STATISTICAL ANALYSIS OF THE RATE OF STRAIN TENSOR IN COMPRESSIBLE HOMOGENEOUS TURBULENCE

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ABSTRACT

Recent analysis of direct numerical simulations of compressible homogeneous shear flow turbulence has unraveled some of the energy transfer mechanisms responsible for the decrease of kinetic energy growth when the flow becomes more compressible. In this complementary study, we focus our attention on the rate of strain tensor. A Helmholtz decomposition of the velocity field leads us to consider a solenoidal and an irrotational rate of strain tensor. Their eigenvalue distributions, eigenvector orientations, and the relative alignment between the eigenvectors and the vorticity and pressure gradient vectors are examined with the use of probability density functions. The irrotational rate of strain tensor is found to have a prefered structure in regions of strong dilatation. This structure depends on the mean shear, and is quite different from that of the solenoidal rate of strain tensor. Compressibility strongly affects the orientation properties of the pressure gradient vector.

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1 Introduction

In recent direct numerical simulations of compressible homogeneous shear flow [1, 2], global properties of the flow were examined to determine the effect of compressibility on the growth of kinetic energy and of dissipation. Conclusions of this study indicate that compressibility effects (associated with increased Mach number) decrease the growth rate of the kinetic energy. This reduction in the growth rate was determined in [1] to be due to enhanced compressible dissipation and pressure-dilatation terms, and due to reduced production by the mean shear in the energy equation. The present study emphasizes the properties of the rate of strain tensor in order to provide insight into the structure of the turbulence at the intermediate and small spatial scales.

It is now accepted that homogeneous incompressible turbulence, although stochastic in nature, contains structures which retain their identity over extended time and/or space. These are the so-called coherent structures, and much research is devoted to isolating and understanding their kinematical and dynamical properties. For example, the proper orthogonal decomposition (POD) [3, 4] constructs a set of mutual orthogonal eigenfunctions that maximizes the kinetic energy content of each successive eigenfunction. Simulations using very few of these modes clearly indicate that they can capture the essential features of the flow. Alternatives to the POD have also been proposed. Vorticity eduction methods [5], standard correlation techniques [6], and wavelet packet techniques [7] are but three of them. Yet another approach has been advanced by Ashurst [8] who proposes to describe turbulence as a statistical ensemble of elemental vortices. The geometry of the vortices and the rules that govern their interaction would then be sufficient to reproduce some of the known small-scale properties of the flow.

Within the last decade, detailed analyses of numerically generated databases have revealed new structural properties of homogeneous turbulence, hitherto unsuspected. Most of the work was motivated by the desire to understand the origins of the intermittent dissipative structures. While it has been known for a long time that vortex stretching is a dominant mechanism for the production of small scales [9, 10, 11], the theories were either quite heuristic, or based on global averages. The advent of powerful supercomputers has permitted detailed visualizations of the flowfields, which suggest that the intermittent structures are associated with extremely high vorticity, affect the energy cascade process, and therefore must be represented by good subgrid scale turbulence models. Some attempts in this direction have already been made by Lundgren [12] who proposes the concept of a spiraling vortex. With a minimum number of assumptions, he provides the mathematical framework from which the Kolgomorov power law is deduced.

Along with more detailed information about the intermittent dissipation regions, several apparent paradoxes were uncovered [13, 14, 15, 16]. For example, numerical evidence suggests that the vorticity vector aligns itself primarily along the eigenvector of the rate of strain tensor which corresponds to the intermediate eigenvalue (which has a tendency to be positive). This is contrary to the expectation that the vorticity grows fastest in the direction of maximum strain, and thus should be aligned in the direction associated with the largest eigenvalue. Further surprises result from the presence of tubular, high intensity, vortex structures in the flow within which two of the eigenvalues of the rate of strain tensor S_{ij} are positive [16, 17, 18, 19]. One would normally expect flat vortex structures to be associated with two positive eigenvalues of S_{ij}. In an attempt to explain some of these results, Cantwell [20], Majda [21], Ashurst, Kerstein, Kerr & Gibson [13], and Vieillefosse [22] show that some of these effects can be predicted from the inviscid equations if the anisotropic component of the pressure Hessian $p_{,ij}$ is neglected. In particular, the inviscid equations predict the observed positiveness of two eigenvalues of S_{ij}, as well as the observed alignment of vorticity. Unfortunately, this model does not predict the preferred shape (eigenvalues in the ratio $-4 : 1 : 3$) of the principal ellipsoid of S_{ij} discussed by Ashurst et al. [13]. This leads to the hypothesis that the effect is either related to viscosity, or to some anisotropic property of the pressure Hessian. With a stochastic model for the viscous terms, Girimaji and Pope [23] reproduce many of the statistical properties of the flow invariants. Data has also been obtained on the alignment properties of the pressure gradient with respect to the eigenvectors of S_{ij}. Several researchers agree that ∇p aligns itself in the direction of maximum compressive eigenvector of S_{ij}, although a plausible explanation has not yet been advanced [13, 15].

This intrinsic turbulent structure, so pervasive in incompressible homogeneous isotropic and shear turbulence, is affected by compressibility. Erlebacher, Hussaini, Kreiss & Sarkar [24] showed that for shockless isotropic turbulence, the dilatational component of the velocity ($\mathbf{v}^C$) decouples from the solenoidal component ($\mathbf{v}^I$) to lowest order[1] More precisely, $\mathbf{v}^I$ satisfies the incompressible Navier-Stokes equations [2] which in turn defines an associated incompressible pressure p^I. The remainder fluctuation pressure, p^C, together with $\mathbf{v}^C$ satisfy a set of equations, which only couple to the nonlinear solenoidal flow through terms of order M_t where M_t is the (assumed small) rms turbulent Mach number. This result is obtained through a multiple time-scale analysis [24] in which the phenomena associated with $(\mathbf{v}^C, p^C)$ vary on a $O(M_t)$ time scale, whereas the solenoidal flow evolves on a $O(1)$ time-scale. In homogeneous shear flow, with moderate to high shear rate S, the situation is more complicated since both $\mathbf{v}^I$ and $\mathbf{v}^C$ now vary on the same $O(S^{-1})$ time-scale, and so are expected to interact more strongly with each other than in isotropic flow.

Very little is currently known about the small scale structure in compressible homogeneous turbulence. Blaisdell, Mansour and Reynolds [25] have done some statistical analysis of results from high resolution direct numerical simulations of homogeneous shear layers up to turbulent Mach numbers of 0.4. A significant result is that the vorticity and the dilatation are statistically independent and are therefore decorrelated. This is borne out in our own simulations.

The main objective of this paper is to initiate a systematic investigation of the small spatial structures present in compressible homogeneous turbulence. This is accomplished through an analysis of the rate of strain tensor calculated both from the solenoidal and from the irrotational components of the velocity field. Statistical properties of the tensors' eigenvalues and eigenvectors are presented. We also consider the relative orientations of the pressure gradient, the vorticity and the eigenvectors of the rate of strain tensors, and how

[1] Although not a Helmholtz decomposition, it is well-defined and unique in homogeneous flows.

[2] The solenoidal component of the velocity derived from a Helmholtz decomposition is denoted by $\mathbf{u}^I$.

these are affected by the mean shear rate. At low shear rates, we expect only slight departure from incompressible statistics. However, as the shear rate is increased, both the solenoidal and the irrotational velocity components will vary in a $O(S^{-1})$ time, determined by the mean shear. This leads to possible interaction mechanisms between the two components. The budgets of enstrophy and dilatational variance are also examined to ascertain the influence of compressibility.

2 Equations

The homogeneous shear flow data used for the ensuing analysis is obtained from the direct numerical simulation (DNS) of the full compressible Navier-Stokes equations subject to assumptions of spatial homogeneity. The mean flow is unidirectional in the x direction, and has uniform shear, S, in the vertical direction:

$$U(y) = Sy. \tag{1}$$

The mean temperature and density are uniform. The velocity components are expressed in perturbation form which leads to coefficients in the equations which depend on the normal coordinate y. A time-dependent transformation is applied to the physical coordinates to replace this y dependence by coefficients which depend on time (t). This transformation was first used by Rogallo [26] in DNS, and is now a standard tool to transform some standard homogeneous flows to a form amenable to Fourier spectral methods.

