Homework 1

Due 08/27/2015

1. In class (see also Example 1.2 in our textbook), we used the forward difference to approximate $f'(x)$, i.e.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \quad \text{where } h > 0.$$

We could have also used the backward difference $\dfrac{f(x) - f(x-h)}{h}$, or - even better - the central difference, which is the average of the two, i.e.

$$f'(x) \approx \frac{1}{2}\left(\frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-h)}{h}\right) = \frac{f(x+h) - f(x-h)}{2h}.$$

(a) Use second order Taylor approximation to derive the absolute error in approximating $f'(x)$ by central difference.

(b) What is the absolute error's rate of convergence as $h \to 0^+$? (Compare with that of Example 1.2).

(c) Use Matlab or Python to make a log-log plot of h vs the computed absolute error (h ranges from 10^{-16} to 1) for approximating $\frac{d}{dx}\sqrt{x}$ at $x = 1$ using both the forward- and central difference approximations. Submit a print out of your plot and submit your code on canvas. *You can refer to the code on the website or in the book on p.8.*

2. Suppose the error $E(h) \to 0$ as $h \to 0^+$, where $h > 0$ is some discretization parameter. If $E(h) = O(h^\alpha)$ as $h \to 0^+$ ($\alpha > 0$), we say the error converges **algebraically/polynomially** at the rate α. In practice, one way of checking whether the error converges algebraically and of estimating the rate α is to plot $\log(h)$ on the x-axis, against $\log(E(h))$ on the y-axis. If for small enough h and some constant $C > 0$,

$$E(h) \approx Ch^\alpha, \text{ then}$$
$$\log(E(h)) \approx \log(Ch^\alpha) = \log(C) + \alpha \log(h).$$

The resulting graph is therefore roughly a straight line and we can then read off the convergence rate α from its slope.

(a) If we're lucky, the error may be **exponential** (some say **geometric**), i.e. $E(h) = O(\beta^{1/h})$, with $0 < \beta < 1$. Devise a similar trick to the one above to check whether $E(h)$ converges geometrically and of estimating β in this case.

(b) The following error expressions $E(n)$ decay either **algebraically, sub-exponentially** (slower than exponential), **exponentially**, or **super-exponentially** (faster than exponential) as $n = 1/h \to \infty$. Here is a useful fact:

$$\lim_{n\to\infty} \frac{E(n)}{E_{ref}(n)} = \begin{cases} \infty & \Rightarrow E(n) \text{ converges slower than } E_{ref}(n) \\ K & \Rightarrow E(n) = O(E_{ref}(n)) \\ 0 & \Rightarrow E(n) = o(E_{ref}(n)) \Rightarrow E(n) = O(E_{ref}(n)) \end{cases},$$

where K is any constant. Use this fact to determine the convergence regime of $E(n)$ (algebraic, sub-exponential, etc..) and the convergence rates when appropriate, i.e. in the case of algebraic or exponential convergence.

i. $E(n) = e^{-n} - \frac{0.01}{n^2}$
ii. $E(n) = n^2 e^{-n}$
iii. $E(n) = 10^{-n^2}$

3. Use the Mean Value Theorem to show that

$$\left|\cos\left(\frac{x_2}{2}\right) - \cos\left(\frac{x_1}{2}\right)\right| \leq \frac{1}{2}|x_2 - x_1|$$

for all $x_1, x_2 \in [0, 2\pi]$ *(Hint: this is a proof by contradiction).*