Homework # 1

Due: Monday, 9/9

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1. (10pts) a. One way to derive difference methods for $y'(t) = f(t, y)$ is to integrate the equation from t_i to t_{i+1} , i.e.,

$$\int_{t_i}^{t_{i+1}} y'(t)\ dt = \int_{t_i}^{t_{i+1}} f(t, y)\ dt$$

and then use a numerical quadrature rule to approximate the integral of f. Derive the backward Euler method by using an appropriate numerical quadrature rule. Be sure and state which quadrature rule you are using.

b. (Graduates only) Use the quadrature rule

$$\int_a^b g(x)\ dx = \frac{b-a}{2}\Big[g(a) + g(b)\Big]$$

to approximate $\int_{t_i}^{t_{i+1}} f(t, y)\ dt$ and derive a difference equation for $y'(t) = f(t, y)$. Is it explicit or implicit?

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2. (10 pts) Identify each method as explicit or implicit. Justify your answer.

a. $Y_{i+1} = Y_i + \frac{\Delta t}{2}\Big[f(t_i, Y_i) + f\big(t_i + \frac{\Delta t}{2}, Y_i + \frac{\Delta t}{2} f(t_i, Y_i)\big)\Big]$

b. $Y_{i+1} = Y_i + \frac{\Delta t}{2}\Big[f(t_i, Y_i) + f\big(t_{i+1}, Y_{i+1}\big)\Big]$

c. $Y_{i+1} = Y_i + \frac{\Delta t}{2}\Big[f(t_i, Y_i) + f\big(t_{i+1}, Y_i + \Delta t f(t_i, Y_i)\big)\Big]$

d. $Y_{i+1} = Y_{i-3} + \frac{4\Delta t}{3}\Big[2f(t_i, Y_i) - f(t_{i-1}, Y_{i-1}) + 2f(t_{i-2}, Y_{i-2})\Big]$

e. $Y_{i+1} = Y_{i-1} + \frac{\Delta t}{3}\Big[f(t_{i-1}, Y_{i-1}) + 4f(t_i, Y_i) + f\big(t_{i+1}, Y_{i+1}\big)\Big]$

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3. (10 its) Consider the specific IVP

$$y'(t) = t^3 + 5t + 1 \quad 0 < t < 4 \qquad y(0) = 1$$

a. Perform one step of the forward Euler method with $\Delta t = .1$ and compute the actual error there (use exact arithmetic). Write the equation of the tangent line to the solution at the origin. Plot the exact solution on $[0, .1]$, the tangent line at the origin and your approximation. Is your approximation on the exact solution curve, the tangent line or neither? Why?

b. We know that the local truncation error in taking a step of the forward Euler method is given by the remaining terms in the series

$$\frac{(\Delta t)^2}{2!} y''(t) + \frac{(\Delta t)^3}{3!} y'''(t) + \frac{(\Delta t)^4}{4!} y^{[iv]}(t) + \cdots$$

which, in general, we are not able to compute exactly because it is an infinite series. However, for our problem the terms involving $(\Delta t)^5$ and higher vanish (why?).

Calculate the local truncation error (using exact arithmetic) using this series when you take one step of length $\Delta t = 0.1$ starting from $t = 0$ and compare with your answer in (a). Is this local error the same as the global error at $t = 0.1$ that you computed before when you use exact arithmetic? Why or why not?

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4. (20pts) Problem #5, Chapter 1
