Editor: Dianne P. O'Leary, oleary@cs.umd.edu

ROBOT CONTROL: SWINGING LIKE A PENDULUM

By Dianne P. O'Leary and Yalin E. Sagduyu

SUPPOSE WE HAVE A ROBOT ARM WITH A SINGLE JOINT, SIMPLY MODELED AS A DAMPED DRIVEN PENDULUM. IT'S AMAZING HOW SUCH A TRIVIAL SYSTEM ILLUSTRATES SO MANY difficult concepts! In this issue's project, we'll study the stability and behavior of this robot arm and develop a strategy to move the arm from one position to another using minimal energy.

The Model

Assume that a pendulum of length ℓ has a bob of mass m. Figure 1 shows the pendulum's position at some time t, with the variable $\theta(t)$ denoting the angle that the pendulum makes with the vertical axis at that time. The angle is measured in radians. The pendulum's acceleration is proportional to the angular displacement from vertical; we model the drag due to friction with the air as being proportional to velocity. This yields a second-order ordinary differential equation (ODE) for $t \geq 0$:

$$m\ell \frac{d^2\theta(t)}{dt^2} + c\frac{d\theta(t)}{dt} + mg\sin(\theta(t)) = u(t)\,, \tag{1}$$

where g is the gravitational acceleration on an object at the Earth's surface, and c is the damping (or frictional) constant. The term $u(t)$ defines the external force applied to the pendulum. In this project, we consider what happens in three cases: no external force, constant external force driving the pendulum to a final state, and then a force designed to minimize the energy needed to drive the robot arm from an initial position to an angle θ_f.

The Robot Arm's Stability and Controllability

The solution to Equation 1 depends on relations among m, ℓ, c, g, and $u(t)$ and ranges from fixed amplitude oscillations for the *undamped* case ($c = 0$) to decays (oscillatory or strict) for the *damped* case ($c > 0$). Unfortunately, there is no simple analytical solution to the pendulum equation in terms of elementary functions unless we linearize the term $\sin(\theta(t))$ in Equation 1 as $\theta(t)$, an approximation that is only valid for small values of $\theta(t)$. Despite the linear approximation's limitations, the linearization helps us find analytical solutions and also apply the results of *linear control theory* to the specific problem of robot arm control. To control the arm in a reasonable way, the system must be stable. Problem 1 considers the stability of a simpler model, valid for small oscillations.

> **Problem 1.** Consider the *undriven damped pendulum* that Equation 1 models when $u(t) = 0$ and $c > 0$. Linearize the second-order nonlinear differential equation using the approximation $\sin(\theta(t)) \approx \theta(t)$. Transform this equation into a first-order system of ODEs of the form $\mathbf{y}' = \boldsymbol{A}\mathbf{y}$, where $\boldsymbol{A}$ is a 2×2 matrix, and the two components of the vector $\mathbf{y}(t)$ represent $y_1(t) = \theta(t)$ and $y_2(t) = d\theta(t)/dt$. Determine the eigenvalues of $\boldsymbol{A}$. Show that the damped system is *stable*—that the real part of each eigenvalue is negative—and that the undamped system is not. Use the eigenvalue information to show how the solutions behave in the damped and undamped systems.

Equation 1's stability is more difficult to analyze than the stability of the linearized approximation to it. *Liapunov's stability* occurs when the total energy of an unforced (or undriven), dissipative mechanical system decreases as the system state evolves in time. Therefore, the *state vector* $\mathbf{y}^T = [\theta(t), d\theta(t)/dt]$ approaches a constant value (or *steady state*) corresponding to zero energy as time increases.[1,2] According to Liapunov's formulation, the equilibrium point $\mathbf{y} = \mathbf{0}$ of a system described by the equation $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ is globally asymptotically stable if $\lim_{t\to\infty} \mathbf{y}(t) = 0$ for any choice of $\mathbf{y}(0)$. Let $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ and let $\bar{\mathbf{y}}$ be a steady-state solution of this differential equation. Terminology varies from text to text, but we will use these definitions:

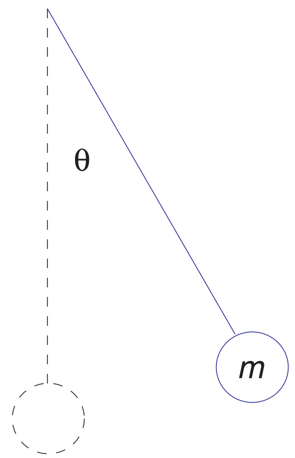

Figure 1. Swinging like a pendulum. We will move this robot arm (or pendulum), shown here at a position $\theta(t)$.

- A *positive definite Liapunov function* v at $\bar{\mathbf{y}}(t)$ is a continuously differentiable function into the set of nonnegative numbers. It satisfies $v(\bar{\mathbf{y}}) = 0$, $v(\mathbf{y}(t)) > 0$, and

$$\frac{d}{dt} v(\mathbf{y}(t)) \leq 0$$

for all $t > 0$ and all $\mathbf{y}$ in a neighborhood of $\bar{\mathbf{y}}$.
- An *invariant set* is a set for which the solution to the differential equation remains in the set when the initial state is in the set.

This version of the Liapunov theorem[3] for global asymptotic stability guides our analysis:

Theorem 1. Suppose v is a positive definite Liapunov function for a steady-state solution $\bar{\mathbf{y}}$ of $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. Then $\bar{\mathbf{y}}$ is stable. If in addition $\{\mathbf{y} : dv(\mathbf{y}(t))/dt = 0\}$ contains no invariant sets other than $\bar{\mathbf{y}}$, then $\bar{\mathbf{y}}$ is asymptotically stable.

Finding a Liapunov function for a given problem can be difficult, but success yields important information. For unstable systems, small perturbations in the application of the external force can cause large changes in the behavior of the equation's solution and, thus, to the pendulum's behavior, so the robot arm might behave erratically. Therefore, in practice, we must ensure that the system is stable (see Problem 2).

> **Problem 2.** Consider the function
>
> $$v(\theta, d\theta / dt) = \frac{(1-\cos\theta)g}{\ell} + \frac{1}{2}\left(\frac{d\theta}{dt}\right)^2$$
>
> for the pendulum Equation 1 describes. Show that it is a valid positive definite Liapunov function for the undriven model. Investigate the stability of the solution $\theta(t) = 0$, $d\theta(t)/dt = 0$ for undamped and damped systems.

Consider the first-order system described by $\mathbf{y}' = \boldsymbol{A}\mathbf{y} + \boldsymbol{B}\mathbf{u}$, where $\mathbf{y}$ is an $n \times 1$ column vector, $\boldsymbol{A}$ is an $n \times n$ matrix, $\boldsymbol{B}$ is an $n \times m$ matrix, and $\boldsymbol{u}$ is an $m \times 1$ column vector. The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ might depend on time t, but in our example, they do not. The system is *controllable* on $t \in [0, t_f]$ if given any initial state $\mathbf{y}(0)$ there exists a continuous function $\mathbf{u}(t)$ such that the solution of $\mathbf{y}' = \boldsymbol{A}\mathbf{y} + \boldsymbol{B}\mathbf{u}$ satisfies $\mathbf{y}(t_f) = 0$. For controllability on any time interval, it is necessary and sufficient that the $n \times nm$ controllability matrix $[\boldsymbol{B}, \boldsymbol{AB}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{B}]$ have rank n on that interval.[1,2]

Controllability of the robot arm means that we can specify a force that will drive it to any desired position. Problem 3 investigates the controllability of the linearized pendulum model.

> **Problem 3.** Consider the linearized version of the driven (or forced), damped pendulum system with constant force term u. Transform the corresponding differential equation to a first-order ODE system of the form $\mathbf{y}' = \boldsymbol{A}\mathbf{y} + \boldsymbol{B}\mathbf{u}$. Specify the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and show that the system is controllable for both the damped and undamped cases.

