Model reduction for compressible fluid-solid coupling and its application to blasting

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Abstract

This work presents the first non-intrusive reduced order model (NIROM) for compressible fluid and structure interactions. The NIROM is constructed through proper orthogonal decomposition (POD) and a radial basis function (RBF) multi-dimensional interpolation method. The novelty of NIROM for blasting modelling lies in reduced order modelling for the unique combination of fluid, solid, air-solid coupling and fracture modelling. In this work, NIROM is applied to fluid modelling where an intensive computational cost is required. A coupling source term is introduced to fluid modelling. A Mohr-Coulomb failure criterion with a tension cut-off is used to judge whether new fractures are generated.

The performance of the NIROM for a structure interacting with compressible fluid flows in the presence of fracture models is illustrated by two complex test cases: a bending beam forced by flows and a blasting test case. The numerical simulation results show that the NIROM is capable of capturing the details of compressible fluid and structure interactions and fractures while the CPU time is reduced by several orders of magnitude. In addition, the issue of whether or not to subtract the mean before applying POD is discussed in this paper. It is shown that solutions of NIROM without mean subtraction before performing POD are better than that NIROM with mean subtraction.

Keywords: ROM, compressible fluid-structure interaction, blasting

1. Introduction

The numerical simulation of fluid/solid interaction has attracted much attention in a wide variety of research areas. This problem is of significance to many fields in engineering such as aerospace engineering, biomedical engineering, wind turbines and blasting. However, the computational expense involved in solving such problems is so

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great that this has hindered development in these areas. In order to address the issue of high computational expense, this paper proposes a non-intrusive reduced order model to solve fluid/solid interaction problems in an efficient manor.

Reduced order modelling is a technique that is capable of reducing the dimensionality of large systems, thus resulting in a considerable increase in computational efficiency. POD is the method most widely used to form reduced order models and it aims to represent a large system with only a relatively small number of optimal basis functions. The POD has been used successfully to various areas such as air pollution [1], ocean modelling [2, 3], fluid mechanics [4, 5, 6], aerospace design [7], neutron photon transport [8], porous media [9, 10] and shape optimization [11]. The large system can be projected onto a reduced system by Galerkin projection to derive the ROMs. However, the use of POD/Galerkin raises numerical instability and non-linearity inefficiency problems [12, 13]. Several methods have been presented to improve the numerical stability of ROM such as calibration [14, 15], Fourier expansion [16], regularisation [17] and Petrov-Galerkin method [2, 18]. In order to enhance the non-linear efficiency, various methods also have been proposed, including empirical interpolation method (EIM) [19], the discrete version of EIM (DEIM) [13], quadratic expansion method [20, 21], a hybrid of DEIM and quadratic expansion (residual DEIM) method [22], Petrov–Galerkin projection method [14], and Gauss–Newton with approximated tensors (GNAT) method [23]. To the best of our knowledge, although there are some works addressing reduced order modelling of fluid-structure interaction problem [24, 25, 26, 27, 28, 29], constructing ROM for compressible fluid and structure interaction problem in a non-intrusive way has not been reported in the literature.

In addition, this paper is the first time to numerically simulate the blast test case using a non-intrusive reduced order model. This is achieved by using a set of RBF interpolation functions (hypersurface) to represent the POD coefficients in reduced space.

The non-intrusive reduced order modelling technology is proposed to tackle the disadvantages of the intrusive ROM such as dependence of the governing equations and difficulties in modifying the source code. The modifications of source code could be impossible in commercial software [30]. A number of NIROM methods have been proposed, such as a black-box stencil interpolation method [30], a POD-RBF method for unsteady fluid flows [31], a Taylor series and Smolyak sparse grid method for the Navier-Stokes equations [32], a two-level NIROM based on POD-RBF method for nonlinear parametrized PDEs [33, 34], a POD-RBF for the Navier-Stokes equations [35]. NIROM has also been applied to realistic problems such as 3D free surface problem [3] and multi-phase flow in porous media problems [10].

This article, for the first time, applies the non-intrusive reduced order modelling method to compressible fluid and structure interaction problems, especially the highly non-linear problem - the blasting test case. This model has been implemented under the framework of a combined finite-discrete element method based solid model (Y2D) and an unstructured mesh finite element model (Fluidity).

In this approach, the results of high fidelity full model are recorded using snapshots method, then a number of POD basis functions are generated from those snapshots through singular value decomposition (SVD) method. A set of hypersurfaces is then constructed to represent the compressible fluid and structure interactions and fracture model using the RBF-POD method. After obtaining the hypersurface, the reduced
system of the compressible fluid and structure interactions and fracture model is solved by inputting earlier time levels’ POD coefficients into the hypersurface.

