# Partial Solution to Last Issue's Homework Assignment

# The Direction-of-Arrival Problem: Coming at You

By Dianne P. O'Leary

We model the sensor measurements as

$$\begin{aligned} \mathbf{x}_1(t) &= A\mathbf{s}(t) + \in_1(t), \\ \mathbf{x}_2(t) &= A\Phi\mathbf{s}(t) + \in_2(t). \end{aligned}$$

Consider the following recipe:

- Find a matrix $B$ of size $d \times m$ so that $BA$ is $d \times d$ and full rank.
- Find a matrix $C$ of size $n \times d$ so that $SC$ is $d \times d$ and full rank.
- Find $d$ vectors $\mathbf{z}_k$ and $d$ values $\lambda_k$ so that $BA\Phi SC\mathbf{z}_k = \lambda_k BASC\mathbf{z}_k$.

**Problem 1**. Show that the eigenvalues $\lambda_k$ are equal to the diagonal entries of $\Phi$.

**Answer:** Let $\mathbf{w}_k = SC\mathbf{z}_k$, and multiply the equation $BA\Phi SC\mathbf{z}_k = \lambda_k BASC\mathbf{z}_k$ by $(BA)^{-1}$ to obtain

$$\Phi\mathbf{w}_k = \lambda_k\mathbf{w}_k, \qquad k = 1, \ldots, d.$$

By the definition of eigenvalue, we see that $\lambda_k$ is an eigenvalue of $\Phi$ corresponding to the eigenvector $\mathbf{w}_k$. Because $\Phi$ is a diagonal matrix, its eigenvalues are its diagonal entries, so the result follows.

**Problem 2**. Suppose that the singular value decomposition (SVD) of $X$ is $U\Sigma W^H$, where $\sigma_i = 0$, $i > d$. Let $\Sigma_1$ be the square diagonal matrix with entries $\sigma_1, \ldots, \sigma_d$, and partition $U$ into

$$U = \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix},$$

where $U_1$ and $U_2$ have $m$ rows and $d$ columns, so that

$$\begin{aligned} X_1 &= AS = U_1[\Sigma_1, O_{d\times(n-d)}]W^H, \\ X_2 &= A\Phi S = U_2[\Sigma_1, O_{d\times(n-d)}]W^H, \end{aligned}$$

where $O_{d\times(n-d)}$ is the zero matrix of size $d \times (n-d)$. Let $\hat{U} = [U_1, U_2]$ have SVD $T\Delta V^H$, and denote the leading $d \times d$ submatrix of $\Delta$ by $\Delta_1$. Partition

$$V = \begin{bmatrix} V_1 & V_3 \\ V_2 & V_4 \end{bmatrix},$$

so that $V_1$ and $V_2$ have dimension $d \times d$. Let

$$B = [\Delta_1^{-1}, O_{d\times(m-d)}]T^H$$

and

$$C = W\begin{bmatrix} \Sigma_1^{-1} \\ O_{(n-d)\times d} \end{bmatrix}.$$

Show that the eigenvalues $\lambda_k$ that satisfy the equation $V_2^H\mathbf{z}_k = \lambda_k V_1^H\mathbf{z}_k$ are $\phi_k$.

**Answer:** Using the SVD of $\hat{U}$, we see that

$$T^H[U_1, U_2] = \Delta V^H = \left[\Delta\begin{bmatrix} V_1^H \\ V_3^H \end{bmatrix}, \Delta\begin{bmatrix} V_2^H \\ V_4^H \end{bmatrix}\right].$$

