Homework # 3

Due: Friday, 9/27

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1. (20 pts) Consider the third order Adams-Bashforth method:

$$Y_{i+1} = Y_i + \frac{\Delta t}{12}\big[23f(t_i, Y_i) - 16f(t_{i-1}, Y_{i-1}) + 5f(t_{i-2}, Y_{i-2})\big]$$

a. Write a computer code to implement this multistep method where you use the third order RK method from Lab # 2 to get your starting values at Y_1 and Y_2. Test your code on the IVP

$$y'(t) = t^2 e^{t^3}, \quad 0 < t < 1 \qquad y(0) = \frac{1}{3}$$

whose exact solution is $y(t) = \frac{1}{3}e^{t^3}$ and verify that it is indeed third order by computing the numerical rate of convergence for $\Delta t = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$.

b. Modify the code from (a) to implement a predictor/corrector pair where we predict with the explicit method in (a) to get Y^p_{i+1} and correct with

$$Y_{i+1} = Y_i + \frac{\Delta t}{12}\big[5f(t_{i+1}, Y^p_{i+1}) + 8f(t_i, Y_i) - f(t_{i-1}, Y_{i-1})\big]\,.$$

Repeat your calculations in (a) using this pair. What do you conclude about the numerical rate and the magnitude of the error?

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2. (8 pts) Write the following fourth order problem as a system of first order IVPs using the unknowns $w_i(t)$, $i = 1, 2, 3, 4$.

$$y''''(t) + 4y''(t) - y'(t) = 4t + y^2(t) \quad 0 < t \le 4 \qquad y(0) = 1, y'(0) = -2, y''(0) = 5, y'''(0) = 0$$

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3. (12pts) Consider the so-called "theta method"

$$Y_{i+1} = Y_i + \Delta t\big[\theta f(t_i, Y_i) + (1-\theta) f(t_{i+1}, Y_{i+1})\big]$$

Note that when $\theta = 1$ we have the forward Euler scheme, when $\theta = 0$ we have the backward Euler scheme and when $\theta = 1/2$ we have the trapezoidal method. We want to apply this scheme to solve the IVP $y'(t) = \lambda y(t)$, $y(0) = 1$.

a. Write the method in the form $Y_{i+1} = \zeta(\lambda\Delta t)Y_i$. Explicitly give the amplification factor $\zeta(\lambda\Delta t)$. Show that your result reduces to the amplification factor we derived for the forward Euler when $\theta = 1$ and the backward Euler when $\theta = 0$.

b. (UG) (i) Assuming that $\lambda < 0$ is real, determine the amplification factor ζ for the trapezoidal rule where $\theta = 1/2$. (ii) Is the trapezoidal rule explicit or implicit? For what value(s) of θ is the general scheme implicit?

c. (G) Assuming that $\lambda < 0$ is real, determine the amplification factor ζ and the region of absolute stability for the trapezoidal rule where $\theta = 1/2$.

d. If we use the forward Euler method to solve $y'(t) = -12y$, $y(0) = 1$ what bound must we have on Δt? Confirm your results by computing the solution for a range of values for Δt in both the stability region and outside this region.

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4. (10 pts) PDEs

a. Determine if each PDE is linear or nonlinear and give its order. If it is a second order linear PDE, classify it as elliptic, parabolic or hyperbolic and justify your answer.

i. $u = u(x,t)$ where $u_t = uu_x$

ii. $u = u(x,t)$ where $u_{tt} - 4u_{xx} = x^4 t$

iii. $u = u(x,y)$ where $-\frac{\hbar}{2m}\Delta u + \nu(x)u = 0$ and $\hbar, m$ are given constants and $\nu(x)$ is a given function

b. (G) Consider the biharmonic equation $\Delta^2 u = \Delta(\Delta u) = f(x,y,z)$ where $u = u(x,y,z)$. This equation occurs in the study of elasticity and fluid flow.

i. First write this equation in terms of the partial derivatives of u; for example if the equation was $-\Delta u = f$ you would write $-(u_{xx} + u_{yy} + u_{zz}) = f(x,y,z)$. Assume that the derivatives are continuous so that e.g., $u_{xy} = u_{yx}$.

ii. Write the biharmonic equation as a system of two second order equations.