During the POD process, the mean of the snapshots is normally subtracted. The problem of mean subtraction was discussed in [36, 37]. In their work, there was not much difference between the results with mean subtraction and results without mean subtraction. In this paper, the solutions with mean subtraction and without mean subtraction are presented and discussed.

The performance of this compressible FSI NIROM without and with mean subtraction has been assessed for two test cases: a bending beam forced by flows and a blasting test case. Comparison between high fidelity full model and the compressible FSI NIROM without mean subtraction using different number of POD bases has been made to validate the accuracy.

The structure of the paper is arranged as follows: section 2 presents the compressible fluid and solid coupling equations; section 3 derives the formulation of a non-intrusive reduced order model for compressible fluid and structure interactions and fracture problems using POD-RBF method; section 4 demonstrates the capability of the derived methodology by two numerical examples: a bending beam forced by flows and a blasting test case. Finally in section 5, the conclusion is drawn.

2. Description of compressible fluid/solid coupling and fracture modelling problem

This work is carried out under the framework of an unstructured mesh multi-phase fluid model (Fluidity) and a combined finite-discrete element solid model (Y2D), therefore, the governing equations, coupling methods and fracture modelling methods used in those models are described in this section.

2.1. Governing equations for compressible fluids under the framework of ”Fluidity”

The governing equations for compressible fluids in Fluidity have the following form,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1}
\]

\[
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u} - \sigma) = \rho \mathbf{F}, \tag{2}
\]

where \(\rho\) denotes the density, \(\mathbf{u}\) is the velocity vector, \(t\) represents the time, \(\sigma\) is the stress tensor, \(\mathbf{F}\) is the volume or internal force per unit mass (e.g., gravity). The density \(\rho\) is calculated by the equation of state:

\[
P = \rho (\gamma - 1) e, \tag{3}
\]

where \(P\) is the pressure, \(\gamma = C_p/C_v\) is a heat capacity ratio (\(C_v\) and \(C_p\) being the specific heat at constant volume and at constant pressure respectively), \(e = C_v T\) is the internal energy per unit mass (“the specific internal energy”).
2.2. Governing equations for solid dynamics

The fluid model, Fluidity, is coupled with Y2D, which resolves the solid dynamics by a finite strain method [38]. The governing equation in solid mechanics is given by:

\[
F_e + F_v + F_p + F_c = m\frac{\partial u_s}{\partial t} + F_i,
\]

where \( F_e \) denotes the external force, \( F_v \) is the viscous force between the solid and fluid, \( F_p \) denotes the pressure between the fluid and solid, \( F_c \) is the contact force among multiple solids, \( m \) is the mass, \( u_s \) is the velocity of the solid and \( t \) is the time. For additional details, see [38].

2.3. Coupling methods between the fluid model (Fluidity) and solid model (Y2D)

2.3.1. Coupling equations

A supplementary equation is introduced to couple the fluid code (Fluidity) and solid code (Y2D), that is,

\[
\frac{\rho_f}{\Delta t} (\hat{u}_f - u_f^{f}) = \frac{\rho_f}{\Delta t} (u_s^{f} - u_f^{f}),
\]

where \( \Delta t \) is the time step and \( \hat{u}_f \) is the bulk velocity (\( \hat{u}_f = \alpha_f u_f^{f} + \alpha_s u_s^{f} = \hat{u}_f^{f} + \hat{u}_f^{s} (\alpha_f \) and \( \alpha_s \) being the volume fractions of fluid and solid, \( \alpha_f + \alpha_s = 1 \)). Subscripts denote the material fields, that is, \( s \) denotes the solid and \( f \) denotes the fluid. Superscripts denote the mesh associated with the material (\( s \) denotes values on the solid mesh and \( f \) denotes values on the fluid mesh). The solid velocity on the solid mesh \( u_s^{f} \) is projected onto the fluid mesh in order to obtain the solutions of the coupled system, then it becomes \( \hat{u}_f^{s} \) [39].

The coupling process is achieved by introducing a source term \( s_c \) into the momentum equation (2), which represents the effects of forces of the solid on the fluid. The momentum equation (2) then has the form of:

\[
\frac{\partial}{\partial t}(\rho f u) + \nabla \cdot (\rho f uu - \sigma) = \rho F + s_c.
\]

The source term \( s_c \) considers an exchange of force between the solid and fluid, and has the form \( s_c = (s_c^{x}, s_c^{y}, s_c^{z})^{T} \). For additional details, see [40].