Now we compute the matrices from Problem 1:

$$\begin{aligned} BASC &= [\Delta_1^{-1}, O_{d\times(m-d)}]T^H U_1[\Sigma_1, O_{d\times(n-d)}]W^H W\begin{bmatrix} \Sigma_1^{-1} \\ O_{(n-d)\times d} \end{bmatrix} \\ &= [\Delta_1^{-1}, O_{d\times(m-d)}]\,\Delta\begin{bmatrix} V_1^H \\ V_3^H \end{bmatrix} \\ &= V_1^H. \end{aligned}$$

$$\begin{aligned} BA\Phi SC &= [\Delta_1^{-1}, O_{d\times(m-d)}]T^H U_2[\Sigma_1, O_{d\times(n-d)}]W^H W\begin{bmatrix} \Sigma_1^{-1} \\ O_{(n-d)\times d} \end{bmatrix} \\ &= [\Delta_1^{-1}, O_{d\times(m-d)}]\,\Delta\begin{bmatrix} V_2^H \\ V_4^H \end{bmatrix} \\ &= V_2^H. \end{aligned}$$

Thus, with this choice of $\boldsymbol{B}$ and $\boldsymbol{C}$, the eigenvalue problem in Problem 1 reduces to $V_2^H \mathbf{z}_k = \lambda_k V_1^H \mathbf{z}_k$.

**Problem 3**. Program the SVD algorithm and experiment with rectangularly windowed data and a window size of $n = 10$. Note that we need to compute $\boldsymbol{U}$ and $\boldsymbol{V}$, but we don't need $\boldsymbol{B}$ or $\boldsymbol{C}$. You can find sample data for $\boldsymbol{X}$ and $\Phi$ at http://computer.org/cise/homework/. Plot the true and computed direction of arrivals (DOAs) as a function of time. Then compute the average absolute error in your DOA estimates (absolute value of true value minus computed value) and the average relative error (absolute error divided by absolute value of true value).

**Answer:** Figure A shows the results. The average error in the angle estimate is 0.62 degrees, and the average relative error is 0.046. The estimated DOAs are quite reliable except when the signals get very close.

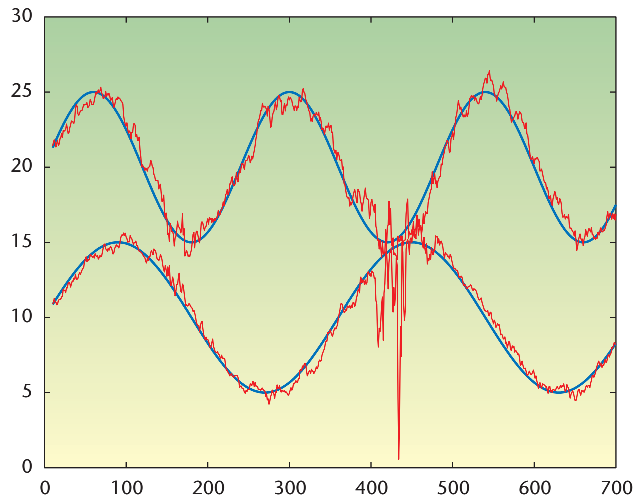


**Figure A. Results of Problem 3. The true direction of arrival (DOA) appears in blue; the DOA estimated by rectangular windowing (in red) is shown as a function of time.**

**Problem 4**. Suppose the matrix $\boldsymbol{X}$ contains the exponential windowing data and that a new data vector $\mathbf{x}$ arrives. Give a formula for the new exponential windowing data matrix and show that the cost of computing it from $\boldsymbol{X}$ and $\mathbf{x}$ is $O(m^2)$ multiplications.

**Answer:**

$$X_{new} X_{new}^H = [fX, x] \begin{bmatrix} fX^H \\ x^H \end{bmatrix} = f^2 XX^H + xx^H.$$

The matrix $XX^H$ has $4m^2$ entries, so multiplying it by $f^2$ requires $O(m^2)$ multiplications. The number of multiplications needed to form $xx^H$ is also $O(m^2)$.

**Problem 5.** Program the Eigen-Esprit algorithm and experiment with exponential windowing for Problem 3's data. Use the forgetting factor $f = 0.9$, and compare the results with those of Problem 3.

**Answer:** Figure B shows the results. The average error in the angle estimate is 0.62 degrees, and the average relative error is 0.046. The results are quite similar to those for rectangular windowing.