For completeness, we present the non-dimensional Navier-Stokes equations in the form in which they are numerically solved:

$$\partial_t \rho + (\rho u_i)_{,i} - St(\rho u_2)_{,1} = 0 \tag{2}$$

$$\begin{aligned} \partial_t(\rho u_i) + (\rho u_i u_j)_{,j} &= -p_{,i} + \frac{1}{Re}\tau_{ij,j} - S\rho u_2 \delta_{i1} \\ &+ St(\rho u_2 u_i)_{,1} + Stp_{,1}\delta_{i2} - St\tau_{i2,1} \end{aligned} \tag{3}$$

$$\begin{aligned} \partial_t p + u_j p_{,j} + \gamma p u_{j,j} &= Stu_2 p_{,1} + \gamma Stpu_{2,1} + (\gamma - 1)\Phi \\ &+ \frac{1}{PrM_0^2 Re}(T_{,jj} - 2StT_{,12} + S^2t^2T_{,11}) \end{aligned} \tag{4}$$

$$\gamma M_0^2 p = \rho T \tag{5}$$

Derivatives in the above formulas are taken in a frame of reference that is moving with the mean flow. The viscous stress tensor τ_{ij} and the viscous dissipation Φ are defined by

$$\tau_{ij} = \mu(u_{i,j} + u_{j,i}) - \frac{2\mu}{3}(u_{k,k})\delta_{ij} \tag{6}$$

and

$$\Phi = \tau_{ij} u_{i,j} \tag{7}$$

where derivatives are taken in the laboratory frame of reference. The Reynolds number is

$$Re = \frac{u^* \rho^* L^*}{\mu^*} \tag{8}$$

while the reference Mach number is

$$M_0 = \frac{u^*}{\sqrt{\gamma R T^*}} \tag{9}$$

Other variables are the fluctuating velocity u_i, the instantaneous density ρ, the pressure p, the temperature T, and the ratio of specific heats, γ.

Length, velocity, density and temperature are respectively non-dimensionalized by a reference length L^*, a reference velocity u^*, and the initial mean density ρ^*, and the initial mean temperature T^*. The viscosity μ^* is constant, and the Prandtl number is set to 0.7, valid for air.

The coordinate-independence of the flow equations in the (moving) computational domain permits an expansion of the solution in a Fourier series. In the new coordinate system, the flow is can be treated as periodic and evolve within a cubic box of non-dimensional length 2π. The equations are solved using spectral collocation. For the simulations presented in this paper, a 2/3 dealiasing procedure is applied to the primitive variables at every time step. It is also applied before and after the regridding procedure. The scheme is fully explicit, and the spatial operator is advanced in time with a low-storage third-order Runga-Kutta method. Details of the algorithm and of the initialization process are given in [1].

Contrary to the dynamics of decaying compressible isotropic turbulence [24, 27], in which the effects of initial conditions are never quite eradicated, we expect that, with proper scaling, the structure of the turbulent shear flow is intrinsic to the flow, i.e., independent of the initial data. The degree of initial compressibility is $\chi = \overline{(u^C)^2}/\overline{u^2}$ where the superscript C refers to the irrotational component of the velocity obtained from the (unique) Helmholtz decomposition of u_i, and $\overline{u^2} = \overline{u_i u_i} = q^2$. The solenoidal component is referred to by a superscript I. Throughout the text, quantities derived from one or the other of these two velocity fields are denoted by one or the other superscript. Further parametrizations of the initial field include the *rms* fluctuations of two of the three thermodynamic variables. The density and the pressure fluctuations are initialized as described in Sarkar et al. [2]. Note that u_i^I results from a true Helmholtz decomposition and is *not* the solution to the incompressible Navier-Stokes equations (Erlebacher et al. [24]).

Intrinsic parameters useful to characterize the turbulent flow include the Taylor length scale $\lambda_T = u_{rms}/\sqrt{\overline{\omega_i \omega_i}}$, the microscale Reynolds number, $R_{\lambda_T} = (\lambda_T u_{rms})/\nu$, the turbulent Mach number $M_t = \sqrt{u_{rms}^2/(\gamma \overline{p}/\overline{\rho})}$, and the nondimensional mean shear rate $S^* = SK/\epsilon$ where ϵ is the total turbulent dissipation $\overline{\Phi}$, and $K = 2q^2$ is the turbulent kinetic energy. The i^{th} vorticity component is denoted by ω_i. In the following, overbars refer to volume averages.

3 Incompressible Homogeneous Shear Flow

In the past several years, there have been a series of new findings related to the small scale structure of incompressible homogeneous turbulence. These findings are the result of investigations into the origin of intermittency in turbulence. It was found [28, 16, 17, 29] that there are localized regions of very intense vorticity magnitude (more than 4 times the rms value), tubular in shape, with a length to diameter ratio of about 6 [30]. These results have been generally obtained from numerical simulations with $R_{\lambda_T} \approx 100$ which is a microscale Reynolds number far in excess of $R_{\lambda_T} \approx 40$ in our shear flow simulations. This might explain why direct evidence of these tubes has not been found in our numerical data. However, the vorticity does align itself with the intermediate principal direction of S_{ij} when the dissipation is high. It was also discovered that the rate of strain ellipsoid has a very unique shape, with eigenvalues in the ratio (-4:1:3), in strongly dissipative regions [13]. The observed tubular structure of vorticity is counter-intuitive, since two positive eigenvalues imply that vorticity should be stretched in two extensional directions thus leading to sheet-like patterns. A theory to explain this has been advanced by Majda [21]. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be the three eigenvalues of S_{ij}. According to Majda's scenario, the vorticity is initially stretched in the direction of λ_3, the largest positive eigenvalue. The evolution equation for S_{ij}, neglecting the effects of pressure and viscosity then implies that the rate of strain ellipsoid is stretched in the direction perpendicular to λ_3. Thus the eigenvalues λ_1 and λ_2 increase in time at the expense of λ_3. At some point the intermediate and expansive eigenvalues interchange their roles while the vorticity retains its previous orientation, which is now along the intermediate eigenvector. While this explanation is certainly attractive, results by She et al. [15] indicates that while λ_2 increases as predicted by Majda, it never exceeds λ_3. An explanation must be sought elsewhere. A simplified model of the Euler equations explains this alignment; however, the physical interpretation of the results is unclear.

The pressure gradient also shows preferred alignment with the principal directions of S_{ij}. Results of Ashurst et al. [13] indicate that ∇p is aligned with the most compressive principal direction, when the pressure gradient is high.

A better understanding of the small scale dynamics (and its relation to coherent structures) will ultimately lead to better turbulent closures, and perhaps to a description of turbulence with far fewer degrees of freedom than the standard $Re^{9/4}$ scaling. Any hope of successfully modeling compressible turbulence thus hinges, in part, on an understanding of how compressibility affects some of the results found in incompressible flows. Are the alignment properties of $\boldsymbol{\omega}$ and ∇p still present in compressible homogeneous flows? Are these properties statistically the same in regions of expansion and compression? If the flow properties are insensitive to compressibility, there is hope that the methodologies developed for incompressible flows will apply to the compressible cases with only minimal modification.

4 Compressible homogeneous shear flow

In this section, we summarize some of the known results of compressible homogeneous shear flow turbulence that are pertinent to this paper. The only extensive direct numerical simulations of compressible homogeneous shear flow to date are those of Sarkar et al. [1] and Blaisdell et al. [25]. They cover different parameter regimes, and obtain results that are generally consistent with each other. Sarkar et al. [1] perform two series of simulations, differentiated solely by the initial conditions. In the first set of simulations, the initial data is incompressible, i.e. $(\rho_{rms})_0 = 0$, $\chi_0 = 0$ while the initial turbulent Mach number $(M_t)_0$ is varied. In the second class, $(M_t)_0 = 0.3$ while $(\rho_{rms})_0$ and $(\chi_{rms})_0$ are varied. In all these runs, $R_{\lambda_T,0} = 24$, $(SK/\epsilon^I)_0 = 7.2$, $k_0 = 12$ where ϵ^I is the solenoidal dissipation and k_0 is the wavenumber where the energy spectrum is maximum. In all our simulations, the initial energy spectrum has the form

$$E(k) = k^4 e^{-2k^2/k_0^2} \tag{10}$$

The primary result from this work is that increased levels of compressibility, whether measured by an increase in M_t, or by an increase in ρ_{rms} (or χ) decreases the growth rate of kinetic energy.