The continuity equation has the form of,

\[
\nabla \cdot \hat{u}_f = 0,
\]

where

\[
\hat{u}_f = \begin{cases} 
  u_f^{f} & \text{if } \alpha_f = 1, \alpha_s = 0 \\
  u_s^{f} & \text{if } \alpha_f = 0, \alpha_s = 1.
\end{cases}
\]
2.3.2. Coupling source term

The coupling process involves the calculation of the source term $s_c$ in equation (6), which is described briefly here (for additional details, see [40]). The viscosity forces $F_{viscosity}^s$ and $F_{pressure}^s$ are calculated by:

$$F_{viscosity}^s + F_{pressure}^s = \int_{\Gamma_{solid}} N_i n \cdot (\tau_{solid} + Ip) d\Gamma,$$

where $N$ is the basis function, $\tau_{solid}$ is the viscous stress term; $\Gamma_{solid}$ is the solid surface, $n$ is the unit normal vector on the solid surface $n = (n_x, n_y, n_z)$. $I$ is the identity matrix.

Once obtaining $F_{viscosity}^s$ and $F_{pressure}^s$, the velocity of solids $\mathbf{u}_s = (u_s, v_s, w_s)$ can be calculated by equation (4). The source term can then be obtained using the following equations:

$$s_{c,x}^f = a_{xx}u_s + a_{xy}v_s + a_{xz}w_s,$$

$$s_{c,y}^f = a_{yx}u_s + a_{yy}v_s + a_{yz}w_s,$$

$$s_{c,z}^f = a_{zx}u_s + a_{zy}v_s + a_{zz}w_s,$$

where $a$ denotes the viscosity coefficients and the subscript $x$, $y$ and $z$ denote the coordinate directions, $\Delta x_{wall}$ is the fluid element length scale around the wall. $\Delta r$ is the thickness of the shell, which is an intermediate thin area between the fluid and solid, and is introduced for calculating the impact of the solid on the fluid [39].

2.4. Fracture modelling

The fracture model in the combined finite-discrete element solid model (Y2D) treats the whole domain as a multi-body system. Each body is discretised into the finite element mesh. The fracture model is comprised of the the finite element formulation and discrete element formulation. The finite element formulation is used to model continuum behaviour (i.e. calculation of stress and strain) before fractures are generated. If the failure criterion is met, the discrete element formulation is then used for modeling discontinuum behaviour (contact forces between discrete bodies and distribution of the contact force to nodes). The combination of the finite element formulation and discrete element formulation ensures the transition from continuum behaviour to discontinuum behaviour can be captured accurately. The method is known as the combined finite-discrete element method (FEMDEM) [41, 42]. A Mohr-Coulomb failure criterion with a tension cut-off is used.

The overall fracture modelling algorithm based on FEMDEM is given in algorithm 1 (for details, see [41, 42]), where $\mathbf{u}_{solid}$ denotes the solid velocity vector at each node at the time step $t$, $\mathbf{u}_{acceleration}$ is the acceleration, $\Delta t$ is the time step, $\mathbf{f}_{external}$ and $\mathbf{f}_{internal}$ are the external and internal forces at each node respectively, and $m$ is the mass of the node.

3. Model reduction

In reduced order modelling, any variable can be expressed as a linear combination of a number of basis functions representing the original high fidelity modelling system.
Algorithm 1: Fracturing simulation

1. Input data.
2. Insert 6-node joint elements between 4-node tetrahedral elements.
3. Calculate stresses using the finite element formulation.
4. Judge whether new fractures are generated.
   if new fractures are generated then
      add new contact forces.
   else
      detect contact forces in DEM domain.
   end if
5. Calculate contact forces in DEM domain.
6. Calculate velocity of each node through explicit time integration.
   \( u_{solid}^{t+1} = u_{solid}^{t} + u_{acceleration} \Delta t \)
   \( u_{acceleration} = \frac{f_{external} - f_{internal}}{m} \)
7. Output data.
9. Stop.

in an optimal sense. It has the following form:

\[
\varphi = \bar{\varphi} + \sum_{i=1}^{m} \alpha_i \Phi_i,
\]  

where \( \varphi \) denotes a variable to be solved (e.g. the velocity, pressure, density and solid concentration), \( \bar{\varphi} \) is the mean of variable solutions over the simulation time period, \( \alpha \) denotes the POD coefficients, \( m \) is the number of POD bases and \( \Phi \) denotes the POD basis functions. Using POD, the basis functions can be calculated from snapshots of variable solutions recorded at regular time intervals. The radial basis function interpolation method is used to calculate the POD coefficients. The procedure of POD is summarized in algorithm 2.

