Ptolemy's Planetary Mean Motions Revisited

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In Ptolemy's tables for determining the positions of the planets, two mean motions are computed for each planet: a mean motion in longitude, representing the uniform angular revolution of the centre of the planet's epicycle around the equant point, and a mean motion in anomaly, representing the uniform revolution of the planet around the epicycle's centre. According to Ptolemy's models, the rate of mean motion of Venus and Mercury in longitude is identical to the sun's rate of mean motion in longitude, whereas for Mars, Jupiter, and Saturn the sum of the rates of mean motion in longitude and anomaly is equal to the sun's mean motion. Ptolemy's mean motion tables are based on very precisely expressed values for the daily mean motions, given to six fractional sexagesimal places (i.e. a precision of about $2° \times 10^{-11}$):

(1)

Planet	*Mean daily motion in longitude*	*Mean daily motion in anomaly*
Saturn	0;2,0,33,31,28,51°	0;57,7,43,41,43,40°
Jupiter	0;4,59,14,26,46,31°	0;54,9,2,46,26,0°
Mars	0;31,26,36,53,51,33°	0;27,41,40,19,20,58°

Venus	0;59,8,17,13,12,31°	0;36,59,25,53,11,28°
Mercury	0;59,8,17,13,12,31°	3;6,24,6,59,35,50°

Ptolemy presents these numbers without their empirical justification in *Almagest* 7.3, preceding his deduction of the eccentricities and epicycle radii in the models. By his reasoning, these deductions only require fair approximations of the mean motions, while the final values of the mean motions are best established after the other parameters of the model are known. Hence at the end of the section concerning the model for each planet, Ptolemy devotes a chapter to the "correction" of its mean motions. He does this by taking two widely-spaced observations of the planet's longitude, one from the third century B.C. and one from his own time in the second century A.D., determining the situation of the model at each date by means of the known properties of the model and the observed longitudes, and deducing the total number of degrees of mean motion in anomaly traveled between the two dates. Each time he does this, he concludes with the following formula:[1]

> And this is pretty well the size of the excess [of motion in anomaly over whole revolutions] that we find in the tables of [the planet's] mean motions that we worked out, since in fact the daily motion was established by us from these [numbers, i.e.] the sum of the degrees arising from the number of [complete] circles and the degrees of the excess, divided by the total number of days of the time between the two observations. (*Almagest* 10.9, 11.3, and 11.7)

So if we take Ptolemy at his word, he originally carried out the deductions of the parameters of each planet's model (eccentricity, apsidal line, epicycle radius, etc.) using approximate values of the mean motions that were slightly different from the ones that appear in the *Almagest*. Then using these parameters, he carried out the analyses of the observation pairs more or less in the same way that we see in the "correction" chapters, but went a step further by actually carrying out the divisions to obtain new values for the daily mean motions. Then, in principle at least, he repeated the entire process of deducing the model parameters using these new values, and this is the version that he "writes up" in *Almagest* 10 and 11.[2] This iterative procedure, in which only the *final* convergent stage is offered to the reader, parallels the way that Ptolemy presents the deduction of the first lunar model's parameters in *Almagest* 4.

So far as we are aware, R. Newton and (less comprehensively) O. Neugebauer independently first pointed out that the mean daily motions that one obtains by dividing the deduced total motion in anomaly between the observations by the total number of days differ from Ptolemy's mean daily motions already in the fourth or fifth sexagesimal place (discrepant digits are in italics):

(2)

planet	*total degrees of motion in anomaly*	*total days between observations*	*quotient*
Saturn	126711;27°	133079;45	0;57,7,43,41,*44*,…
Jupiter	124305;45°	137732;57,30	0;54,9,2,*45*,…
Mars	69181;43°	149881;40	0;27,41,40,19,*28*,…

Venus	92138;25°	149452	0;36,59,25,*49*,…
Mercury	456726;53°	147013;33,45	3;6,24,6,*58*,…

