# *A Year Length Hidden in Ancient Planetary Mean Motions*

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The mean motions in anomaly for the five planets as determined in the *Almagest* are all too small. The differences (real minus *Almagest*) in units of degrees per day are

| | real – *Alm* | *Almagest* |
|---|---|---|
| Saturn | 0;0,0,0,36 $^{0/d}$ | 0;57,7,43,42 |
| Jupiter | 0;0,0,1,5 | 0;54,9,2,46 |
| Mars | 0;0,0,0,9 | 0;27,41,40,19 |
| Venus | 0;0,0,2,44 | 0;36,59,25,53 |
| Mercury | 0;0,0,6,37 | 3;6,24,7,0 |

In the *Almagest* the mean motion in anomaly for the outer planets is defined as the mean motion of the Sun minus the mean motion in longitude of the planet, both relative to the same fixed direction. For the inner planets it is the mean motion of the planet on the epicycle with respect to the epicycle apogee, or alternatively, the mean motion of the planet on the epicycle with respect to a fixed direction minus the mean motion of the Sun relative to the same fixed direction. Since the position in anomaly is the difference of two longitudes, precession plays no direct role.

The actual numerical values given in *Almagest* IX 3 are known, despite some confusing remarks by Ptolemy, to be derived from the ratio of numbers that result from simple period relations with fractional corrections.[1] However, in *Almagest* IX 10, X 4, X 9, XI 3, and XI 7 Ptolemy purports to derive the mean motions in anomaly empirically by giving for each planet a pair of observations, one modern and one ancient, separated in time by about 400 years. The mean motion in anomaly is computed directly from the total motion in anomaly accomplished in the total number of days between the two observations of each planet. In order to determine the position in anomaly of a planet on a given date it is necessary to know, among a number of other items, the mean motion of the Sun, which is of course directly related to the length of the solar year. As is often the case in the *Almagest*, Ptolemy's determinations of these mean motions are carefully arranged to produce a targeted result,[2] and as we shall now see, the values he was targeting appear to have been derived in an earlier analysis using a year length that was different from the *Almagest* year length.

All of the ancient observations that are used in the anomaly determinations are sidereal in nature, i.e. the position of the planet is given with respect to some star. Ptolemy reduces these positions to tropical coordinates using his precession constant of 1° per 100 years, but for our purposes it is simpler to work in sidereal coordinates and to consider how the determination of the mean motion in anomaly is affected by changes in the assumed length of the sidereal year.

Variation of the year length affects the determinations of the mean motions in anomaly for the outer and inner planets in different ways. If the assumed sidereal year length is too long, compared to reality, then the mean motion of the Sun is too slow. For the outer planets, the motion of the planet around the epicycle is, in fact, the motion of the Sun, and so for a given number of elapsed days between the ancient and modern observations the distance traveled on the epicycle will be too short, and the inferred mean motion in anomaly will be too slow. For Saturn, Jupiter, and Mars, and assuming Ptolemy's sidereal year length of about $365;15,24,31,32^d \simeq 365 + \frac{1}{4} + \frac{1}{147}^d$, the shortfall in mean longitude of the Sun will be about 0.174°, 0.181°, and 0.196°, respectively, in the number of days Ptolemy gives for each planet, compared to what the Sun's mean longitude would be had he used an accurate sidereal year length of about $365;15,22,47,5^d \simeq 365 + \frac{1}{4} + \frac{1}{159}^d$. Repeating Ptolemy's reductions of the ancient observations in *Almagest* X 9, XI 3, and XI 7 using mean longitudes of the Sun reduced by those amounts reduces the distance traveled in anomaly by 0.155°, 0.197°, and 0.223°, respectively, and changes the resulting mean motions in anomaly by the following amounts:

| | Implied error | *Almagest* |
|---|---|---|
| Saturn | 0.155°/133,079.75$^d$ = 0;0,0,0,15 $^{\circ/d}$ | 0;0,0,0,36 $^{\circ/d}$ |
| Jupiter | 0.197°/137,733.00$^d$ = 0;0,0,0,19 | 0;0,0,1,5 |
| Mars | 0.223°/149,881.67$^d$ = 0;0,0,0,19 | 0;0,0,0,9 |

These are qualitatively comparable to the error pattern in the *Almagest* values, but they are on average too small, suggesting that the underlying sidereal year actually used in the reduction of the observations was longer than Ptolemy's. We can estimate the underlying year length by simply adjusting it to get the best overall agreement between the errors given in the *Almagest* and the model errors that result using the adjusted year length. The result is that if the original analysis of the outer planets used the same sidereal year length for all the planets, then most likely it fell within the range 365;15,24$^d$ – 365;15,29$^d$.