The radial basis function interpolation is used to determine the POD coefficients in (11). Commonly used RBFs are plate spline, multi-quadric, inverse multi-quadric and Gaussian. RBFs have been widely used in the context of multidimensional interpolation. An interpolation function \( f(x) \) representing a physical problem can be approximated through a linear combination of the RBF \( \phi \) centred at \( N \) points. In this work, the Gaussian RBF is used to construct the interpolation function \( f(x) \). The Gaussian RBF has a form of \( \phi(r) = e^{-r/\sigma^2} \) (\( r \) being the radius and \( \sigma \) being the shape parameter). In the following paragraph, a set of interpolation functions or hypersurfaces is derived through POD-RBF method. The POD-RBF NIROM method was firstly presented by Xiao, et al. [35]. In this work this method is used to derive NIROM for the compressible fluid/solid problem and fracture problem. The form of the equations used for
Algorithm 2: Proper Orthogonal Decomposition

1. Compute solution of the coupled compressible fluid and solid system at time levels 1, ..., Ns;
2. Retrieve snapshots matrix A from the solutions obtained;
3. Subtract the mean of snapshots matrix A, i.e. $A' = A - A_{\text{mean}}$;
4. Perform Singular Value Decomposition (SVD) to snapshots matrix A or $A'$, i.e. $A = E \Sigma F^T$;
5. Choose the dimension of ROM, $m$ ($m < N_s$);
6. Obtain POD basis functions $\Phi_i = E_{:,i}$ for $i \in \{1, 2 \ldots m\}$;

solving the reduced system is:

\[
\begin{align*}
\alpha_{n-u,j} &= f_{u,j}(\alpha_{n-1,u}, \alpha_{n-1,p}, \alpha_{n-1,d}, \alpha_{n-1,c}) \quad (12) \\
\alpha_{n-p,j} &= f_{p,j}(\alpha_{n-1,u}, \alpha_{n-1,p}, \alpha_{n-1,d}, \alpha_{n-1,c}) \quad (13) \\
\alpha_{n-d,j} &= f_{d,j}(\alpha_{n-1,u}, \alpha_{n-1,p}, \alpha_{n-1,d}, \alpha_{n-1,c}) \quad (14) \\
\alpha_{n-c,j} &= f_{c,j}(\alpha_{n-1,u}, \alpha_{n-1,p}, \alpha_{n-1,d}, \alpha_{n-1,c}), \quad (15)
\end{align*}
\]

where $\alpha$ denotes POD coefficients, subscripts $u$, $p$, $d$ and $c$ denote velocity, pressure, density and solid concentration components, subscript $j$ is the $j$th POD coefficient of a complete set of POD coefficient $(\alpha_u, \alpha_p, \alpha_d, \alpha_c)$, $n$ is time step, $f$ is a set of hypersurface functions representing the reduced system.

The hypersurface functions are constructed using POD-RBF method, as described in algorithm 3, where $N$ denotes the number of data points. $m$ denotes the number of POD basis functions. $A$ is the matrix associated with the data point and centre $c$ and $A_{i,j} = \phi(||(\alpha_{i,u}, \alpha_{j,p}, \alpha_{j,d}, \alpha_{j,c}) - c||)$.
Algorithm 3: Constructing a set of hypersurface using POD-RBF

(1) Generate a number of snapshots over the time period \([0, T]\) by solving the compressible fluid/solid interaction problem and fracture model;

(2) Calculate POD basis functions \(\Phi_u, \Phi_p, \Phi_d, \Phi_c\) through a truncated SVD of the snapshots matrix;

(3) Obtain the functional values \(y_i\) at the data point \(\alpha^i_u, \alpha^i_p, \alpha^i_d, \alpha^i_c\) via the solutions from the high fidelity full model, where \(t \in \{1, 2, \ldots, T\}\);

(4) Obtain a set of hypersurfaces through the following loop:

\[
\text{for } j = 1 \text{ to } m \text{ do }
\]

(i) Calculate the weights \(w_{i,j}\) by solving (16);

\[
Aw_{i,j} = y_{i,j}, \quad i \in \{1, 2, \ldots, N\},
\]

(ii) Obtain a set of hyper surfaces \((f_{u,j}, f_{p,j}, f_{d,j}, f_{c,j})\) by substituting the weight values obtained in the above step into equation (16);

\[
f_{u,j}(\alpha_u, \alpha_p, \alpha_d, \alpha_c) = \sum_{i=1}^{N} w_{i,j} \phi_j \left(\left(\alpha^i_u, \alpha^i_p, \alpha^i_d, \alpha^i_c\right) - \left(\alpha^j_u, \alpha^j_p, \alpha^j_d, \alpha^j_c\right)\right),
\]

\[
f_{p,j}(\alpha_u, \alpha_p, \alpha_d, \alpha_c) = \sum_{i=1}^{N} w_{i,j} \phi_j \left(\left(\alpha^i_u, \alpha^i_p, \alpha^i_d, \alpha^i_c\right) - \left(\alpha^j_u, \alpha^j_p, \alpha^j_d, \alpha^j_c\right)\right),
\]