Neugebauer, in the order of his treatment, writes the division for Venus as if it results in the mean motion in anomaly given in (1), notes that the quotient for Mercury is wrong in the final digits, and for the outer planets simply quotes the leading two digits of the quotient, perhaps suggesting that while he was aware of the discrepancies, it was not an issue meriting further discussion.[3] Newton, however, remarks on the discrepancies for all the planets, and infers that Ptolemy's mean daily motions were *not* derived from the cited observations, but instead adopted from the works of unknown predecessors.[4] If this is the case, Newton correctly points out, the circumstance that Ptolemy's analysis of each pair of observations leads to total motions in anomaly and longitude closely agreeing with his mean motion tables, generally within about 0;1°, is cause for suspicion. Newton supposes that at least one in each pair of observations is not genuine.

In an appendix to his translation of the *Almagest*, Toomer revisits the problem of the mean motions. He first suggests an indirect route by which Ptolemy might have used his analyses of pairs of observations to obtain daily mean motions, and shows that in the cases of Saturn, Venus, and Mercury this route would lead to Ptolemy's numbers; but for Jupiter and Mars it "fails miserably," leading Toomer to doubt the validity of his hypothesis. He therefore proposes that Ptolemy derived his mean motions from *other* observations, and merely intended his analyses in the *Almagest* to confirm them; the details of the true origin of Ptolemy's numbers is, Toomer maintains, beyond confident recovery.

Rawlins, agreeing with Newton that Ptolemy owed his mean motions to earlier astronomers, has found periodicities of the planets that could have been discovered from pairs of observations and that can give rise to Ptolemy's numbers.[5] Three of these are the period relations for Saturn, Venus, and Mercury that Toomer also proposed but then rejected (they are in fact in Ptolemy's text); however, unlike Toomer, Rawlins contended that these relations were not discovered by Ptolemy but adopted by him from his predecessors. While it is not clear how widely accepted Rawlins's other reconstructions are, there appears to be consensus that Ptolemy seriously misrepresented the way he obtained his mean motions.

But Toomer was unjustly dismissive of his own first hypothesis reconciling the observations in the *Almagest* with the mean daily motions. One part of Toomer's reconstruction (which we will discuss presently) is certainly correct for Saturn, Venus, and Mercury, and with an adjustment is also certainly correct for Jupiter and Mars. The other part involves a small discrepancy for Mars but yields full agreement for the remaining planets, so that at first glance it seems also tenable. We shall show, however, on different grounds that it is true for at most three of the five planets.

Before explaining Toomer's method, it is necessary to draw attention to the fact that *Almagest* 9.3, the chapter that introduces the mean motions, presents them in three forms. First, Ptolemy gives a period relation asserting that a certain whole number of complete revolutions of the planet in anomaly is equal to a certain whole number of tropical years plus or minus a certain number of days, and also equal to a certain number of degrees of mean motion in longitude:

(3)

planet	*revolutions in anomaly*	*time*	*revolutions in longitude*
Saturn	57	59y + $(1+\frac{1}{2}+\frac{1}{4})$d	$2^r + (1+\frac{2}{3}+\frac{1}{20})°$
Jupiter	65	71y – $(4+\frac{1}{2}+\frac{1}{3}+\frac{1}{15})$d	$6^r - (4+\frac{1}{2}+\frac{1}{3})°$
Mars	37	79y + $(3+\frac{1}{6}+\frac{1}{20})$d	$42^r + 3\frac{1}{6}°$
Venus	5	8y – $(2+\frac{1}{4}+\frac{1}{20})$d	$8^r - 2\frac{1}{4}°$
Mercury	145	46y + $1\frac{1}{30}$d	$46^r + 1°$

The numbers of anomalistic revolutions in these periods are not arbitrary: they are well known as Babylonian Goal-Year Periods, anomalistic intervals less than a century long that are close to a whole number of years. Ptolemy tells us that he took over these periods in a preliminary form (i.e. presumably with different corrections to the whole numbers of years and revolutions in longitude) from Hipparchus, who in turn might have learned them from Babylonian astronomy. Ptolemy gives us no indication of these preliminary values, since in the subsequent working out of the planetary models he assumes their final values.

