Computer Lab Assignment # 1

Computational Methods for Continuous Problems

Lab: Wednesday, 9/4 Due: 8:30 a.m., Wednesday, 9/11

Submit write up, code and any auxiliary files to Dave Witman, dw11d@my.fsu.edu

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Goals: To investigate modeling using exponential growth and logistic growth and to use the forward Euler method for numerical approximations to these models.

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1. (10 pts) Write a code to implement the forward Euler method. As input you need to specify the initial and final times, the time step, and the initial condition. You should have separate functions for the given derivative in the problem (i.e., $f(t, y)$) and the exact solution. Make sure that your code handles problems where the final time minus the initial time is not an integer multiple of the time step. Test your code on each of the following problems by calculating the solution for $\Delta t = 1/4, 1/8, \ldots, 1/64$ and calculating the numerical rate of convergence from equation (1.15). Tabulate the solution and the error at the final time for each value of Δt along with the calculated numerical rate. Compare your numerical rate with the theoretical result we obtained in class.

$$\text{a.} \quad y'(t) = t^3 y \quad 0 < t \le 1 \qquad y(0) = 0.5$$

$$\text{b.} \quad y'(t) = \sin t \quad \frac{\pi}{2} < t \le 2\pi \qquad y(\frac{\pi}{2}) = 1$$

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2. (20 pts) Suppose you are interested in modeling the growth of the Bread Mold Fungus, *Rhizopus stolonifer*, and comparing your numerical results to experimental data that is taken by measuring the number of square inches of mold on a slice of bread over a period of several days. Assume that the slice of bread is a square of side 5 inches.

a. To obtain a model describing the growth of the mold you first make the hypothesis that the growth rate of the fungus is proportional to the amount of mold present at any time with a proportionality constant of k. Assume that the initial amount of mold present is 0.25 square inches. Let $p(t)$ denote the number of square inches of mold present on day t. Write an initial value problem for the growth of the mold and determine its exact solution which will be in terms of k.

b. Assume that the following data is collected over a period of ten days and that k is a constant. (i) Use the data at day one to determine k. (ii) Use your forward Euler method with Δt a fourth, an eighth and a sixteenth of a day to obtain numerical estimates for each day of the ten day period; tabulate your results and compare with the experimental data. Plot the experimental data along with your numerical results on the same graph. When do the results become physically unreasonable? Why?

$t = 0$	$p = 0.25$	$t = 1$	$p = 0.55$
$t = 2$	$p = 1.1$	$t = 3$	$p = 2.25$
$t = 5$	$p = 7.5$	$t = 7$	$p = 16.25$
$t = 8$	$p = 19.5$	$t = 10$	$p = 22.75$

c. The difficulty with the exponential growth model is that the bread model grows in an unbounded way. To improve the model for the growth of bread mold, we want to incorporate the fact that the number of square inches of mold can't exceed the number of square inches in a slice of bread. Write a logistic differential equation which models this growth using the same initial condition and growth rate as before.

d. Use the forward Euler method with Δt a fourth, an eighth and a sixteenth of a day to obtain numerical estimates for the amount of mold present on each of the ten days using your logistic model. Tabulate your results as in (b) and compare your results to those from the exponential growth model. Plot your logistic and exponential results for Δt=1/16 of a day on the same plot.

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3. (20 pts) In this problem we want to model the contamination of a a body of water. There are many assumptions made in this model, e.g., we will not consider evaporation, we assume the pollutant is uniformly distributed, etc.

a. Assume that a pond contains 10 million gallons of water and that water is coming into the pond at the rate of 5 million gallons/year and the mixture in the pond flows out at the same rate. Also assume that the water coming into the pond contains a contaminant. Let $\delta(t)$ represent the concentration of the chemical contaminant (in grams per gallon) in the inflow water at any time t, i.e., assume that it varies in time and is described by the function $\delta(t)$. Let $Q(t)$ represent the amount (in grams) of pollutant in the pond at any time t and assume that initially there is no contaminant, i.e., $Q(0) = 0$. We want to write an IVP for $Q(t)$ which will allow us to approximate the amount of contaminant at any time t. The rate of change of $Q(t)$ is governed by

$$\frac{dQ}{dt} = \text{contaminant rate in} - \text{ contaminant rate out}$$

so

$$\frac{dQ}{dt} = (5 \times 10^6)\delta(t) - (5 \times 10^6)\frac{Q}{10 \times 10^6} = (5 \times 10^6)\delta(t) - \frac{Q}{2}$$

and $Q(0) = 0$. We have used the fact that in a pond of size 10^6 the fraction of contaminant present at any time t is $Q(t)/10^6$. To avoid the issue of such a large coefficient of δ we perform a change of variables and let $q = (10^{-6})Q$. Multiplication of the differential equation by 10^{-6} gives the final IVP

$$\frac{dq}{dt} = 5\delta(t) - \frac{q}{2} \quad 0 < t \leq T, \quad q(0) = 0\,.$$

Now we choose the specific pollutant concentration

$$\delta(t) = 2 + \sin(2t)$$

We want to solve this problem using the implicit backward Euler method. Note that our differential equation is linear in the unknown $q(t)$ so a nonlinear equation doesn't have to be solved at each time step so it is an easy modification to your forward Euler method to incorporate backward Euler for this problem. Use the backward Euler method to approximate the solution using decreasing values of Δt starting with $\Delta t = 2$. Decrease Δt by cutting it in half until you have two successive solutions which differ by no more than 0.05 at the end of 10 years. What is your final value of Δt? Plot your result on $[0, 20]$ and discuss your approximation to $Q(t)$ based on the model.

b. In this problem we modify our model slightly and then use it to determine the amount of time for a percentage of the pollutant to be flushed from the body of water if the contamination is stopped completely. Suppose we have a lake of volume V in km^3 and we once again let $Q(t)$ represent the amount of pollutant present at time t. Then the concentration $c(t)$ at any time t is Q/V. Assume that the volume remains fixed so that the water entering and leaving is at a rate r in km^3 per year. In addition, assume that the contaminant comes from two sources; first the water coming in has a pollutant concentration of k in grams per volume and a pollutant is added at a constant rate of P in grams per year. We want to write an IVP for $c(t)$ and then estimate the time when the concentration becomes zero if the inflow of contaminant is stopped. Once again dc/dt is found by subtracting the concentration of contaminant leaving from that coming in. We have

$$\frac{dc}{dt} = \frac{r}{V}\left[k + \frac{P}{r}\right] - \frac{r}{V}c\,.$$

Assume that the volume of Lake Superior is 12,200 km^3, $r = 65.2$, $k = 145$, and $P = 17$. (i) Use the forward Euler scheme to estimate the concentration in the lake after 50 years to four digits of accuracy (i.e., 0.abcd $\times 10^n$). Do not find the exact solution but rather compute solutions by halving Δt until your result is changing in the fifth digit; start with $\Delta t = 1$. (ii) After 50 years assume that all pollution is stopped. Determine (to the nearest year) the number of years it will take for the contaminant to reach 50% and 10% of its 50 year high. Use your final Δt from (i).