For the inner planets the effect of a slow mean Sun is to move the position of the epicycle itself. If the planet is between greatest evening elongation and greatest morning elongation, i.e. broadly in the vicinity of the perigee of the epicycle, then the effect is once again that the inferred mean motion in anomaly is too small, but if the planet is outside this range, i.e. broadly on the superior part of the epicycle, then the inferred mean motion in anomaly will be too large and the sign of the error as defined here will be negative. Assuming Ptolemy's sidereal year the shortfall in the longitude of the mean Sun will be about 0.19° for Mercury and Venus, and using Ptolemy's ancient observations as found in *Almagest* IX 10 and X 4 shows that the changes in anomaly are about 0.56° for Mercury and –0.66° for Venus (the negative sign resulting from the fact that Venus is past greatest morning elongation). Since, however, the mean motion in anomaly for Venus as given in the *Almagest* is in fact slower than reality, it is perhaps appropriate to repeat Ptolemy's analysis using an observation of Venus before greatest morning elongation. An appropriate choice would be the occultation of λ Gem by Venus on –271 Jul 25, some 79 days earlier than Timocharis' observation as quoted by Ptolemy. For this observation there is the same shortfall in the longitude of the mean Sun of 0.19°. and now this results in a change in mean anomaly of 0.25°. These change the mean motions in anomaly as follows:

| | Implied error | *Almagest* |
|---|---|---|
| Mercury | $0.557°/147{,}013.56^d = 0;0,0,0,49^{°/d}$ | $0;0,0,6,37^{°/d}$ |
| Venus | $-0.66°/149{,}452^d = -0;0,0,0,57$ | 0;0,0,2,44 |
| Venus(2) | $0.25°/149{,}531^d = 0;0,0,0,22$ | 0;0,0,2,44 |

The *Almagest* Mercury and adjusted Venus results are qualitatively comparable to the error patterns in the *Almagest* values, but they are also on average too small, suggesting again that the underlying sidereal year actually used in the reduction of the observations was longer than Ptolemy's. Estimating the underlying year length as before shows that if the original analysis of the inner planets used the same sidereal year length for both planets, then most likely it fell within the range $365;15,35^d$ – $365;15,36^d$.

Combining the results for the outer and inner planets yields an estimate for the underlying sidereal year in the range $365;15;25^d$ – $365;15;35^d$. Therefore, the overall pattern of the errors in both sign and magnitude for mean motion in anomaly is explained by using a year length that is too long. The error in the year length used appears to be larger than the error in the *Almagest* sidereal year, suggesting that the original analysis that Ptolemy adapted was actually made using a year length other than the ones Ptolemy mentions. An alternative way to estimate the underlying year length is to choose the year length that minimizes the sum of squares of the differences of the errors given in the *Almagest* and the model errors, as shown in Figure 1, and the resulting underlying year length is $365;14,34^d$. The residuals result from other sources, such as ordinary measurement errors.

All of these estimates must, of course, be tempered by the fact that we have no particular reason to believe that there is in fact a single year length underlying all the planets. The alternative view, that Ptolemy somehow collected together results from a variety of sources, cannot be ruled out. That being said, year lengths near the specified interval are not unknown. For example

(a) a year length of $365;15,25,56^d$ is found in the Babylonian System B′ scheme,
(b) a year length of $365;15,30^d$ is found in the *Pancasiddhantika* of Varahamihira (*ca.* A.D. 560), which is probably derived from some (unknown) Greco-Roman source and is called "Babylonian" by al-Battani, and
(c) a year length of $365;15,32^d$ is a value that al-Biruni attributes to the "Persians".