Because of the relation between the planetary mean motions and the sun's mean motion, the small corrections in longitude should in principle be obtainable by multiplying the corrections in time by the sun's mean daily motion, 0;59,8,17,13,12,31°/d. The actual numbers that Ptolemy gives do not preserve this proportionality very accurately because they have been approximated by round fractions or whole numbers of minutes.

Ptolemy then restates the equations between anomalistic motion and time (but leaving out longitudinal motion), expressing the numbers purely in degrees and days respectively. In converting tropical years to days (using 1y = 365;14,48d), Ptolemy rounds to the first sexagesimal place, with discrepancies of 0;1d for Mercury and Mars.[6]

(4)

planet	*mean motion in anomaly*	*days (Ptolemy)*	*days (accurate)*
Saturn	20520°	21551;18	21551;18,12
Jupiter	23400°	25927;37	25927;36,48
Mars	13320°	28857;43	28857;42,12
Venus	1800°	2919;40	2919;40,24
Mercury	52200°	16802;24	16802;22,48

Only now does Ptolemy give the six-sexagesimal-place mean daily motions in anomaly, indicating that they come from dividing the degrees in (4) by the corresponding numbers of days. This is not consistent with the strict letter of what Ptolemy says in the later chapters on the "correction" of the mean motions. However, if the intervals of time and mean longitude in the period relations (3) were themselves derived from Ptolemy's observation pairs, a derivation of the mean daily motions from the period relations would be *mathematically* equivalent to obtaining them directly from the observation pairs, except for the effects of roundings and inaccuracies of computation. Toomer's hypothesis is that Ptolemy did in fact follow this "perverse" route from (2) to (1) by way of (3) and (4).

Let us first consider the question whether the mean daily motions were obtained by dividing out the period relations (4). We repeat Toomer's calculations:

(5)

planet	*mean motion in anomaly*	*days*	*quotient*
Saturn	20520°	21551;18	0;57,7,43,41,43,39,41,…
Jupiter	23400°	25927;37	0;54,9,2,*42,55,52*,…
Mars	13320°	28857;43	0;27,41,40,*11,44,37*,…
Venus	1800°	2919;40	0;36,59,25,53,11,27,36,…
Mercury	52200°	16802;24	3;6,24,6,59,35,49,55,…

First Rawlins, and later Toomer, noticed that for Saturn, Venus, and Mercury the quotients, rounded to six sexagesimal places, are *identical* to Ptolemy's daily motions (1), while for the other planets disagreement starts in the fourth place. The failure for Jupiter and Mars leads Toomer to doubt the significance of the three matches, and subsequently he entertains the possibility that the period relations (3) were actually obtained by multiplying up the daily motions. But this cannot be the case for Saturn, Venus, and Mercury. The times in (3) are given to one sexagesimal place (or an equivalently imprecise fraction), which means that even for the longest of the periods, that for Mars, the implied precision of the mean daily motion in anomaly is about $\pm 3^\circ \times 10^{-7}$, or about

±0;0,0,0,4°. The six-place mean daily motions are roughly 10^4 times more precise. Exact agreement to all six places can only mean that the daily motions come from the period relations, not *vice versa*.

In the case of Mars and Jupiter, for no obvious reason, Ptolemy used a different method. He first calculated the mean daily motion in *longitude* by dividing the progress in mean longitude as given in (3) by the number of days as given in (4):

(6)

planet	*progress in mean longitude*	*days*	*quotient*
Jupiter	$6\times360° - 4\frac{5}{6}°$	25927;37	0;4,59,14,26,46,31,23,,…
Mars	$42\times360° + 3\frac{1}{6}°$	28857;43	0;31,26,36,53,51,32,55,…

These quotients, rounded to the sixth place, again agree *exactly* with the six-place mean motions in (1). The mean motion in anomaly was then obtained by subtracting this quotient from the sun's mean motion. Again it is certain that the six-place sexagesimals were obtained from the period relations rather than the other way around because the implied precision of the period relations is much rougher than that of the daily motions.