\[
f_{d,j}(\alpha_u, \alpha_p, \alpha_d, \alpha_c) = \sum_{i=1}^{N} w_{i,j} \phi_j \left(\left(\alpha^i_u, \alpha^i_p, \alpha^i_d, \alpha^i_c\right) - \left(\alpha^j_u, \alpha^j_p, \alpha^j_d, \alpha^j_c\right)\right),
\]

\[
f_{c,j}(\alpha_u, \alpha_p, \alpha_d, \alpha_c) = \sum_{i=1}^{N} w_{i,j} \phi_j \left(\left(\alpha^i_u, \alpha^i_p, \alpha^i_d, \alpha^i_c\right) - \left(\alpha^j_u, \alpha^j_p, \alpha^j_d, \alpha^j_c\right)\right),
\]

\[
\text{endfor}\]
Algorithm 4: Online NIROM calculation for compressible fluid and solid interaction and fracture model

(1) Initialisation.
   for \( j = 1 \) to \( m \) do
   | Initialize \( a_{u,0}^0, a_{p,0}^0, a_{d,0}^0 \) and \( a_{c,j}^0 \);
   endfor

(2) Calculate solutions at current time step:
   for \( n = 1 \) to \( T \) do
   for \( j = 1 \) to \( m \) do
   | Solving fluid process:
      | (i) Evaluate the hypersurfaces \( f \) at the previous time step \( n - 1 \) by using the complete set of POD coefficients \( a_{u,j}^{n-1}, a_{p,j}^{n-1}, a_{d,j}^{n-1} \) and \( a_{c,j}^{n-1} \):
      | \[
      f_{u,j} \leftarrow (a_{u,0}^{n-1}, a_{v,0}^{n-1}, a_{d,0}^{n-1}, a_{c,0}^{n-1}), \]
      | \[
      f_{p,j} \leftarrow (a_{u,0}^{n-1}, a_{v,0}^{n-1}, a_{d,0}^{n-1}, a_{c,0}^{n-1}), \]
      | \[
      f_{d,j} \leftarrow (a_{u,0}^{n-1}, a_{v,0}^{n-1}, a_{d,0}^{n-1}, a_{c,0}^{n-1}), \]
      | \[
      f_{c,j} \leftarrow (a_{u,0}^{n-1}, a_{v,0}^{n-1}, a_{d,0}^{n-1}, a_{c,0}^{n-1}), \]
      | (ii) Calculate the POD coefficients \( a_{u,n}, a_{p,n}, a_{d,n} \) and \( a_{c,n} \) at the current time step \( n \) using the following equations:
      | \[
      a_{u,j}^n = \sum_{i=1}^{N} w_i \phi_{i,j}(r), \quad a_{p,j}^n = \sum_{i=1}^{N} w_i \phi_{i,j}(r), \tag{16}
      
      a_{d,j}^n = \sum_{i=1}^{N} w_i \phi_{i,j}(r), \quad a_{c,j}^n = \sum_{i=1}^{N} w_i \phi_{i,j}(r),
      
      \]
   endfor

   Calculate the solution \( u^n, p^n, d^n \) and \( c^n \) on the full space at current time step \( n \) by projecting \( a_{u,j}^n, a_{p,j}^n, a_{d,j}^n \) and \( a_{c,j}^n \) onto the full space.
   \[
   u^n = \sum_{j=1}^{m} a_{u,j}^n \Phi_{u,j}, \quad p^n = \sum_{j=1}^{m} a_{p,j}^n \Phi_{p,j}, \quad d^n = \sum_{j=1}^{m} a_{d,j}^n \Phi_{d,j}, \quad c^n = \sum_{j=1}^{m} a_{c,j}^n \Phi_{c,j},
   \]

   Solving solid-fluid coupling:
   \[
   F_{viscosity} + F_{pressure} = \int_{solid} N_i n \cdot (\sigma_{solid} + I p) d\Gamma,
   \]
   obtain \( s^f = (s_{x,x}^f, s_{x,y}^f, s_{x,z}^f) \) using (10).
   endfor
4. Application to compressible fluid and solid problems

The FSI NIROM has been implemented under the framework of an advanced 3D unstructured mesh multi-phase fluid model (Fluidity) and a combined finite-discrete element solid model (Y2D). The compressible FSI NIROM is validated by a bending beam forced by flows first, then further validated by a more complex case: blasting.

4.1. Case 1: a bending beam forced by flows

The first case is a bending beam forced by flows. In this case, a solid beam is embedded in fluids with subject to a high pressure wave. The domain consists of a rectangle of non-dimensional size of $4 \times 2$ with 7500 nodes. The beam is located at the bottom center and has a size of $0.286 \times 1$. The initial pressure at the area ($0 < x < 1.5$) with a density of 8 is set as 116.5 Pa and the rest of the domain with a density of 1.5 is set as 1 Pa. A slip boundary condition is applied on the left, bottom and the top sides. The momentum out boundary condition is applied on the right side. The density of the solid is 100. The high fidelity full model was simulated during the time period $[0, 0.8]$ with a time step size of $\Delta t = 0.001$. 800 snapshots were taken at a regularly spaced time interval of 0.001.

