(t)$, and $R(t)$ on a single graph. Report the proportion of the population that became infected and the maximum difference between $I(t) + S(t) + R(t)$ and 1.

Answer: We've posted sample programs at www.computer.org/cise/homework. Figure A shows the results; 95.3 percent of the population becomes infected.

Problem 2. Instead of using the equation $dR/dt = I/k$, we could have used the conservation principle

$$I(t) + S(t) + R(t) = 1$$

for all time. Substituting this for the dR/dt equation gives us an equivalent system of *differential algebraic equations* (DAEs); we will call this Model 2.

Redo Problem 1 using Model 2 instead of Model 1. To do this, differentiate the conservation principle and express the three equations of the model as $My' = f(t, y)$, where M is a 3×3 matrix.

Answer: Figure A shows the results, which, as expected, are indistinguishable from those of Model 1.

Problem 3.

a. Redo Problem 1 using Model 3

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \tau I(t-k)S(t-k),$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t),$$

$$\frac{dR(t)}{dt} = \tau I(t-k)S(t-k),$$

instead of Model 1. For $t \le 0$, use the initial conditions

$I(t) = 0$, $S(t) = 1$, $R(t) = 0$,

and let $I(0) = 0.005$, $S(0) = 1 - I(0)$, and $R(0) = 0$.

Note that these conditions match our previous ones at $t = 0$. Compare the results of the three models.

Answer: Figure B shows the results; 94.3 percent of the population becomes infected, slightly less than in the first models. The epidemic dies out in roughly half the time.

Problem 4. Let S, I, and R depend on a spatial coordinate (x, y) as well as t, and consider the model

$$\frac{\partial I(t,x,y)}{\partial t} = \tau I(t,x,y)S(t,x,y) - I(t,x,y)/k + \delta\left(\frac{\partial^2 I(t,x,y)}{\partial x^2} + \frac{\partial^2 I(t,x,y)}{\partial y^2}\right)S(t,x,y),$$

$$\frac{\partial S(t,x,y)}{\partial t} = -\tau I(t,x,y)S(t,x,y) - \delta\left(\frac{\partial^2 I(t,x,y)}{\partial x^2} + \frac{\partial^2 I(t,x,y)}{\partial y^2}\right)S(t,x,y),$$

$$\frac{\partial R(t,x,y)}{\partial t} = I(t,x,y)/k.$$

To solve this problem, we will *discretize* and approxi-

Authorized licensed use limited to: to IEEExplore provided by Virginia Tech Libraries. Downloaded on November 6, 2008 at 09:30 from IEEE Xplore. Restrictions apply.

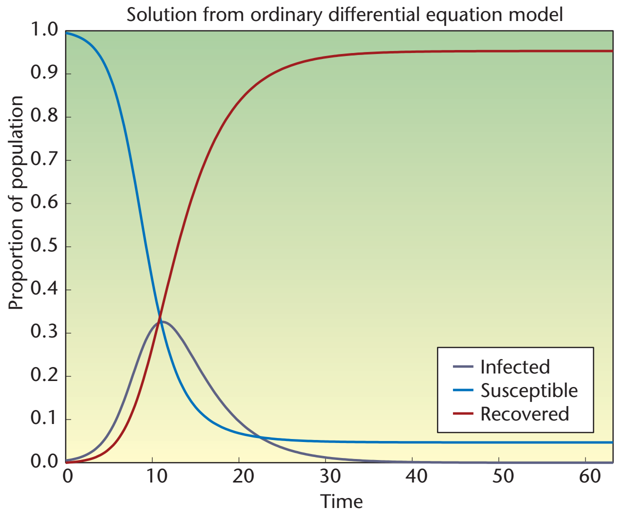

Figure A. Proportion of individuals infected by the epidemic from the ODE Model 1 or the DAE Model 2.

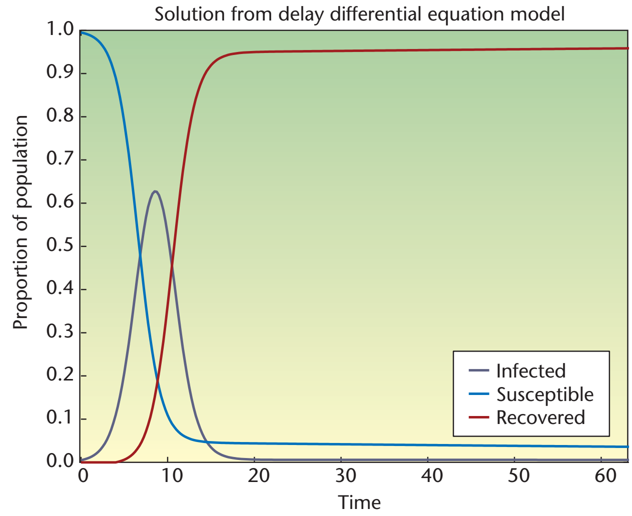

Figure B. Proportion of individuals infected by the epidemic from the DDE Model 3.