The next question, then, is whether, as Toomer first hypothesized, the period relations (3) and (4) were themselves derived by Ptolemy from the observation pairs cited in the *Almagest*. In each "correction" chapter Ptolemy arrives at an interval in mean anomalistic motion and an interval in days. Repeating Toomer's calculations, we scale down the number of days in (2) to find the time corresponding to the Goal-Year numbers of complete revolutions in anomaly:

(7)

planet	*revolutions in anomaly*	*time*
Saturn	57	21551;17,59,55,…
Jupiter	65	25927;36,42,19,…
Mars	37	28857;40,45,50,…
Venus	5	2919;40,5,19,…
Mercury	145	16802;24,1,…

In four cases out of five, the resulting times, when rounded to the first sexagesimal place, match the times in both forms of the period relations, (3) and (4). For Mars there are small discrepancies (0;1 day and 0;2 days respectively), which could be explained away as errors of calculation or rounding.

This appears to be an almost complete vindication not only of Toomer's hypothesis but also of Ptolemy's veracity: he may have chosen an unnecessarily roundabout method of getting the corrected daily mean motions from the observation pairs, and in the process he may have allowed roundings and small errors to affect the lowest-order places of the quotients, but there remains no argument here refuting his claims at the end of the "correction" chapters to have obtained his final mean motions from the observations in question.

Ptolemy, however, has left other traces of the earlier stages of his planetary theory. In *Almagest* 4.9, after describing how he had revised certain of his earlier values for parameters of the lunar model, continues:

> We have done a similar thing in the case of the models for Saturn and Mercury, changing some of our former assumptions that were not very accurately made because we had got our hands subsequently on more indisputable observations.

It is likely that Ptolemy is alluding in this passage to certain parameters of his models that he published in his *Canobic Inscription*, erected in A.D. 146/7.[8] In particular, the apogee of Mercury and the eccentricities of both Mercury and Saturn are significantly different in the *Canobic Inscription* from the values deduced in the *Almagest*:

(8)

	Canobic Inscription	*Almagest*
Mercury *A*	186	190
Mercury *e*	2;30	3
Saturn *e*	3;15	3;25

The mean daily motions of all five planets, however, are identical in the *Inscription* and in the *Almagest*. In addition, the epoch values for anomaly and mean longitude for all planets except Saturn are the same when adjusted for the change from era Nabonassar to era Augustus. At least one of the epoch values for Saturn must be textually corrupt, since they do not sum to the mean longitude of the Sun at epoch.[9]

Now in Ptolemy's "correction" chapters the parameters *e*, *r*, and *A* of the planetary models are used to determine the configurations of the models at the time of one or both of the observations, so that a change in the parameters should affect the configurations and thus the resulting corrected mean motion in anomaly. In other words, if Ptolemy's observation pairs combined with the *Almagest* parameters lead to the mean motions of Mercury and Saturn in (1), these same observations combined with the earlier *Canobic Inscription* parameters should not have done so, and the same conclusion follows for the epoch values in anomaly and mean longitude.

For example, in *Almagest* 9.10, the "correction" chapter for Mercury, Ptolemy calculates for two observations the arc of the epicycle between the planet and the apogee of the epicycle, i.e. the planet's position in anomaly. For the earlier of these observations, dated November 15, 265 B.C., Ptolemy computes that the arc is 212;34°, while accurate computation gives 212;32°. Recomputation using the *Inscription* parameters results in 211;55°, a change of over half a degree. For the later observation, dated May 17, A.D. 139, Ptolemy computes that this arc is 99;27°, while accurate computation gives 99;34°, and recomputation using the *Inscription* parameters gives 104;40°, a change of about 5°. Between the two observations, therefore, Ptolemy would have found a total motion in anomaly of 456732;44° (instead of 456726;53°, or 456727;2° computed accurately).[10] Scaling this number to the 46-year Goal-Year period of (4), we find that the time interval corresponding to 145 synodic periods is 16802;11 days (instead of 16802;24 days). This would affect the mean daily motion in anomaly already in the third sexagesimal place, and the epoch values in anomaly and mean longitude in the first sexagesimal place.[11]