4.1.1. Results with mean subtraction before performing POD

The FSI NIROM was first formed with mean subtraction before performing POD. In this case, 30 POD basis functions were chosen to form the FSI NIROM. The singular eigenvalues associated to the chosen 30 POD bases are given in figure 1. It can be seen that 30 leading POD basis functions are capable of capturing almost 99.5% of energy in the original dynamic system.

The pressure results from both the Fidelity model and FSI ROM are shown in figure 2. It is illustrated that these FSI NIROM results are not good in comparison with those from the high fidelity model. To further assess the quality of the FSI NIROM with mean subtraction before performing POD, the error analysis is carried out. The root mean square error (RMSE) and correlation coefficient of results between the FSI NIROM and the fidelity model are shown in figure 3. It can be seen that the accuracy of FSI ROM results is low and need to be improved.
Figure 1: case 3: the figure shows the singular eigenvalues in order of decreasing magnitude under the condition that no mean is involved in SVD process.

Figure 2: Bending beam forced by flows case with mean subtraction: Pressure solutions comparison between the full model and FSI NIROM with 30 POD bases at time instances $t = 0.3$ and $t = 0.8$. 
Figure 3: Bending beam forced by flows case with mean subtraction: The correlation coefficient and RMSE of pressure solutions between the high fidelity and FSI NIROM with 30 POD bases.
4.1.2. Results without mean subtraction before performing SVD

In this subsection, the FSI NIROM results without mean subtraction before performing SVD are given and discussed. Figure 4 presents the singular eigenvalues in a decreasing magnitude order. The figure 4 shows that the first 12 singular eigenvalues decrease much more rapidly than other singular eigenvalues. The first 12 POD bases associated with the first 12 singular eigenvalues have already captured 87.59% of the total energy.

Figure 5 shows the pressure solutions comparison between the full model and FSI NIROM with 12, 18 and 30 POD bases at time instances $t = 0.3$ and $t = 0.8$. We can see that the results are much better than those with mean subtraction in figure 2. The results with mean subtraction being in all probability averaged by mean. It also shows that FSI NIROM using 12 POD bases performs well and the results of FSI NIROM are improved by increasing the number of POD bases, especially at the front (see figure 5(a), (c), (e), (g)). In order to see the effects of improvements by increasing number of POD bases, the absolute error of pressure between high fidelity model and FSI NIROM with different number of POD bases at time instances $t = 0.3$ and $t = 0.8$ is given in figure 6. The figure clearly shows that the error of the FSI NIROM relative to the high fidelity model become smaller as a larger number of POD bases is used.

In order to further validate the accuracy of the FSI NIROM without mean subtraction, correlation and RMSE are used, see figure 6. It is shown in this figure that the FSI NIROM is very close to high fidelity model even when only 12 POD bases are used and the error is decreased as the number of POD bases are increased.

![Figure 4: The figure shows the singular eigenvalues in order of decreasing magnitude without mean subtraction before performing SVD.](image-url)
Figure 5: Bending beam forced by flows case without mean subtraction: Pressure solutions comparison between the full model and FSI NIROM with 12, 18 and 30 POD bases at time instances $t = 0.3$ and $t = 0.8$. 

(a) full model, $t = 0.3$  
(b) full model, $t = 0.8$  
(c) SVD without mean subtraction (12 POD bases), $t = 0.3$  
(d) SVD without mean subtraction (12 POD bases), $t = 0.8$  
(e) SVD without mean subtraction (18 POD bases), $t = 0.3$  
(f) SVD without mean subtraction (18 POD bases), $t = 0.8$  
(g) SVD without mean subtraction (30 POD bases), $t = 0.3$  
(h) SVD without mean subtraction (30 POD bases), $t = 0.8$
Figure 6: Bending beam forced by flows case without mean subtraction: Error between the high fidelity model and FSI NIROM with 12, 18 and 30 POD bases at time instances $t = 0.3$ and $t = 0.8$. 
Figure 7: Bending beam forced by flows case without mean subtraction: RMSE and correlation coefficient of pressure solutions between the high fidelity and FSI NIROM with 12, 18 and 30 POD bases.
4.2. Blasting test case

In the second example a blasting-induced fracture test case is resolved. The computational domain is presented in figure 15 which shows a solid square block with a size of $2 \times 2$ m embedded within a compressible gas rectangle area with a size of $3 \times 3$ m. The explosion point lies at the center of the computational domain with a diameter of 0.2 m and a very high initial pressure. The initial high pressure of the explosion point is set to be $10^8$ Pa and the initial high temperature is set to be 1000 Kelvin. The background area (excluding the explosion point) has an initial pressure of 101325 Pa and an initial temperature of 273.26 Kelvin. The viscosity $\mu$ is 0.1 Pa $\cdot$ s. The solid with a density of 2340 kg/m$^3$ has a penalty number of $2.0 \times 10^{10}$ and a Youngs modulus $E$ of $2.66 \times 10^{10}$. The tensile strength and the shear strength are $4 \times 10^6$ Pa and $1.4 \times 10^7$ Pa respectively. The energy decrease rate is 200.