mate the solution at the points of a grid of size $n \times n$. Let $h = 1/(n-1)$ and let $x_i = ih$, $i = 0, \ldots, n-1$ and $y_j = jh$, $j = 0, \ldots, n-1$. Our variables will be our approximations $I(t)_{ij} \approx I(t, x_i, y_j)$ and similarly for $S(t)_{ij}$ and $R(t)_{ij}$.

a. Use Taylor series expansions to show that we can approximate

$$\frac{d^2 I(t, x_i, y_j)}{dx^2} = \frac{I(t)_{i-1,j} - 2I(t)_{ij} + I(t)_{i+1,j}}{h^2} + O(h).$$

We can derive a similar expression for $d^2 I(t, x_i, y_j)/dy^2$.

b. Form a vector $\hat{I}(t)$ from the approximate values of $I(t)$ by ordering the unknowns as $I_{00}, I_{01}, \ldots, I_{0,n-1}; I_{10}, I_{11}, \ldots, I_{1,n-1}, \ldots, I_{n-1,0}; I_{n-1,1}, \ldots, I_{n-1,n-1}$. In the same way, form the vectors $\hat{S}(t)$ and $\hat{R}(t)$ and derive the matrix $\boldsymbol{A}$ so that our discretized equations become Model 4:

$$\frac{\partial \hat{I}(t)}{\partial t} = \tau \hat{I}(t). * \hat{S}(t) - \hat{I}(t)/k + \delta(\boldsymbol{A} I(t)). * \hat{S}(t),$$

$$\frac{\partial \hat{S}(t)}{\partial t} = -\tau \hat{I}(t). * \hat{S}(t) - \delta(\boldsymbol{A}\hat{I}(t)). * \hat{S}(t),$$

$$\frac{\partial \hat{R}(t)}{\partial t} = \hat{I}(t)/k,$$

where the notation $\hat{I}(t). * \hat{S}(t)$ means the vector formed from the product of each component of $\hat{I}(t)$ with the corresponding component of $\hat{S}(t)$. To form the approximation near the boundary, assume that the (Neumann) boundary conditions imply $I(t, -h, y) = I(t, h, y)$, $I(t, 1+h, y) = I(t, 1-h, y)$ for $0 \le y \le 1$, and similarly for S and R. Make the same type of assumption at the two other boundaries.

Answer:

a. Since Taylor series expansion yields

$$I(t)_{i-1,j} = I(t, x, y) - hI_x(t, x, y) + \frac{h^2}{2} I_{xx}(t, x, y) - \frac{h^3}{6} I_{xxx}(t, x, y) + O(h^4)$$

and

$$I(t)_{i+1,j} = I(t, x, y) + hI_x(t, x, y) + \frac{h^2}{2} I_{xx}(t, x, y) + \frac{h^3}{6} I_{xxx}(t, x, y) + O(h^4),$$

we see that

$$\frac{I(t)_{i-1,j} - 2I(t)_{ij} + I(t)_{i+1,j}}{h^2} = \frac{h^2 I_{xx}(t, x, y) + O(h^4)}{h^2} = I_{xx}(t, x, y) + O(h^2).$$

b. The matrix $\boldsymbol{A}$ can be expressed as

$$\boldsymbol{A} = T \otimes I + I \otimes T,$$

where

$$T = \frac{1}{h^2}\begin{bmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{bmatrix}$$

Authorized licensed use limited to: to IEEExplore provided by Virginia Tech Libraries. Downloaded on November 6, 2008 at 09:30 from IEEE Xplore. Restrictions apply.

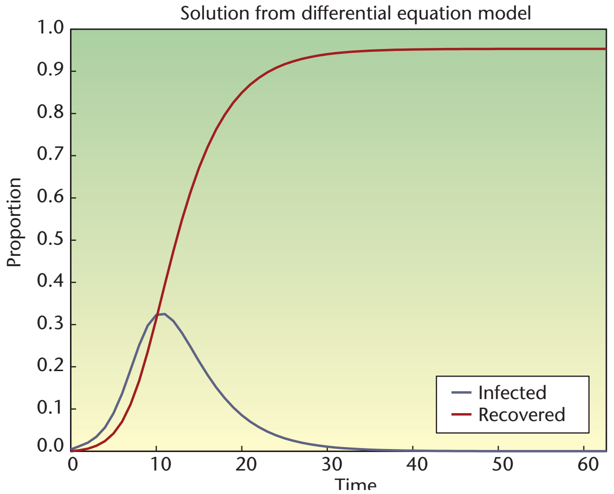

Figure C. Proportion of individuals infected by the epidemic from the differential equation of Model 5a.