Sarkar et al. [1, 2] compute the probability distribution functions (pdfs) of velocity, vorticity and dilatation. As expected, the velocities (related to the larger scales) display a Gaussian distribution (characterized by a skewness $\mathcal{S}k \approx 0$, and a flatness $\mathcal{F} \approx 3$). The single exception to this was for the transverse velocity with a slightly higher flatness of 3.3. Results also indicate that the dilatation field is more intermittent than the vorticity field, and that regions of compression are more likely to occur than regions of expansion. This is true even in the absence of shocks. No evidence of shocks were found in these simulations because R_λ and M_t are too low. Blaisdell [25] finds similar results, including at least one flow with shocklets.

5 Methodology for determining statistics

We consider data from DNS on 128^3 grids. A Helmholtz decomposition is applied to the velocity field. In turn, the resulting solenoidal and irrotational velocity vectors are used to compute associated solenoidal and irrotational rate of strain tensors. To simplify the subsequent analysis of the irrotational rate of strain tensor, it is made deviatoric by subtraction of $\Theta\delta_{ij}/3$. The derivatives of the velocity are calculated with a 6^{th} order compact scheme rather than with spectral collocation to reduce internal storage requirements. A description of the algorithm can be found in [31]. Given the components of the rate of strain tensor S_{ij}, we compute its (real) eigenvalues ($\lambda_i, i = 1, 2, 3$) which are ordered from smallest to largest:[3]

$$\lambda_1 \leq \lambda_2 \leq \lambda_3. \tag{11}$$

Along with the eigenvalues, the three principal eigenvectors of S_{ij} are also computed and normalized to unit length. The i^{th} eigenvector is labeled $\mathbf{f}_i$, $i = 1, 2, 3$. The absolute value

[3] Note that many authors order the eigenvalues from largest to smallest.

of the direction cosines is taken before calculating their statistics.

Superscripts C or I appear over variable names depending on whether they are derived from irrotational (compressible) or solenoidal (incompressible) velocity fields. No superscript naturally implies that the full perturbation velocity field is used. For example, $\mathbf{f}_i^C$ refers to the i^{th} eigenvector of the irrotational rate of strain tensor S_{ij}^C.

In what follows, we consider two types of diagnostics. The first type is a simple one-dimensional pdf, sometimes weighted by a scalar. Given the pdf of a 3-D function $g(\mathbf{r})$:

$$\mathcal{P}_g(g_0)dg = \frac{\int_{g_0<g<g_0+dg} g(\mathbf{r})d^3\mathbf{r}}{\int g(\mathbf{r})\, d^3\mathbf{r}} \tag{12}$$

we define $\mathcal{P}_g^w$, the weighted pdf of $g(\mathbf{r})$ with respect to a positive weight function $w(\mathbf{r})$ by

$$\mathcal{P}_g^w(g_0) = \mathcal{P}_{wg}(g_0) \tag{13}$$

As a special case, if the weight function is of the form

$$w(\mathbf{r}) = \delta(g - g_0), \tag{14}$$

the weighted pdf becomes

$$\mathcal{P}_g^w(g) = \delta(g - g_0) \tag{15}$$

which clearly demonstrates that the weighting function biases the probability distribution function towards regions where the weight is strongest.

The second type of diagnostic is a simplified form of conditional averaging. Given two variables $\mathcal{U}_1$ and $\mathcal{U}_2$, we define $\mathrm{Cond}(\mathcal{U}_1,\mathcal{U}_2)$ as a function of $\mathcal{U}_2$. The notation $\mathrm{Cond}(\mathcal{U}_1,\mathcal{U}_2)$ should be interpreted as the average of $\mathcal{U}_1$ over the regions where $\mathcal{U}_2$ lies within a small band $\Delta\mathcal{U}_2$. For all plots, the range of the independent variable is forced to lie within $\pm 5\ rms$ from its mean value. This range is then divided into 25 equal bins which determine the discrete pdf. All pdfs are normalized so that their integral over the independent variable is unity.

To reduce computer memory requirements when performing the diagnostics, we only use sample points from the lower left-hand 96^3 points in the domain. This provides more than 800,000 sample points. In the plots that follow, a lack of sample points is characterized by oscillations in the distributions. To clearly show the lack of sample points (when it occurs), these distributions are not smoothed.

6 Results

We present results from one compressible simulation (R-I) with initial data $S = 20$, $S^* = SK/\epsilon = 3.6$, $M_{t0} = 0.2$, $Re = 150$, $R_{\lambda_T,0} = 13.5$, $k_0 = 18$ and incompressible initial conditions[4] . Most of the plots are shown at $St = 9$ when $R_{\lambda_T} = 23.4$, $M_t = 0.27$, and $S^* = 6.05$. During the course of the simulation, S^* varies from 4.1 at $St = 3$ to 6.7 when

[4] This simulation is different from the ones presented in [1].

$St = 13$. We will have occasion to briefly refer to two other simulations. The first, labeled R-II is an incompressible shear flow simulation with the same initial conditions as R-I. R-II is used in the enstrophy budget calculations to compare against R-I. Finaly, R-III refers to a compressible decaying isotropic simulation with initial conditions $Re = 125$, $R_{\lambda_T,0} = 27.2$.

In this section, we study several properties of the rate of strain tensor to help identify characteristics of the flow which are solely the result of compressibility. First, we investigate the time-dependent budgets of enstrophy and of $\overline{\Theta^2}$. Relative magnitudes of physically relevant flow variables, and growth rates give a global picture of the flow dynamics.

Then we concentrate on the structure of the rate of strain tensor. Since the flow is anisotropic, it is useful to consider characteristics of the flow both with respect to the fixed laboratory coordinate system, and with respect to a coordinate system attached to, and rotating with an individual fluid element. Global orientation properties of S_{ij} are established by studying how the principal directions of S_{ij} are distributed in the laboratory coordinate system. Intrinsic small scale properties, which are perhaps more independent of the large scale conditions, are characterized by considering the relative orientation of vectors such as vorticity and scalar gradients with respect to the principal directions of S_{ij}.

6.1 Enstrophy and Dilatation

While vorticity has long been viewed as a fundamental quantity in incompressible turbulent flows, the flow dilatation is perhaps the single most important measure of compressibility of the velocity field. The properties of the dilatation field must be well understood to extract the relevant leading order compressibility effects. The dilatation, $\Theta = u_{i,i}$, describes to leading order the departure of the flow from an incompressible state. The transport equation for Θ is

$$\frac{D\Theta}{Dt} = (-S_{ij}S_{ji} + \frac{1}{2}\omega^2) - 2S\,S_{12} - S\omega_3 + (\rho^{-1}P_{ij,i})_{,j} \tag{16}$$

where P_{ij} is the combined pressure and viscous stresses,

$$P_{ij} = -p\delta_{ij} + \frac{2}{Re}\overset{\circ}{S}_{ij} \tag{17}$$

and $\overset{\circ}{S}_{ij}$ is the deviatoric part of S_{ij}, where S_{ij} is the rate of strain tensor $(u_{i,j} + u_{j,i})/2$ and $\overset{\circ}{S}_{ij} = S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}$. The time derivative D/Dt represents the material derivative, following a fluid element. The variance of Θ satisfies

$$\begin{array}{ccccccccc} \frac{1}{2}\frac{\partial\overline{\Theta^2}}{\partial t} & = & -\frac{1}{6}\overline{\Theta^3} & -\overline{S^I_{ij}S^I_{ij}\Theta} & -\overline{\overset{\circ}{S}{}^C_{ij}\,\overset{\circ}{S}{}^C_{ij}\Theta} & -2\overline{S^I_{ij}\,\overset{\circ}{S}{}^C_{ij}\Theta} & +\frac{1}{2}\overline{\omega^2\Theta} & -2S\,\overline{\Theta\, S^I_{12}} & -2S\,\overline{\Theta\;\overset{\circ}{S}{}^C_{12}} \\ & & 28(II) & \approx 0(II) & +96(II) & \approx 0(II) & +34(II) & \approx 0(I) & 355(I) \\ & & -S\overline{\omega_3\Theta} & -\overline{\Theta(\rho^{-1}p_{,j})_{,j}} & -\frac{4}{3}\,\nu\,\overline{\Theta_{,j}\Theta_{,j}} & & & & \\ & & 69(I) & 33(III) & -292(IV) & & & & \end{array} \tag{18}$$