The only conclusion to be drawn from this is that Ptolemy did not get his final mean motions for Mercury by analysis of the two observations used in *Almagest* 9.10. Moreover, when he revised the eccentricity of his model for this planet, he chose not to recalculate the mean motions in the light of this change, but instead manipulated either the reports of those two observations or the reduction and analysis of them (or both) so as to obtain as close agreement as possible with the mean motions to which he had already committed himself. Ptolemy's reduction of the older observation report is not straightforward; as Toomer remarks, it "is difficult to see how Ptolemy arrives at this position from his data." And the later one, which he claims to have made himself, would be a very poor choice if Ptolemy had really intended to calibrate the mean motion in anomaly from it, since according to his model the planet is so close to the edge of its epicycle as seen from the earth that a tiny change in the data brings about a ludicrously large change in the deduced position in anomaly.

In *Almagest* 11.7, the "correction" chapter for Saturn, the older observation is a Babylonian observation of Saturn passing by a "Normal Star," the report of which Ptolemy can hardly have tampered with, although it is likely that the date he assigns to it, March 1, 229 B.C., is one day too late.[12] His calculation of 183;17° as the planet's position in anomaly from the report has no significant arithmetical inaccuracies. If we recompute using the *Canobic Inscription* eccentricity (3;15) instead of the *Almagest* eccentricity (3;25), the position in anomaly comes to 182;57°. This change of a mere third of a degree in the position in anomaly for one of a pair of observations separated by

133079;45 days changes the rescaled days for Saturn in (4) from 21551;18 to 21551;14, and changes the resulting mean motion in the 3rd sexagesimal place. Since the value of mean anomaly at epoch is based on the computed anomaly of the ancient observation, we would expect the *Inscription* and *Almagest* epoch values to be inconsistent for Saturn, but using 182;57° for the anomaly does not lead to epoch values at all resembling the partially corrupt transmitted ones. Still more troubling is the fact that the later observation that Ptolemy uses in this chapter is the third of the three mean oppositions of Saturn (that of July 8, A.D. 136) from which he deduced Saturn's eccentricity in *Almagest* 11.5, and the position in anomaly that he uses in the "correction" was found in the course of that earlier computation. This pair of observations seems unlikely to have given Ptolemy the empirical basis for the mean motions of Saturn that he published in the *Canobic Inscription*, before he had "got his hands… on more indisputable observations."

It therefore seems most likely that, as Newton surmised, the precise circumstances of the observations used in the correction chapters were adjusted to be consistent with the assumed period relations and their resulting mean motions. If Ptolemy's mean motions for two of the planets had nothing to do with the analysis of observations in the "correction" chapters, the same may well be true of the remaining three for which we do not have the evidence of parameters changed since the *Canobic Inscription*. In any event we have one more demonstration that Ptolemy's deductions in the *Almagest* cannot be taken as a historical account, even when he says outright that they can. The question whether it was Ptolemy himself or an earlier astronomer (or indeed several) who arrived at these numbers remains open. Now that we know that the six-place mean motions are merely a consequence of the much less precisely expressed relations (3) and (4), the prospect of discovering a unique plausible empirical derivation for them that would illuminate the research methods of either Ptolemy or his predecessors is, alas! greatly diminished.

BIBLIOGRAPHY

Hamilton, N. T., Swerdlow, N. M., Toomer, G. J.

1987: “The Canobic Inscription: Ptolemy’s Earliest Work”, in Berggren, J. L. and Goldstein, B. R., eds: *From Ancient Omens to Statistical Mechanics,* Copenhagen.

Heiberg, J. L.

1898-1903: *Claudii Ptolemaei opera quae exstant omnia. Volumen I. Syntaxis Mathematica.* Leipzig, 2 vols.

1907: *Claudii Ptolemaei opera quae exstant omnia. Volumen II. Opera astronomica minora.* Leipzig.

Neugebauer, O.

1975: *A History of Ancient Mathematical Astronomy*, Berlin, 3 vols.

Newton, R.

1982: *The Origin of Ptolemy’s Astronomical Parameters*, Baltimore.