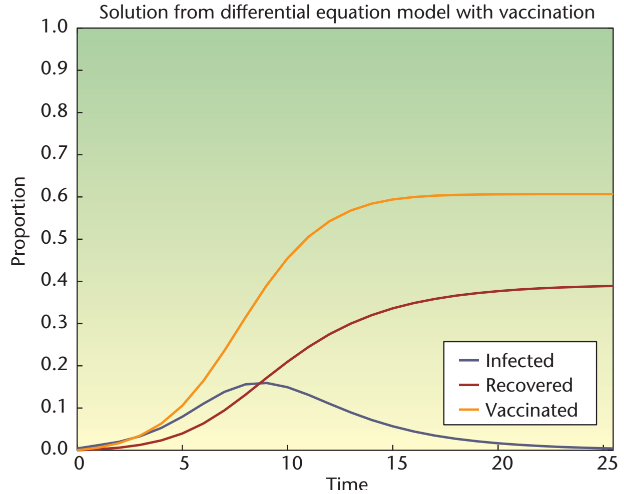

Figure D. Proportion of individuals infected by the epidemic from the differential equation of Model 5b, including vaccinations.

and $\boldsymbol{T}$ and $\boldsymbol{I}$ are matrices of dimension $n \times n$. (The notation $C \otimes D$ denotes the matrix whose (i, j)th block is $c_{ij}D$. The Matlab command to form this matrix is `kron(C,D)`, which means Kronecker product of C and D.)

Look to the Future

IEEE Internet Computing reports emerging tools, technologies, and applications implemented through the Internet to support a worldwide computing environment.

In the next year, we'll look at
- Business Processes for the Web
- Seeds of Internet Growth
- Internet-Based Data Dissemination
- the Wireless Grid
- Measuring Performance
- Homeland Security

... and more!

IEEE Internet Computing

www.computer.org/internet

Problem 5.

a. Set $n = 11$ (so that $h = 0.1$), $k = 4$, $\tau = 0.8$, and $\delta = 0.2$ and use an ODE solver to solve Model 4. For initial conditions, set $S(0, x, y) = 1$ and $I(0, x, y) = R(0, x, y) = 0$ at each point (x, y), except that $S(0, 0.5, 0.5) = I(0, 0.5, 0.5) = 0.5$. (For simplicity, you need only use I and S in the model, and you may derive $R(t)$ from these quantities.) Stop the simulation when the average value of either $\hat{I}(t)$ or $\hat{S}(t)$ drops below 10^{-5}. Form a plot similar to that of Problem 1 by plotting the average value of $I(t)$, $S(t)$, and $R(t)$ versus time. Compare the results.

b. Let's vaccinate the susceptible population, at a rate $\nu S(t, x, y)\, I(t, x, y)/(I(t, x, y) + S(t, x, y))$. This rate is the derivative of the vaccinated population $V(t, x, y)$ with respect to time, and this term is subtracted from $\partial S(t, x, y)/\partial t$. Run this model with $\nu = 0.7$ and compare the results with those of Model 4.

Answer: Figure C shows the results of Problem 5a, and Figure D shows those for Problem 5b. The infection rate without vaccination is 95.3 percent (very similar to Model 1), but with vaccination, it drops to 38.9 percent. Vaccination also significantly shortens the epidemic's duration.

Acknowledgments

I'm grateful to G.W. Stewart for helpful comments on this project.

Dianne P. O'Leary is a professor of computer science and a faculty member in the Institute for Advanced Computer Studies and the Applied Mathematics Program at the University of Maryland. Her interests include numerical linear algebra, optimization, and scientific computing. She received a BS in mathematics from Purdue University and a PhD in computer science from Stanford. She is a member of SIAM, ACM, and AWM. Contact her at oleary@cs.umd.edu; www.cs.umd.edu/users/oleary/.

Authorized licensed use limited to: to IEEExplore provided by Virginia Tech Libraries. Downloaded on November 6, 2008 at 09:30 from IEEE Xplore. Restrictions apply.