What's Inside

In this second installment of Your Homework Assignment, we pose a problem on the motion and control of a robot arm, using tools from linear algebra, ordinary differential equations, and optimization. The solution will appear here in the next issue.

By now, perhaps you've deblurred the image from the last issue's homework assignment. One implementation is now available on the Web site (http://computer.org/cise/homework/v5n3.htm), and the solutions to the individual problems appear in this issue at the end of this article.

Tools

For Problem 1, consult an ODE text[1] for converting second-derivative equations to a system of equations involving only first derivatives. An elementary linear algebra text[2] will discuss computation of the eigenvalues and eigenvectors of a 2×2 matrix; the ODE text will explain how to solve linear ODE systems once the eigensystem is known.

An ODE text[1] also can serve as a reference on Liapunov stability, which is used in Problem 2. The theorem is taken from Beltrami's book.[3] Control theory texts[4,5] discuss stability plus the concept of controllability used in Problem 3.

Problem 4 requires an ODE solver for initial value problems (for example, Matlab's `ode45`). Problem 5 uses the ODE solver with a nonlinear equation solver (such as Matlab's `fzero`) and a linear equation solver (such as Matlab's backslash) that uses the LU decomposition or some other numerically stable method. Consult a standard numerical analysis textbook[6,7] for information on such packages. Problem 6 relies on an ODE solver and a function for minimization of a function of a single variable under bound constraints (such as, Matlab's `fminbnd`).

References

1. W. Leighton, *Ordinary Differential Equations*, Wadsworth Publishing, 1966.
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Numerical Solution of the Initial Value Problem

Next, we develop some intuition for the behavior of the original and the linearized models by comparing them under various experimental conditions. For the numerical investigations in Problems 4 through 6, assume that $m = 1$ kg, $\ell = 1$ m, and $g = 9.81$ m/sec^2, with $c = 0$ for the undamped case and $c = 0.5$ kg-m/sec for the damped case. First we investigate the effects of damping and of applied forces in Problem 4.

Problem 4. For the initial conditions $\theta(0) = \pi/4$ and $d\theta(0)/dt = 0$, use an ODE solver to find the numerical solutions on the interval $t = [0, 30]$ for the nonlinear model in Equation 1 for

1. an undamped ($c = 0$), undriven ($u = 0$) pendulum,
2. a damped ($c > 0$), undriven ($u = 0$) pendulum, and
3. a damped ($c > 0$), driven pendulum with the applied forces $u = mg \sin(\theta_f)$, where $\theta_f = \pi/8, \pi/4, \pi/3$.

Repeat the same experiments for the pendulum's linearized model and discuss the difference in behavior of the solutions. It will help if you plot the $\theta(t)$ results for the corresponding linear and nonlinear models in the same figure.

Missing Data: Solution of the Boundary Value Problem

In Problem 4, we solved the *initial value problem*, in which values of θ and $d\theta/dt$ were given at time $t = 0$. In many cases, we don't have the initial value for $d\theta/dt$, because this value might not be *observable*. The missing initial condition prevents us from applying standard methods to solve initial value problems. Instead, we might have the value $\theta(t_B) = \theta_B$ at some other time t_B. Next we investigate two solution methods for this *boundary value problem*: the shooting method and the finite-difference method.

The idea behind the *shooting method* is to guess at the missing initial value $z = d\theta(0)/dt$, integrate Equation 1 using our favorite method, and then use the results to improve the guess. To do this systematically, we use a nonlinear equation solver to solve the equation $\rho(z) \equiv \theta_z(t_B) - \theta_B = 0$, where $\theta_z(t_B)$ is the value reported by an initial value problem ODE solver for $\theta(t_B)$, given the initial condition $z = d\theta(0)/dt$.