Each of these is effectively from a tradition that was unaware of precession, and of course there is no known connection between any of these and the pre-*Almagest* value discussed here, if for no other reason than that the ultimate origin of that value is unknown.

Most likely we have another source of information on the errors in mean motion in anomaly used in the pre-*Almagest* era, namely the errors in mean longitudes of the planets given in the early Indian astronomy texts. The error patterns in these are similar to what we would expect from the mean anomaly errors, which is consistent with the fact that there is good reason to suspect that the Indian mean longitudes were ultimately derived from extrapolating even more ancient mean motions in anomaly forward to the

5th and 6th centuries.[3] The large error in the mean motion in anomaly for Mercury is mirrored in the abnormally large negative errors in mean longitude for Mercury that are seen in all the ancient Hindu texts. The fact that in the same texts the mean longitudes of the other planets are relatively accurate could mean that someone was using a somewhat more accurate year length than the one derived above from the Almagest errors.

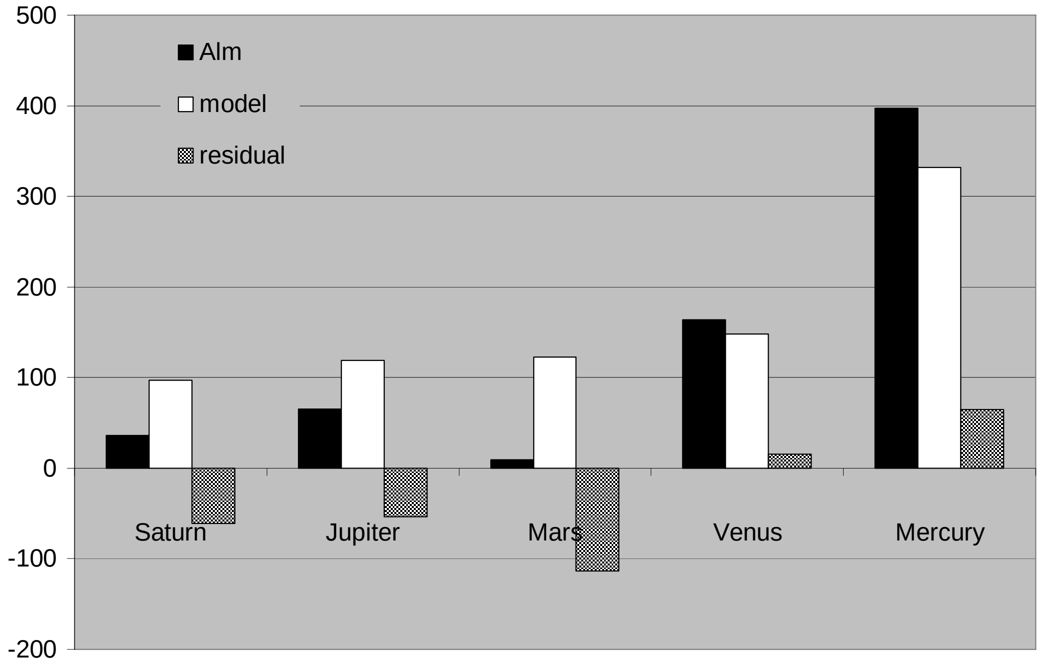


Figure 1. The errors, in units of degrees per day times $60^4$, as found in the *Almagest* and as computed in the model with a year length of $365;15;34^d$. The residual (in the same units) is the difference in the two errors and results from other sources.

REFERENCES

[1] Alexander Jones and Dennis Duke, "Ptolemy's Planetary Mean Motions Revisited", *Centaurus,* 47 (2005) 226-235.

[2] G. J. Toomer, *Ptolemy's Almagest* (1984)*,* 467 fn 104, 479 fn 21, 504 fn 65, 524 fn 19, 543 fn 36; Dennis W. Duke, "Ptolemy's Treatment of the Outer Planets", *Archive for History of Exact Sciences*, 59 (2005), 173, 175–6.

[3] Dennis W. Duke, "Mean Motions and Longitudes in Indian Astronomy, *Archive for History of Exact Sciences*, 62 (2008) 489-509.