The numbers beneath Eq. (18) represent the budget of $\overline{\Theta}^2$ at $St = 9$ obtained from simulation R-I. Note that the thermodynamic term (which involves pressure and density gradients) is on the same order as the nonlinear interaction term between the compressible rate of strain tensor and the dilatation. While the nonlinear term evolves almost monotonically, the thermodynamic term has strong oscillations. The terms in Eq.(18) divide naturally into four categories, labeled I through IV which are respectively: (I) Interaction terms driven by the mean shear, (II) nonlinear interaction terms between the dilatation field and the fluctuation rate of strain tensor, (III) influence of the thermodynamic variables, and (IV) viscous dissipation. From the budget shown above, it is clear that interaction terms between solenoidal and irrotational fields are much smaller than between the irrotational components themselves. To further interpret the budget, we plot the time evolution of the various terms, and the total budget (normalized by $\overline{S\Theta^2/2}$) in Figure 1. Several features are immediately apparent. Both linear (linear in the shear rate) and nonlinear (category II) terms arise from the divergence of the convection term in the dilatation equation. It is clear that the high shear leads to a dominant linear process, whose growth rate steadily decreases in time. This is compensated by a continuous increase in the growth of $\overline{\Theta}^2$ due to the nonlinear terms. By $St = 13$, mean shear effects (I) still dominate. The pressure and density fluctuations also contribute to the growth of dilatation rms, and are of the same order of magnitude as the nonlinear contribution, increasing in time, but subject to large-scale oscillations. Viscosity dissipates the dilatation away at increasing rates. However, the combination of terms generates a positive growth of dilatation, with oscillations of decreasing magnitude. Unfortunately, the data does not extend sufficiently far in time to draw definitive conclusions for $St \to \infty$.

The vorticity satisfies the transport equation

$$\frac{D\omega_i}{Dt} = \omega_j S_{ij} - \Theta(\omega_i - S\delta_{i3}) - S\,S_{3i} + \frac{S}{2}\,(\omega_1\delta_{i2} + \omega_2\delta_{i1}) + \epsilon_{ijk}(\rho^{-1}P_{kp,p})_{,j}. \tag{19}$$

It is obvious that compressibility introduces new terms. The baroclinic torque along with the interaction of dilatation and vorticity are expected to influence the enstrophy generation budget. To ascertain the degree to which this is true, we consider the evolution equation for the turbulent enstrophy:

$$\frac{1}{2}\frac{D\omega^2}{Dt} = (\omega_i S_{ij}\omega_j - \omega^2\Theta) + S(\omega_1\omega_2 + \omega_3\Theta - S_{3j}\omega_j) + \omega_i\epsilon_{ijk}(\rho^{-1}P_{kp,p})_{,j}. \tag{20}$$

The terms on the right-hand side of Eq. (20) represent three different mechanisms of enstrophy variation. The first term generates enstrophy through the stretching of the perturbation vorticity lines. Betchov [10] showed that in incompressible isotropic flow, the spatial average of this term and $\overline{(u_{1,1})^3}$ have the opposite sign. The velocity derivative skewness is defined as

$$\mathcal{S}k = \frac{\overline{u_{1,1}^3}}{\overline{(u_{1,1})^2}^{3/2}}. \tag{21}$$

From experimental measurements, $\mathcal{S}k < 0$, from which one deduces that (at least for isotropic turbulence) the vortex stretching term causes production of enstrophy [10]. The $(-\omega^2\Theta)$ term is a destruction term in expansion regions, and a production term in regions of compression. The second group of terms in brackets on the right-hand side of Eq. (20) represents the generation of enstrophy by interaction of the mean shear with the perturbation flow field. Note the extra effect of dilatation. The third term is the baroclinic torque combined with viscous effects.

Invoking a Helmholtz decomposition of the velocity field, one can separate out the effects of compressibility from those solely due to the intrinsic solenoidal turbulent component of the flow:

$$
\begin{array}{cccccccc}
\frac{1}{2}\frac{\partial\overline{\omega^2}}{\partial t} = & +\overline{\omega_i S^I_{ij}\omega_j} & +\overline{\omega_i \overset{\circ}{S}{}^C_{ij}\omega_j} & -\frac{2}{3}\overline{\omega^2\Theta} & +S\overline{\omega_1\omega_2} & +\frac{2}{3}S\overline{\omega_3\Theta} & -S\,\overline{S^I_{3j}\omega_j} & +S\,\overline{\overset{\circ}{S}{}^C_{3j}\omega_j} \\
 & +4124 & -26 & -45 & +4555 & -46 & 0 & 0 \\
 & +6699 & & & +5301 & & & \\
 & -\overline{\omega_i\epsilon_{ijk}(\rho^{-1}p_{,k})_{,j}} & -\nu\overline{\omega_{i,j}\omega_{i,j}} & & & & & \\
 & 1.1 & -6355 & & & & & \\
 & +0 & -8453 & & & & &
\end{array} \tag{22}
$$

The vortex stretching term has been explicitly rewritten in terms of $\overset{\circ}{S}{}^C_{ij}$ and S^I_{ij}. The first component is now the incompressible vortex stretching. The two sets of numbers below Eq. (22) refer to two simulations. The topmost row of numbers correspond to the previous run discussed (R-I), while the lower set results from an incompressible simulation with identical parameters except that $\rho_{rms} \equiv 0$ and $M_t \equiv 0$ (R-II). The total rate of enstrophy increase is +2,208 for the compressible case, and +3,547 for the incompressible run. Both the production terms and the dissipation terms are larger when the flow is incompressible. Note that compressibility has little direct effect; all the terms that involve the dilatation remain rather small (less than 1% of the total amount of enstrophy generation). However, the compressible terms implicitly decrease the enstrophy growth rate by altering the balance between vortex stretching and viscous destruction. We also found (numerically) that $\overline{S^I_{3j}\omega_j}$ and $\overline{S^C_{3j}\omega_j}$ are identically zero. This was later proven rigorously for general homogeneous flows [32].

The enstrophy budget is plotted in Figure 2. Comparing the overall growth rate (solid curve) to that of $\overline{\Theta^2}$, it is apparent that the enstrophy grows at the slower rate. Extrapolating the curve to $St \to \infty$, the enstrophy growth rate tends to 0.25, which is three times less that what is found for $\overline{\Theta^2}$. There are some notable differences in the dynamical evolution of these two quantities. While the growth of enstrophy is initially dominated by terms linear in S by $St = 9$, the nonlinear terms eventually take over. This is the direct result of vortex stretching, an effect not present in the dilatation equation.

6.2 Correlations

Based on DNS numerical databases, Blaisdell [25] showed that in high shear homogeneous flows, not only are Θ^2 and $\boldsymbol{\omega}^2$ uncorrelated which implies that

$$\overline{\Theta^2\boldsymbol{\omega}^2} = \overline{\Theta^2}\ \overline{\boldsymbol{\omega}^2}, \tag{23}$$

but they are also statistically independent which implies that

$$\mathcal{P}(\Theta^2\boldsymbol{\omega}^2) = \mathcal{P}(\Theta^2)\,\mathcal{P}(\boldsymbol{\omega}^2) \tag{24}$$

where $\mathcal{P}(x)$ is the pdf of the variable x. We investigated the pdf of vorticity conditioned on Θ and found that the distribution of vorticity is approximately independent of whether the fluid is compressing ($\Theta < 0$) or expanding ($\Theta > 0$). This implies that the average of ω^2 within regions whose dilatation is in a specified range is independent of that range. Furthermore, high vorticity and high dilatation regions are distributed independently of each other.

One must be careful when extrapolating small correlations to zero correlations. Unless the correlation is *exactly* zero, Eq. (23) is only approximately satisfied. When two variables $\mathcal{F}$ and $\mathcal{G}$ are found to be nearly uncorrelated, it is useful to consider the fraction

$$\mathcal{C}(\mathcal{F}\mathcal{G}) = \frac{\overline{\mathcal{F}\mathcal{G}}}{\overline{\mathcal{F}}\ \overline{\mathcal{G}}}. \tag{25}$$

This fraction should be very close to unity for uncorrelated variables. We find that $\mathcal{C}(\Theta^2, \omega^2) = 0.99$, so that the the dilatation and the vorticity are quite uncorrelated. Other pairs of variables are found to be uncorrelated, for example $(\omega^2, \overset{\circ}{S}{}^C_{ij}\ \overset{\circ}{S}{}^C_{ij})$ and $(\overset{\circ}{S}{}^C_{ij}\ \overset{\circ}{S}{}^C_{ij}, S^I_{ij}S^I_{ij})$.