The high fidelity full model was simulated with a finite element mesh of 48600 nodes during the time period $[0, 0.2]$ with a time step size of $\Delta t = 0.00008$. 250 snapshots were taken at a regularly spaced time intervals of $\Delta t = 0.0008$.

4.2.1. Results with mean subtraction before performing SVD

In this section, results from NIROM with mean subtraction before performing the SVD are presented. Figure 8 presents the velocity solutions from the high fidelity full model and FSI NIROM with 100 POD bases at time instances $t = 0.04$ and $t = 0.16$. It is shown that the structure of flows obtained from the FSI NIROM is similar to that from the high fidelity model, but there are some large errors in velocity values. Figure 9 shows the pressure solutions from the high fidelity full model and FSI NIROM with 100 POD bases at time instances $t = 0.04$ and $t = 0.16$. It again shows that the results from FSI NIROM need to be improved.
Figure 8: Blasting case with mean subtraction: Velocity solutions comparison between the full model and FSI NIROM with 100 POD bases at time instances $t = 0.04$ and $t = 0.16$. 

(a) full model, $t = 0.04$
(b) full model, $t = 0.16$
(c) (100 POD bases), $t = 0.04$
(d) (100 POD bases), $t = 0.16$
Figure 9: Blasting case with mean subtraction: Pressure solutions comparison between the full model and FSI NIROM with 100 POD bases at time instances $t = 0.04$ and $t = 0.16$. 

(a) full model, $t = 0.04$

(b) full model, $t = 0.16$

(c) (100 POD bases), $t = 0.04$

(d) (100 POD bases), $t = 0.16$
4.2.2. solutions without mean subtraction before performing SVD

In this section, solutions without mean subtraction before performing SVD are given. Figure 10 shows the singular eigenvalues in order of decreasing magnitude without mean subtraction before performing SVD. In this example the rate of decrease in the singular values is fast, the fifteenth singular value is considerably smaller than the first singular value. According to the decreasing rate of the singular values as shown in figure 10, 6, 12 and 50 POD basis functions were chosen to illustrate the capability of the FSI NIROM.

Figure 11 shows the velocity solutions comparison between the full model and FSI NIROM with 6, 12 and 50 POD bases at time instances $t = 0.04$ and $t = 0.16$. It is evident that the FSI NIROM with only 6 POD basis functions perform well when mean was not subtracted before SVD, even better than solutions from FSI NIROM with 100 POD basis functions when the mean was subtracted before SVD – as shown in figure 8. For velocity solutions, there is no obvious difference between the high fidelity model and FSI NIROMs. The figure 11 also shows that shock front of the blast wave is captured very well by increasing the number of POD bases from 6 to 50. There is no visible difference between the high fidelity model and FSI NIROM with 50 POD bases. In order to see clearly the effects of choosing larger number of POD bases, the errors between the high fidelity model and FSI NIROM with 6, 12 and 50 POD basis functions at time instances $t = 0.04$ and $t = 0.16$ are presented in figure 12. It is evident that greater accuracy is obtained by choosing larger number of POD bases.

Figure 13 presents the pressure solutions comparison between the full model and FSI NIROM with 6, 12 and 50 POD bases at time instances $t = 0.04$ and $t = 0.16$. The pressure solutions from FSI NIROM using 6 POD bases are not as good as velocity solutions from FSI NIROM using the same number of POD bases. In this case, there are visible differences between the high fidelity model and FSI NIROM using 6 and 12 POD bases, which is evident at the time instance $t = 0.16$. The errors between the high fidelity model and FSI NIROM with 6, 12 and 50 POD basis functions at time instances $t = 0.04$ and $t = 0.16$ are plotted in figure 14. It is evident that the error is decreased by choosing more POD basis functions.

In order to assess the performance of the FSI NIROM, velocity solutions obtained from the high fidelity model and FSI NIROMs at a point $(x = 1.5, y = 1.6333)$ near the explosion point over the simulation time period is plotted in figure 15. One reason that we choose the point around the explosion centre is that there is an abrupt change around the explosion point. The figure 15 illustrates that FSI NIROM with less number of POD basis functions perform well when there is no abrupt changes, whereas FSI NIROM with 50 POD basis functions captures the abrupt changes very well.