The *finite-difference method* is an alternate method to solve a boundary value problem. Choose a small time increment $h > 0$ and replace the first derivative in the linearized model of Equation 1 by

$$\frac{d\theta(t)}{dt} \approx \frac{\theta(t+h) - \theta(t-h)}{2h}$$

and second derivative by

$$\frac{d^2\theta(t)}{dt^2} \approx \frac{\theta(t+h) - 2\theta(t) + \theta(t-h)}{h^2}.$$

Let $n = t_B/h$, and write the equation for each value $\theta_j \approx \theta(jh)$, $j = 1, \ldots, n-1$. The *boundary conditions* can be stated as $\theta_0 = \theta(0)$, $\theta_n = \theta_B$. This method transforms the linearized version of the second-order differential Equation 1 to a system of $n-1$ linear equations with $n-1$ unknowns. Assuming the solution to this linear system exists, we then use our favorite linear system solver to solve these equations. We try these two methods in Problem 5.

Problem 5. Consider the linearized model of Equation 1 with constant applied force $u(t) = mg\sin(\pi/8)$ and damping constant $c = 0.5$. Suppose that we have the boundary conditions $\theta(0) = \pi/32$ and $\theta(10) = \theta_B$, where θ_B is the value of the solution when $d\theta(0)/dt = 0$. Apply the shooting method to find the solutions to the damped, driven, linearized pendulum equation on the time interval $t = [0,10]$. For the shooting method, use a nonlinear equation solver and an ODE solver for initial value problems. Try different initial guesses for $d\theta(0)/dt$ and compare the results.

Now use the finite-difference method to solve this boundary value problem with $h = 0.01$. Use your favorite linear system solver to solve the resulting linear system of equations.

Compare the results of the shooting and finite-difference methods with the solution to the original initial value problem.

Controlling the Robot Arm

Finally, in Problem 6, we investigate how to design a forcing function that drives the robot arm from an initial position to some other desired position with the least expenditure of energy. We measure energy as the integral of the absolute force applied between time 0 and the convergence time t_c when the arm reaches its destination.

$$e_f = \int_0^{t_c} |u(t)| dt .$$

Problem 6. Consider the damped, driven pendulum with applied force

$$u(t) = mg\sin(\theta_f) + mlb\, d\theta(t)/dt,$$

where $\theta_f = \pi/3$. This force is a particular *closed-loop control* with control parameter b, and it drives the pendulum position to θ_f. The initial conditions are given as $\theta(0) = \pi/4$ and $d\theta(0)/dt = 0$. Assume $c = 0.5$ as the damping constant, $t_c = 5$ seconds as the time limit for achieving the position θ_f, and $h = 0.01$ as the time increment for numerical solutions.

We will call a parameter b successful if the pendulum position satisfies $|\theta(t) - \theta_f| < 10^{-3}$ for $5 \le t \le 10$. Approximate the total energy by

$$\hat{e}_f \approx \sum_{k=1}^{5/h} |u(kh)| h .$$

Write a function that evaluates $\hat{e}_f$. The input to the function should be the control parameter b and the output should be the approximate total consumed energy $\hat{e}_f$.

For stability of the closed-loop control system, we impose the constraint $b < c/(ml)$, which makes the real parts of the eigenvalues (of the linearized version) of the system strictly negative. Now use your favorite constrained minimization solver to select the control parameter b to minimize the energy function $\hat{e}_f(b)$. Display the optimal parameter and graph the resulting $\theta(t)$.

In this homework assignment, we studied one simple mathematical model of a robot arm. The numerical solution couples techniques borrowed from linear algebra, ordinary differential equations, optimization, and control theory. Control of our system involves cost trade-offs: energy versus time of convergence. It's worth noting that stability and control studies of most physical systems (such as underground seepage, muscle movement, and combustion reactions) require a combination of analytical and computational tools, even for quite simple mathematical models. The solution to this homework will appear here in the next issue and on the Web page, http://computer.org/cise/homework/v5n4.htm.

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References

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