The enstrophy correlates rather strongly with $S^I_{ij}S^I_{ij}$. Table 1 lists various pairs of variables, their correlation function

$$\mathrm{Corr}(\mathcal{F}, \mathcal{G}) = \frac{\overline{(f-\overline{f})(g-\overline{g})}}{\sqrt{\overline{(f-\overline{f})^2}\ \overline{(g-\overline{g})^2}}} \tag{26}$$

and the fraction $\mathcal{C}$. The dilatation correlated with any variable constructed from the solenoidal velocity produces correlation coefficients less than 10^{-2}.

$\mathcal{F}$	$\mathcal{G}$	$\mathrm{Corr}(\mathcal{F}, \mathcal{G})$	$\mathcal{C}(\mathcal{F}, \mathcal{G})$
ω_x^2	$S^I_{ij}S^I_{ij}$	0.14	1.2
ω_y^2	$S^I_{ij}S^I_{ij}$	0.48	1.6
ω_z^2	$S^I_{ij}S^I_{ij}$	0.68	1.9
ω_x^2	$\overset{\circ}{S}{}^C_{ij}\ \overset{\circ}{S}{}^C_{ij}$	0.01	1.03
ω^2	$\overset{\circ}{S}{}^C_{ij}\ \overset{\circ}{S}{}^C_{ij}$	0.008	1.01
$S^I_{ij}S^I_{ij}$	$\overset{\circ}{S}{}^C_{ij}\ \overset{\circ}{S}{}^C_{ij}$	—	0.99

Table 1: Correlation factors between several pairs of flow variables.

Note that ω_z correlates strongly with $S^I_{ij}S^I_{ij}$. On the other hand, the streamwise and normal vorticity have a much lower correlation with the solenoidal dissipation. The magnitude of the three vorticity components are within 10% or each other.

6.3 Intrinsic structure of S_{ij}

As mentioned earlier, in regions of high dissipation, the rate of strain tensor has a preferred shape when the flow is homogeneous and incompressible.

The perturbation rate of strain S_{ij} retains its preferred shape in the presence of moderate compressibility and mean shear. To see this, we plot in Figure 3 the pdf of λ_2/λ_1 with and without weighting (unit weight) for the shear flow simulation R-I. In the absence of weighting, the ratio of eigenvalues peaks at $\lambda_2/\lambda_1 = -0.2$. However, when the pdf is weighted with the solenoidal dissipation $\overset{\circ}{S}_{ij} \overset{\circ}{S}_{ij}$, the peak of the distribution is sharper, and has shifted to a ratio closer to $\lambda_2/\lambda_1 = -0.25$, which corresponds to the eigenvalues λ_i in the ratio $(-4 : 1 : 3)$. This result matches the findings of Ashurst et al. [13], even though we considered the eigenvalue statistics based on the total perturbation rate of strain tensor. This eigenvalue ratio is almost constant across the range of mean shear considered in the DNS. The third curve in Figure 3 corresponds to a weighting with respect to the dilatation squared. As expected from the lack of correlation between vorticity and dilatation, there is no substantial difference with the unweighted case.

To clearly separate out the effects of the irrotational rate of strain tensor, we consider next the structure of $\overset{\circ}{S}{}^{C}_{ij}$. We ask whether $\overset{\circ}{S}{}^{C}_{ij}$ also has a preferred shape in regions of high irrotational strain, as measured by the magnitude of $\overset{\circ}{S}{}^{C}_{ij} \overset{\circ}{S}{}^{C}_{ij}$. The results are shown in Figure 4 where the contrast between weighted and unweighted distributions is much more pronounced (simulation R-I). The maximum value of the weighted pdf is more than twice the maximum of the unweighted distribution, thus indicating that regions of strong compressible dissipation impose a structure different than in the rest of the flow. The pdf peaks at an eigenvalue ratio of $(-2.2 : 1 : 1.2)$ which suggests that highly compressible structures (i.e. those structures that are associated with regions of high dilatation) are primarily sheet-like, and parallel to the plane formed by $\mathbf{f}_2^C$ and $\mathbf{f}_3^C$. In a later section, we will consider the orientation of these sheets in the laboratory frame of reference. The weighted pdf has a secondary peak of smaller amplitude at an eigenvalue ratio of $-1 : -0.7 : 1.7$ showing that regions with one large expansion strain are also likely to be associated with high dilatation. Note that weighting of λ_2^C/λ_1^C with respect to either Θ^2 or $\overset{\circ}{S}{}^{C}_{ij} \overset{\circ}{S}{}^{C}_{ij}$ produces nearly identical distributions. Statistics of λ_2^I/λ_1^I are similar to those of λ_2/λ_1, mainly because the irrotational velocity has a much smaller magnitude than the solenoidal component.

We have not been able to establish a definite preferred ratio of λ_2^C/λ_1^C, independent of the mean shear rate, contrary to what is found for λ_2/λ_1. In Figure 5, the pdfs of λ_2^I/λ_1^I corresponding to a compressible, shockless, isotropic simulation ($S = 0$) clearly show that its behavior with respect to weighting by $S^I_{ij}S^I_{ij}$ is identical to that found at $S = 20$ (Figure 3). However, weighting of λ_2^C/λ_1^C with respect to $\overset{\circ}{S}{}^{C}_{ij} \overset{\circ}{S}{}^{C}_{ij}$ (Figure 6) does not significantly affect its pdf when the flow is isotropic. This clearly indicates that there are no particular structures associated with regions of strong dilatation in isotropic turbulence. This is consistent with our view of shockless isotropic turbulence as a collection of randomly oriented “quasi-linear” acoustic waves superimposed on an essentially solenoidal turbulent field. Thus, the mean shear must play an essential role in the creation of quasi-two-dimensional structures in regions of high dilatation in strongly sheared flows. Unfortunately, the shear can influence the statistics either directly (through linear processes), or indirectly, through the pressure or viscous terms. Clearly, further study is necessary to make a definitive choice. Note that for isotropic turbulence, exact solutions to approximations of the incompressible Euler equations have established that the preferred shape of S^I_{ij} is due to either the viscous stress terms, or the properties of the pressure Hessian ($p_{,ij}$) [20].

To help understand the evolution of $\mathring{S}^C_{ij}$, we computed time histories of the unweighted pdfs of both λ_2/λ_1 and λ^C_2/λ^C_1. We found that, after an initial transient ($St < 9$), the pdfs of both variables reach a quasi-steady-state. This indicates that, although the turbulent kinetic energy grows close to exponentially, the small scale structure of the flow maintains some form of self-similarity.

6.4 Extrinsic Orientation of S_{ij}

By definition, in isotropic turbulence, the principal directions of the rate of strain tensor are randomly oriented in the laboratory coordinate system. However, homogeneous shear turbulence is anisotropic, and therefore we expect the rate of strain ellipsoid to have a preferential orientation. This is obvious, since the mean shear itself imposes a constant strain on the flow, oriented at 45° to the x and y coordinate directions.

Consider the angle θ_{ij} between eigenvector $\mathbf{f}_i$ and the unit coordinate axis vector $\mathbf{x}_i$ from which the pdfs of $|\cos\theta_{ij}|$ are computed. The cosine of θ_{ij} is chosen rather than θ_{ij} so that the probability density function of a Gaussian distribution curve is flat. Because the independent variable is the cosine of an angle, one must be careful when interpreting the magnitude of the pdf maxima. A strong peak in the pdf of $|\cos\theta|$ does not necessarily imply a corresponding strong peak in the pdf of θ. The angle at which this peak occurs must also be taken into account. For example, in Figure 7d the pdf reaches values of 1.25 in regions where $|\cos\theta| \approx 0.7$ ($\theta \approx 45°$), while in Figure 9b, it reaches values of 4 for $|\cos\theta| \approx 1.0$ ($\theta \approx 0°$). Since $d\theta = -d(\cos\theta)/\sin(\theta)$, a given $d(\cos\theta)$ corresponds to a much larger angular spread near $\theta = 0°$ than near $\theta = 45°$.