Rawlins, D.

1987: “Ancient Heliocentrists, Ptolemy, and the equant”, *American Journal of Physics*, 55 (1987) 235-9.

1997: “Unpublished Letters”, *Dio*, 7.1 (1997) 25-33.

Sachs, A. J. and Hunger, H.

1989: *Astronomical Diaries and Related Texts from Babylonia II: Diaries from 261 B. C. to 165 B. C.,* Vienna

Toomer, G. J.

1984: *Ptolemy's Almagest*, London.

NOTES

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[1] The translations from the *Almagest* are by Alexander Jones; text in Heiberg 1898-1903.

[2] "In principle," because it is possible that, at the precision with which Ptolemy was working, the revision of the mean motions made no difference to the actual numbers used in working out the model's other parameters. Since, however, the reader of the *Almagest* is not told what the provisional mean motions were, Ptolemy clearly intends the mean motions that go into the deduction of the parameters to be understood as taken from his final tables.

[3] Neugebauer 1975, v. 1, 157 (Venus), 168 (Mercury), 182 (outer planets).

[4] Newton 1977, 320-321 and 325-327.

[5] The first published mention of Rawlins' results for Saturn, Venus, and Mercury is in Newton 1982, 103-109. Rawlins' proposed derivation for Mars and Jupiter is first published in Rawlins 1987, 237 and 239 fn. 27, respectively.

[6] The number of days for Mars in the manuscripts of the *Almagest* is 28857;53. Toomer emends to 28857;43, assuming a scribal misreading of the Greek numeral *M* (40) as *N* (50). This emendation, of which Toomer was uncertain, is proved correct by the derivation from it of Ptolemy's values for the daily mean motions shown below.

[8] The fullest and most accurate discussion of the *Canobic Inscription* and of its relationship to the *Almagest* is Hamilton et. al. 1987. Text of the *Inscription* in Heiberg 1907, 149-155.

[9] Neugebauer 1975 v. 2, 915 note 14 mistakenly reports that one manuscript of the *Canobic Inscription* has a reading for Saturn's epoch in anomaly that is approximately consistent with the *Almagest* epoch. Hamilton et. al. propose emending *both* Saturn's

epoch positions in the *Inscription* to bring them into near agreement with the *Almagest*. Given the uncertainties about the changes Ptolemy made in Saturn's model, we think it is more prudent to leave the numbers as they have been transmitted.

[10] The significance of the differences between the parameters of Mercury in the *Canobic Inscription* and the *Almagest* was pointed out, with few numerical details, in Rawlins 1987, 236-237.

[11] Hamilton et al. 1987, 65-67 entertain suspicions that at the time of the *Canobic Inscription* Ptolemy did not yet have the special model for Mercury with the rapidly rotating eccentre, or the theory that the planets' apsidal lines are sidereally fixed. Moreover the epicycle radius for Mercury is reported in the manuscripts of the *Canobic Inscription* as 22 ¼. Since all the other parameters in the inscription are expressed as sexagesimals and the Greek for "¼" (delta) is an easy misreading of "30" (lambda) in capitals, this is likely a textual error, although it is worth noting that in the later *Planetary Hypotheses* (Heiberg 1907, 88) Mercury's epicycle radius is given as 22 ¼ with no likelihood of textual error. We have here assumed that the *Canobic Inscription* model differed from that of the *Almagest* only in *A* and *e*; assuming other differences would result in comparable or even larger discrepancies in the "corrections." For an argument that the *Canobic Inscription* model for Mercury was structurally the same as that of the *Almagest* see Rawlins 1987, 236-237 and 239 notes 23-24. The confirmation of this adduced in Rawlins 1997, 29 appears to us to be invalid.

[12] Ptolemy gives the original Babylonian date (in thin Macedonian disguise) as Seleucid Era 82 XII 5. From an extant Babylonian Diary text (No. -229 in Sachs-Hunger 1989) we know that this date was February 29/March 1, not March 1/2. Ptolemy probably did not

have documentary means of knowing the exact beginnings of the Babylonian months in the third century B.C.