**Problem 6.** Suppose we have a matrix $\boldsymbol{X}$ of size $m \times n$, $m \le n$ and that each element of $\boldsymbol{X}$ is normally distributed with mean 0 and standard deviation $\psi$.

a. Show that the random variable equal to the sum of the squares of the entries of $\boldsymbol{X}$ is equal to the sum of the squares of the singular values of $\boldsymbol{X}$.

b. Show, therefore, that for rectangular windowing of this data, the expected value of $\sigma_1^2 + \ldots \sigma_m^2$ is $\psi^2 mn$, where $\sigma_j$ is a singular value of $\boldsymbol{X}$.

c. Using a similar argument, show that for exponential windowing, the expected value of $\sigma_1^2 + \ldots \sigma_m^2$ is approximately $\psi^2 f^2 m/(1 - f^2)$, where $\sigma_i$ is a singular value of $\boldsymbol{FX}$. Here, $\boldsymbol{F}$ is a diagonal matrix, with the $j$th entry equal to $f^j$.

**Answer:**

a. The sum of the squares of the entries of $\boldsymbol{X}$ is the square of the *Frobenius norm* of $\boldsymbol{X}$, and this norm is invariant under multiplication by an orthogonal matrix. Therefore,

$\|\boldsymbol{X}\|_F^2 = \|\Sigma\|_F^2 = \sigma_1^2 + \ldots \sigma_m^2$.

b. The expected value of the square of each entry of $\boldsymbol{X}$ is $\psi^2$, so the sum of these $mn$ values has expected value $\psi^2 mn$.

c. The expected value is now

$$\sum_{k=1}^{m}\sum_{j=1}^{n} f^{2j} E(x_{kj}^2) = m \sum_{j=1}^{n} f^{2j}\psi^2 \to \frac{mf^2\psi^2}{1-f^2}$$

for large $n$, where $E$ denotes expected value.

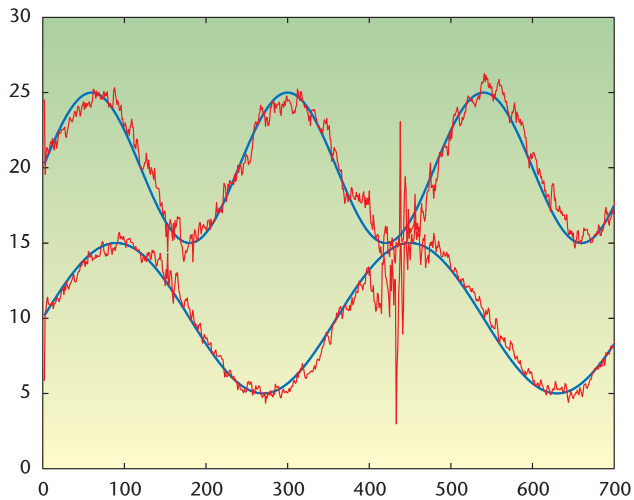


**Figure B. Results of Problem 5. The true direction of arrival (DOA) appears in blue; the DOA estimated by exponential windowing (in red) is shown as a function of time.**

**Problem 7**. Modify the programs to determine $d$, and explore the methods' sensitivity to the choice of $n$, $f$, and $\kappa$.

**Answer:** The software at http://computer.org/cise/homework/ varies $\kappa$ between 2 and 6. For rectangular windowing, a window size of 4 produced fewer $d$-failures than window sizes of 6 or 8 at a price of increasing the average error to 0.75 degrees. As $\kappa$ increased, the number of $d$-failures also increased, but the average error when $d$ was correct decreased.

For exponential windowing, the fewest $d$-failures (8) occurred for $f = 0.7$ and $\kappa = 2$, but the average error in this case was 1.02. As $\kappa$ increased, the number of $d$-failures increased, but, again, the average error when $d$ was correct decreased.

We have seen that matrix-based algorithms are powerful tools for signal processing, but they must be used in light of statistical theory and the problem's geometry. CiSE


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