Figures 7a-c show the pdfs of $|\cos\theta_{ij}|$ at $St = 9$. (The first subscript to θ refers to the eigenvector, while the second subscript refers to the coordinate direction; the lack of a subscript I of C on variables implies that the variable is based on the perturbation velocity.) It is evident that the directions of minimum extensional ($i = 1$) and maximum extensional ($i = 3$) strain have qualitatively similar distributions. These two eigenvectors have preferred alignments in the range of 45° to 50° to the x and y axes. However, the 50° orientation with respect to the streamwise direction is strongest. This suggests that the mean strain plays the role of a polarizer which forces these two directions to align with itself. The pdf of $|\cos\theta_{2j}|$ shows a tendency for the intermediate eigenvector $\mathbf{f}_2$ to align at 30° to the x axis. When the pdfs of $\mathbf{f}_j$ are weighted with respect to $S^I_{ij}S^I_{ij}$, $\mathbf{f}_1$ is strongly aligned at 45° to the x and y axes in high dissipation regions. This is illustrated in Figure 7d. Statistics of $\mathbf{f}_3$ are similar to those of $\mathbf{f}_1$ and are not shown. The influence of compressibility on the structure of the rate of strain tensor is most easily brought out by weighting the pdfs of $\mathbf{f}_i$ with Θ^2. Except for the three cases displayed in Figures 8a-c, the dilatation has a small effect on the shape of the pdf. Figure 8 superimposes the pdf's of $|\cos\theta_{ij}|$ corresponding to the total rate of strain tensor without weighting, and with a weight of Θ^2. Whereas the most compressed ($\mathbf{f}_1$) and the most expanded ($\mathbf{f}_3$) eigenvectors have similar orientations in the absence of weighting, or when the weighting is with respect to the solenoidal dissipation, the same is not true in regions of strong dilatation. From Figure 8a, it appears that in regions of strong dilatation, $\mathbf{f}_1$ has a tendency to align normal to the mean flow direction. On the other hand, the third eigenvector ($\mathbf{f}_3$) loses its preferred orientation with respect to the normal direction (y). Together, Figures 8a-b indicate that strong compression and/or expansion occurs primarily along the y coordinate axis. This scenario is also supported by Figure 8c which shows a reduced tendency for alignment of the most compressed eigenvector with respect to the spanwise direction, in favor of the y direction. We also found that the dilatation does not affect the pdf's of the solenoidal component, which is consistent with a lack of correlation between solenoidal and irrotational rate of strain tensors.

The strong influence of the dilatation on the orientation of the certain principal directions of S_{ij} is more easily seen by considering statistics based on the irrotational rate of strain tensor $\mathring{S}^{C}_{ij}$. The pdfs of $|\cos\theta^C_{ij}|$ in Figures 9a-c are qualitatively different from those of $|\cos\theta_{ij}|$. Figures 9 and 10 compare the pdf's of $|\cos\theta^C_{ij}|$ without and with the weighting function Θ^2. The greatest effect of strong dilatation is to greatly increase the probability that the most compressive eigenvector ($\mathbf{f}^C_1$) is aligned with the normal direction (y). This confirms the trends discussed in the previous paragraph. The intermediate eigenvector ($\mathbf{f}^C_2$) no longer has a bipolar orientation in either the x or z directions (Figure 9b,10b), but rather, is aligned in the spanwise direction (z). Alongside with the shift of alignment of $\mathbf{f}^C_2$, the most expansive eigenvector $\mathbf{f}^C_3$ aligns itself either along the streamwise (x) or the normal (y) direction (Figure 10c). However, a close comparison between the unweighted pdfs (Figure 9c) and the weighted pdfs (Figure 10c) reveals that strong dilatation strongly increases the probability that $\mathbf{f}^C_3$ is aligned in the streamwise direction, while the alignment with the y axis is independent of dilatation. Therefore, in light of our earlier finding that the turbulent structures in regions of strong dilatation are oriented in the $\mathbf{f}^C_2 - \mathbf{f}^C_3$ plane, we deduce that the sheets are oriented along the $x-z$ plane. A later analysis of the pressure gradient orientation will show that this sheet is formed by compression along the y coordinate direction.

The above statistical analysis has been repeated using data from several turbulent simulations at $M_t = 0.2, 0.4$, and shear levels of $S = 10, 20, 30$; the general conclusions remain unchanged. The only noticeable effect of weighting the pdfs of $|\cos\theta^C_{ij}|$ with $\mathring{S}^{C}_{ij}\,\mathring{S}^{C}_{ij}$ is to sharpen the peaks of the pdfs. The qualitative features of the distributions are not affected by the weighting. Thus, $\mathring{S}^{C}_{ij}\,\mathring{S}^{C}_{ij}$ and Θ^2 affect the orientation of the irrotational rate of strain tensor differently, although their effect on the pdf of λ^C_2/λ^C_1 is similar.

In summary, the most extensional and most compressive principal directions of S^I_{ij} tend to lie between the coordinate directions, while the principal directions of $\mathring{S}^{C}_{ij}$ tend to align along the coordinate directions.

Why are the properties of $\mathring{S}^{C}_{ij}$ so different from those of S^I_{ij}? Although the dynamics of S^I_{ij} are somewhat understood, this is not the case for $\mathring{S}^{C}_{ij}$. When the flow is incompressible, the vortex stretching term in the enstrophy transport equation is $\omega_i S^I_{ij} \omega_j$. The vorticity stretching is initially in the direction of the maximum mean strain, i.e. at 45° to the streamwise direction. This in turn initially aligns the extensional eigenvector of S^I_{ij} in that direction, so that $\mathbf{f}^I_3$ is aligned with $\boldsymbol{\omega}$. However, the change in u^I_i induces a change in the vorticity, whose interaction with S^I_{ij} serves to reorient S^I_{ij} relative to $\boldsymbol{\omega}$. This process is described by the analytic formulations of Vieillefosse [22], Majda [21], and Cantwell [20], which predict that the vorticity eventually aligns itself with $\mathbf{f}^I_2$. The evolution of $\mathring{S}^C_{ij}$, on the other hand, is driven by one of two major effects. Either the structure of S^I_{ij} acts as a source term or as spatially dependent coefficients in equations linear in $\mathring{S}^C_{ij}$, or else the mean shear is primarily responsible for the structure of $\mathring{S}^C_{ij}$. A more detailed set of diagnostic tools must first be developed before a clear cut choice between these two scenarios is established.

The fine-scale orientation properties of the turbulence tend to equilibrate. This is seen from Figure 11 which shows the time evolution of the distribution function of the most compressive principal direction ($\mathbf{f}_1$) orientation with respect to the x coordinate axis. Past $St = 7$, an equilibrium structure has been established. Of course some of the $|\cos\theta_{ij}|$ equilibrate faster than others. For example, the distribution of the orientation of $\mathbf{f}_1$ with respect to the normal direction (Figure 12) and that of $\mathbf{f}_2$ with respect to the spanwise direction (Figure 13) are still evolving, albeit slowly, by $St = 15$.

6.5 Orientation of the vorticity vector

As expected from previous investigations [13, 15], the vorticity orients itself along the intermediate principal direction of the rate of strain tensor ($\mathbf{f}_2$) in regions of high solenoidal dissipation. If the angle between $\boldsymbol{\omega}$ and $\mathbf{f}_k$ is denoted by $\theta^{\omega}_{f_k}$, this trend is clearly seen by conditioning $\cos^2\theta^{\omega}_{f_k}$ on $\mathring{S}_{ij}\,\mathring{S}_{ij}$ to obtain the distribution $\mathrm{Cond}(\cos^2\theta^{\omega}_{f_k}, \mathring{S}_{ij}\,\mathring{S}_{ij})$ (Figure 14). In regions of high dissipation, $\boldsymbol{\omega}$ aligns itself strongly in the direction of $\mathbf{f}_2$, and normal to the other two principal directions, $\mathbf{f}_1$ and $\mathbf{f}_3$ as reported by Ashurst et al. [13] and She et al. [15]. The vorticity is aligned at and above 45° to the x axis. However, results from the equivalent incompressible simulation RII show the vorticity vector aligned at around 35° to the x axis, in qualitative agreement with the results of Rogers and Moin [33]. They find that as time progresses, the turbulent enstrophy increases, and the angle that the vorticity forms with the streamwise direction decreases from 45° to approximately 30°. However, the vorticity vector remains close to 45° in the compressible simulation R-I.

The compressible dissipation has no influence on the relative orientation of $\boldsymbol{\omega}$ and $\mathbf{f}^I_k$. Indeed, the distribution of $\mathrm{Cond}(|\cos\theta^{\omega}_{f_k}|, \Theta)$ is almost completely flat (not shown here). Again, this is consistent with a lack of correlation between the dilatation and the vorticity.