The accuracy of the FSI NIROM is validated by RMSE and correlation coefficients, which is shown in figure 16. It is shown that the RMSE of pressure results decreases as the number of POD bases increases. The correlation coefficients from FSI NIROMs are over 0.935, indicating that the high fidelity model and FSI NIROMs are highly correlated. The FSI NIROM gets more closer agreement to high fidelity model as the number of POD basis functions increases.
Figure 10: Blasting case: The singular eigenvalues in order of decreasing magnitude without mean subtraction before performing SVD.
Figure 11: Blasting case without mean subtraction: Velocity solutions comparison between the full model and FSI NIROM with 6, 12 and 50 POD bases at time instances $t = 0.04$ and $t = 0.16$. 

(a) full model, $t = 0.04$
(b) full model, $t = 0.16$
(c) (6 POD bases), $t = 0.04$
(d) (6 POD bases), $t = 0.16$
(e) (12 POD bases), $t = 0.04$
(f) (12 POD bases), $t = 0.16$
(g) (50 POD bases), $t = 0.04$
(h) (50 POD bases), $t = 0.16$
Figure 12: Blasting case without mean subtraction: Velocity error between the high fidelity model and FSI NIROM with 6, 12 and 50 POD bases at time instances $t = 0.04$ and $t = 0.16$. 
Figure 13: Blasting case without mean subtraction: pressure solutions comparison between the full model and FSI NIROM with 6, 12 and 50 POD bases at time instances \( t = 0.04 \) and \( t = 0.16 \).
Figure 14: Blasting case without mean subtraction: Pressure error between the high fidelity model and FSI NIROM with 6, 12 and 50 POD bases at time instances $t = 0.04$ and $t = 0.16$. 

(a) error (6 POD bases), $t = 0.04$

(b) error (6 POD bases), $t = 0.16$

(c) error (12 POD bases), $t = 0.04$

(d) error (12 POD bases), $t = 0.16$

(e) error (50 POD bases), $t = 0.04$

(f) error (50 POD bases), $t = 0.16$
Figure 15: Blasting case: Velocity comparison at a point \((x = 1.5, y = 1.6333)\).
Figure 16: blasting case without mean subtraction: The correlation coefficient and RMSE of pressure solutions between the high fidelity and FSI NIROM with 6, 12 and 50 POD bases.
4.3. Efficiency of the NIROM

In this section, we compare the computational cost required for running the high fidelity full model with the online NIROM computation involved in algorithm 4. The simulations were carried out on a 12 cores (Intel(R) Xeon(R) X5680) workstation with 48GB RAM. During the simulations, only one core with 3.3GHz is used. The process of constructing a set of hypersurfaces involved in algorithm 3 is precomputed, therefore, it is not listed in the table. As shown in table 1, the computational time required for running the NIROM is decreased drastically. In blasting test case with 48600 nodes, the NIROM obtained a CPU speed-up of 5 orders of magnitude.

Table 1: Comparison of the online CPU cost (dimensionless) required for running the high fidelity model and NIROM during one time step.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Model</th>
<th>Assembling and Solving</th>
<th>Projection</th>
<th>Interpolation</th>
<th>Total</th>
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<td>0</td>
<td>4.95120</td>
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<td>0.0001</td>
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<tr>
<td>Blasting</td>
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<td>0</td>
<td>0</td>
<td>224.47059</td>
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<tr>
<td></td>
<td>NIROM</td>
<td>0</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.00040</td>
</tr>
</tbody>
</table>

5. Conclusion

A POD-RBF NIROM has been, for the first time, applied to a structure interacting with compressible fluid flows and fracture models and implemented under the framework of a combined finite-discrete element method based solid model (Y2D) and an unstructured mesh finite element model (Fluidity). The model is independent of the governing equations, therefore, it is easy to modify and implement. The performance of the NIROM for compressible fluid/solid interactions and fracture models is numerically illustrated for two test cases: a bending beam forced by flows and a blasting case. The mean subtraction issue before performing POD is discussed by comparison between solutions with and without subtraction. An error analysis was carried out to validate and assess the newly NIROM. The numerical results show that the NIROM performs well and exhibits a good agreement with high fidelity model. The front around the beam is captured well using only a few number of POD bases without mean subtraction beforehand (see figure 5). The computational cost of the NIROM is compared against high fidelity full model. The CPU cost required for NIROM can be reduced by a factor of several orders of magnitude. In addition, the CPU cost is independent of number of nodes on computational mesh which is a particular important factor affecting simulation time of high fidelity full model. Future work includes extending this model to problems with variable material properties.

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References