6.6 Orientation of the pressure gradient vector

Also of interest are the properties of the pressure gradient. Referring back to Eq. (16) for the transport of Θ and its higher moments, it is clear that the properties of the pressure gradient and pressure Hessian must be understood before there is any hope of correctly modeling the subtle relationships between dilatation and vorticity. As shown in Erlebacher et al. [24], the total pressure (in isotropic turbulent flows) can be thought of as a combination of an elliptical (incompressible) component p^I which together with $\mathbf{v}^I$ satisfy the incompressible Navier-Stokes equations, while the difference between the pressure and p^I, i.e. the compressible pressure p^C, is representative of compressible effects. An alternate decomposition of the pressure field is considered in Sarkar [34], in which the incompressible pressure retains all the effects which involve the mean density. The latter generalization of the pressure decomposition is found useful for the analysis of inhomogeneous turbulent flows. Both decompositions have similar characteristics regarding the properties of the pressure components. The incompressible pressure is elliptic, i.e. information in one part of the flow is immediately transmitted throughout the flowfield. In contrast, the compressible pressure, p^C is mostly hyperbolic, i.e. the pressure and dilatational fields are responsible for waves traveling at the speed of sound within the flow. It has a small elliptic component which serves to cancel the ellipticity of p^I; since the total pressure has a finite propagation speed, it does not act on the whole flowfield instantaneously.

We begin by considering the alignment properties of ∇p both with respect to the principal directions of the rate of strain tensor $\mathbf{f}_i$ and with respect to the coordinate directions (Figures 15-17). The angle between the pressure gradient and the i^{th} eigenvector is denoted by $\theta_{f_i}^{\nabla p}$, while the orientation of ∇p with respect to the i^{th} coordinate direction is labeled $\theta_i^{\nabla p}$. From Figure 15, we observe that there is a preferred orientation of ∇p at about $40°$ to both $\mathbf{f}_1$ and $\mathbf{f}_3$. Although the alignment is strongest in the most compressive direction ($\mathbf{f}_1$), there is also a relatively strong alignment along the spanwise direction (z), seen in Figure 17b.

Ashurst et al. [35] find that in incompressible homogeneous shear flow turbulence, the pressure gradient tends to align along the most compressive strain direction, $\mathbf{f}_1$, and very weakly along the most expansive direction, $\mathbf{f}_3$. In light of their result, we have considered the solenoidal part of the pressure gradient (∇p^I) instead of ∇p. However, the result was identical to that using the full pressure field.

Although ∇p and ∇p^I are similarly oriented with respect to the eigenvectors of S_{ij}, the magnitudes of ∇p and ∇p^I behave differently. To bring this out, we compute the conditional average of ∇p and ∇p^I with respect to their alignment with $\mathbf{f}_i$. Specifically, we plot $\mathrm{Cond}((\nabla p)^2, |\cos\theta_{f_i}^{\nabla p}|)$ and $\mathrm{Cond}((\nabla p^I)^2, |\cos\theta_{f_i}^{\nabla p^I}|)$ in Figures 16a-b. The most striking feature of Figure 16 is that $(\nabla p)^2$ is maximum when the pressure gradient is aligned with the most compressive eigenvector of S_{ij} ($\mathbf{f}_1$), while the solenoidal pressure gradient magnitude peaks in a plane perpendicular to the intermediate eigenvector, in a direction approximately $45°$ to both $\mathbf{f}_1$ and $\mathbf{f}_3$. Figure 16b has the same features found by Ashurst et al. [35]. The absolute orientation of ∇p^I in the laboratory frame of reference is shown in Figure 17b as an unweighted pdf of $|\cos\theta_i^{\nabla p^I}|$. It is evident that the solenoidal pressure gradient (∇p^I) is tilted in the spanwise direction. Combining this with the earlier observation that $\mathbf{f}_1$ and $\mathbf{f}_3$ are preferentially aligned at $45°$ to the x axis (Figure 7), we conclude that $\mathbf{f}_1$ and $\mathbf{f}_3$ do not lie in the $x-y$ plane (since that would contradict Figure 16b), but instead form approximately equal angles with the 3 coordinate direction, with a $45°$ tilt with respect to the x coordinate axis.

Returning to the full pressure, we find that ∇p is oriented primarily along the normal (y) coordinate direction (Figure 17a). Earlier, we observed that the compression of the flow is strongest along y, which naturally suggests that this compression and the pressure gradient form an oscillatory system. It is interesting to note that the pressure is dominated by the compressible component. At $St = 9$, p^I and p^C have respective rms values of 2.6 and 4.2. This difference in magnitude is amplified when the pressure gradients are considered. While the three components of ∇p^I have approximately equal rms values of 11.1, 13.7 and 15.5 in the x, y and z directions, the compressible pressure gradient is much more anisotropic with rms values of 15.7, 54.8 and 21.8 along the three coordinate axes. The dominance of ∇p^C along y had already been observed from the various distribution functions. In contrast, the irrotational velocity components are about 5-10 times smaller in magnitude than their solenoidal counterparts. For comparison purposes, the rms values of the three components of u_i^I are $(1.9, 0.75, 1.1)$ while that of u_i^C are $(0.18, 0.4, 0.15)$. Note the dominance of the normal component of u^C over the other two components.

Finally, the influence of the dilatation on the orientation of the pressure gradient with respect to $\mathbf{f}_i$ is shown in Figure 18 where we plot $\mathrm{Cond}(\cos^2 \theta_{f_i}^{\nabla p^C}, \Theta)$. In regions of expansion ($\Theta > 0$), the pressure gradient is aligned with the most expansive principal direction of $\mathring{S}_{ij}$, while in regions of compression ($\Theta < 0$), the alignment is along its most compressive direction ($\mathbf{f}_1$). Note that we had previously found that $\mathbf{f}_1$ was aligned along y when in regions of strong dilatation. Therefore, we find that when the dilatation is high, $\mathbf{f}_1$ is aligned along the normal direction only when the flow is compressing. The intermediate direction shows no preferred alignment.

7 SUMMARY

The purpose of this work was to determine the leading order effects of compressibility on the small spatial scales in homogeneous compressible shear flow turbulence. To this end, the logical candidates to study are the rate of strain tensor, the vorticity, the dilatation and the pressure gradient. In addition, the Helmholtz decomposition was used to further separate out differences between the solenoidal and the irrotational components of the velocity and associated pressure fields.

The data analysis was primarily based on the properties of the eigenvectors and eigenvalues of the rate of strain tensor. From a statistical analysis of the eigenvalues and eigenvector orientations of S_{ij}, we deduced information on the small-scale structure of turbulence in regions of strong dissipation. By weighing the various distribution functions with respect to the dissipation associated with the irrotational velocity field, we deduced the presence of structures whose shapes and orientations were quite different from those found in incompressible flows.

We studied the budget of the transport equations for turbulent enstrophy and the mean square dilatation in homogeneous shear flows. From two shear flow simulations with identical initial conditions, one compressible and the other incompressible, we found that the influence of the explicit compressibility terms is negligible, and that differences in overall growth were due to the altered balance between nonlinear and dissipative effects in the two cases. Both the dissipation of turbulent enstrophy and the nonlinear interaction terms are larger when the flow is incompressible. Furthermore, the total enstrophy growth rate is larger for the incompressible case. While the linear terms are responsible for the growth of enstrophy at early time, the nonlinear terms eventually take over. In the absence of shocks, the baroclinic torque has a negligible influence on the enstrophy budget.

On the other hand, the terms linear in the mean shear dominated the growth of Θ^2 for all time, although the long-time asymptotic state is not clear. Oscillations in the total growth rate of Θ^2 are primarily due to the coupling between the pressure and density gradients which are correlated since the flow is nearly polytropic.

We have confirmed results by Blaisdell et al. [25] that the vorticity and the dilatation are uncorrelated and statistically independent. More generally, there is very little correlation between pairs of variables when one is based on the gradient of solenoidal velocity components, and the other is based on the gradient of irrotational velocity components.

From the statistics of the eigenvalue ratio λ_2/λ_1 (intermediate to compressive), we conclude that the eigenvalues of $\overset{\circ}{S}_{ij}$ are close to the preferred ratio of $-4 : 1 : 3$ discussed by Ashurst et al. [13], but that the eigenvalues of the irrotational rate of strain tensor has a different structure. In fact, the pdf of λ_2^C/λ_1^C indicates that in regions of strong dilatation, the flow forms sheet-like structure in the $x - z$ plane. The strain rates are extensional in the plane of the sheet and strongly compressive normal to the sheet. These sheets are absent from the isotropic compressible turbulence databases we looked at, which indicates the leading role played by the mean shear. Note that the non-zero extensional strain in the $x - z$ plane could be an artifact of considering the deviatoric rate of strain tensor. For example, a simple planar shock has a nonzero rate of strain in only one direction: normal to itself. However, the deviatoric rate of strain tensor will show a structure similar to what we find in our simulations. A more careful study of isolated structures in regions of high dilatation is clearly indicated.

The anisotropy of a homogeneous shear flow leads to preferred orientations of the eigenvectors of S_{ij}^I and $\overset{\circ}{S}{}_{ij}^C$ in a laboratory frame of reference. The principal directions of S_{ij}^I tend to orient themselves at angles of $30° - 50°$ to each of the three coordinate directions (with the strongest alignments with respect to the x axis), while the eigenvectors of $\overset{\circ}{S}{}_{ij}^C$ are primarily aligned along the axes of laboratory coordinate system. Histograms taken at different points in time show that they attain a steady distribution.

The orientation of these "dilatational" sheets in the laboratory frame is determined by examining the structure of the irrotational rate of strain tensor with and without dilatational weighing. We find that, in regions of strong dilatation, the most compressive eigenvector $\mathbf{f}_1^C$ is aligned along y, $\mathbf{f}_2^C$ along z or x, and $\mathbf{f}_3^C$ along y and x. However, the alignment of $\mathbf{f}_3$ with y is independent of the dilatation, so clearly, the orientation of the dilatational sheets is in the $x - z$ plane.

These results are further corroborated by an examination of the orientation of the pressure gradient. In the presence of strong mean shear, the pressure gradient statistics are dominated by the compressible component, even in the absence of weighing. Its orientation with respect to the eigenvectors of S_{ij} strongly depends on the intensity of the dilatation. If the dilatation is close to zero, ∇p^C shows no preferred alignment. However, in regions of strong expansion, the compressible pressure gradient is aligned along the most expansive direction of S_{ij}. When the dilatation is strongly compressive, the alignment of ∇p^C is along the most compressive eigenvector. The solenoidal pressure gradient on the other hand, shows a strong alignment in the spanwise direction. The different alignment properties of ∇p and ∇p^I is heuristically understood once we recognize that the wavelike compressible pressure dominates the pressure field. On the other hand, p^I is the solution to a Poisson equation, and is primarily elliptical. Finally, the pressure gradient shows a strong tendency to align in a direction normal to the mean flow, i.e. the y direction, which is consistent with the properties of $\overset{\circ}{S}{}_{ij}^{C}$ mentioned above.

The results presented in this paper should motivate the development of a theoretical model to explain the observed trends. This will be the object of future research.

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9 Figures

Figure 1: Budget of the evolution equation for Θ^2 (Eq. (18)) as a function of time, normalized by $S\overline{\Theta^2}$. The terms have been grouped into linear, nonlinear, viscous and thermodynamic contributions.

Figure 2: Budget of the evolution equation for enstrophy (Eq. (22)) as a function of time. Terms in the equation are normalized with respect to $S\overline{\omega^2}$. The terms have been grouped into linear, nonlinear, viscous and thermodynamic contributions. The lines correspond to the compressible run (R-I), while the symbols are based on the incompressible simulation (R-II).

Figure 3: Pdf of λ_2/λ_1, the ratio of the intermediate to the minimum eigenvalue of the rate of strain tensor (S_{ij}). The three curves correspond to the pdf weighed with unity, $\overset{\circ}{S}_{ij}\ \overset{\circ}{S}_{ij}$, and Θ^2. Case R-I.

Figure 4: Pdf of λ_2^C/λ_1^C, the ratio of the intermediate to the minimum eigenvalue of the irrotational rate of strain tensor ($\overset{\circ}{S}{}_{ij}^C$). The three curves correspond to the pdf weighed with unity, $\overset{\circ}{S}{}_{ij}^C\ \overset{\circ}{S}{}_{ij}^C$, and Θ^2. Case R-I.

Figure 5: Pdf of λ_2^I/λ_1^I, the ratio of the intermediate to the minimum eigenvalue of the solenoidal rate of strain tensor (S_{ij}^I). The data is based on a compressible isotropic simulation (R-III). The pdf are weighed with unity (solid line), and $S_{ij}^I S_{ij}^I$.

Figure 6: Pdf of λ_2^C/λ_1^C, the ratio of the intermediate to the minimum eigenvalue of the irrotational rate of strain tensor (S_{ij}^C). The data is based on an isotropic simulation. The pdf are weighed with unity (solid line), and $\overset{\circ}{S}{}_{ij}^C\ \overset{\circ}{S}{}_{ij}^C$. Initial conditions are $S = 0$, $Re = 150$.

Figure 7: Pdf of the orientation of the eigenvectors of S_{ij} with respect to the three coordinate axes. $\cos\theta_{ij}$ refers to the orientation of the i^{th} eigenvector of S_{ij} with respect to the j^{th} coordinate direction. (a) Most compressive eigenvector $\mathbf{f}_1$, (b) intermediate eigenvector $\mathbf{f}_2$, and (c) most expansive eigenvector $\mathbf{f}_3$, (d) pdf of $\mathbf{f}_1$ weighted with $S_{ij}S_{ij}$. The pdfs in (a)-(c) are unweighted.

Figure 8: Pdf of the orientation of selected eigenvectors of S_{ij} with respect to a subset of the coordinate axes. Each pdf has been computed with unit weight (solid line), and a weight equal to Θ^2 (dotted line). (a) Orientation of $\mathbf{f}_1$ with respect to y, (b) orientation of $\mathbf{f}_3$ with respect to y, and (c) orientation of $\mathbf{f}_1$ with respect to z.

Figure 9: Unweighted pdfs of the orientation of the eigenvectors of S_{ij}^C with respect to the three coordinate axes. $\cos\theta_{ij}^C$ refers to the orientation of the i^{th} eigenvector with respect to the j^{th} coordinate direction. (a) Most compressive eigenvector, (b) intermediate eigenvector, and (c) most expansive eigenvector.

Figure 10: Pdfs of the orientation of the eigenvectors of S_{ij}^C with respect to the three coordinate directions. $\cos\theta_{ij}^C$ refers to the orientation of the i^{th} eigenvector with respect to the j^{th} coordinate direction. (a) Most compressive eigenvector, (b) intermediate eigenvector, and (c) most expansive eigenvector. The pdfs are weighted with Θ^2.

Figure 11: Unweighted pdf of the orientation of $\mathbf{f}_1$ with respect to the streamwise direction (x) at $St = (7, 9, 11, 13, 15)$.

Figure 12: Unweighted pdf of the orientation of $\mathbf{f}_2$ with respect to the normal direction (y) at $St = (7, 9, 11, 13, 15)$.

Figure 13: Unweighted pdf of the orientation of $\mathbf{f}_2$ with respect to the spanwise direction (z) at $St = (7, 9, 11, 13, 15)$.

Figure 14: Average of the angle between the vorticity and the eigenvectors of S_{ij}, conditioned on normalized $\mathring{S}_{ij}\ \mathring{S}_{ij}$. This curve is referred to in the text as $\mathrm{Cond}(\cos^2\theta^{\omega}_{f_k}, \mathring{S}_{ij}\ \mathring{S}_{ij})$.

Figure 15: Unweighted pdf of the orientation of the pressure gradient ∇p with respect to the eigenvectors of S_{ij}.

Figure 16: Average of $(\nabla p)^2$ conditioned on the orientation of the pressure gradient with respect to the eigenvectors of S_{ij}. (a) Based on the total pressure p, (b) based on the solenoidal pressure p^I.

Figure 17: Unweighted pdf of the orientation of the pressure gradient with respect to the three coordinate directions. (a) Based on the total pressure p, (b) based on the solenoidal pressure p^I.

Figure 18: Average of the orientation of the irrotational pressure gradient (∇p^C) with respect to the eigenvectors of S_{ij} conditioned on the normalized dilatation (Θ). This curve is referred to in the text as $\mathrm{Cond}(\cos^2\theta^{\nabla p^C}_{f_i}, \theta)$.