Ptolemy's Treatment of the Outer Planets

Dennis Duke, Florida State University

A primary theme of Ptolemy's *Almagest*,[1] virtually unique among known classical sources, is that one can use empirical observations to determine numerical values for the parameters of geometrical planetary models. The principal examples are:

1. an autumn equinox, a spring equinox, and a summer solstice, each of which is combined with an ancient observation to determine the length of the tropical year. Also, the resulting season lengths are used to determine the eccentricity and the direction of apogee for the Sun.
2. a trio of lunar eclipses which determine the lunar parameters at syzygy. Virtually identical lunar parameters result from Ptolemy's reports of three earlier lunar trios.
3. several lunar elongations are used to determine the parameters of the lunar model away from syzygy.
4. two pairs of observations of Mercury that determine the longitude of apogee in Ptolemy's time, and six additional ancient observations that determine it some 400 years earlier.
5. two pairs of observations of Venus that determine the longitude of apogee in Ptolemy's time.

Detailed analysis[2] has shown that each of these sets of observations share a common, conspicuous set of qualities. First, they are in most cases redundant. While using redundant sets of data is in general a good idea, real concern arises in Ptolemy's cases when the redundant sets invariably result in essentially identical values for the parameters being determined. Second, the parameters so determined are sometimes rather far from their optimum values. Examples include the dates of the equinoxes and solstices, which are all about a day late, and the direction of Mercury's apogee, which is some 30° off. Third, the deduced parameters are sometimes too close to their optimum values, even though the quality of the underlying observations is too poor to expect such good agreement. Altogether, we can safely conclude that at least for the Sun, the Moon and the inner planets Mercury and Venus, there is a substantial gap between Ptolemy's description of events and what actually happened.[3]

The situation for the outer planets is, however, different. Ptolemy abandons his custom of quoting redundant observations, and adopts instead a rather minimalist style, quoting precisely the number of observations required to determine his parameters, and no more. Given the *Almagest* bisected-equant model, exactly five observations are required. Three mean oppositions determine the eccentricity, the longitude of apogee, and the mean longitude at some time. A fourth observation[4] determines the radius of the epicycle, and a fifth observation, from about 400 years earlier, provides a high precision determination of the mean motion in anomaly. The assumption that the radius of the epicycle always points toward the mean Sun effectively allows determination of (a) the angle of the epicycle with the epicycle apogee at some time, and (b) the mean motion in longitude from the mean motion in anomaly and the length of the tropical year, thus completing the determination of all the model parameter values needed to compute the position of the planet at any moment in time. Ptolemy's minimalist style extends to treating each planet identically, to the extent of using virtually identical prose throughout the three discussions of parameter derivations.

Suspicions that Ptolemy's account might be less than candid are not new. Newton raised the issue in light of (a) the failure of Ptolemy's data to reproduce exactly the mean motions in anomaly that Ptolemy uses in his tables, (b) the fact that for the fourth observation for each planet, redundant determinations of the longitude with respect to a star and the Moon agree exactly with each other, and (c) the fact that Ptolemy's data for Mars lead to almost exactly a round value for Mars' eccentricity.[5] Thurston pointed out that the sequence of successively rounder values that Ptolemy finds for 2*e* for each planet are not what one expects from an iterative solution, suggesting that Ptolemy was in fact working backward from those round parameters.[6]

The purpose of this paper is to investigate whether for the outer planets Ptolemy followed his otherwise consistent custom of describing a scenario that did not happen as he says, or whether, at least for the outer planets, he left us a more accurate rendition of events. The detailed reconstructions of Ptolemy's calculations that follow show that, at least in the *Almagest*, Ptolemy is a writer with consistent habits when it comes to observations. We begin by reviewing, with minimal editorial comment, Ptolemy's calculations for each planet.

Mars

The input data for the Mars analysis are three oppositions with the mean Sun and two additional observations, one from several hundred years before Ptolemy's time. Ptolemy reports times and the corresponding longitude of Mars at those times:

	Ptolemy	Model
t_1	1768888.54167	
t_2	1770418.37500	
t_3	1771974.41667	
t_4	1771977.35903	
t_5	1622092.75000	
λ_1	81;00°	80;58,54°
λ_2	148;50°	148;46,24°
λ_3	242;34°	242;32,04°
λ_4	241;36°	241;35,12°
λ_5	212;15°	212;16,32°
$\lambda_1(S)$		260;58,54°
$\lambda_2(S)$		328;50,22°
$\lambda_3(S)$		62;31,44°

Ptolemy typically gives the times to the nearest hour and does not correct for the equation of time, which would be a negligible effect in the present context. I have converted his times to standard astronomical Julian day numbers relative to noon at Alexandria. The third column gives the longitudes predicted for Mars and the mean Sun using the final *Almagest* models at Ptolemy's specified times.

Ptolemy's first task is to use three mean oppositions to determine the eccentricity and longitude of apogee. The oppositions cannot, of course, be directly observed, and so must be estimated from a set of data. Ptolemy refers to this twice. In the preface to the analysis of the Mars trio of oppositions he writes[7]

> Just as, in the case of the moon, we took the positions and the times of three lunar eclipses and demonstrated geometrically the ratio of anomaly and the position of the apogee, in the same we here too, using three acronychal oppositions with respect to the mean course of the sun for each of these planets, we observed as accurately as possible the positions by means of astrolabic instruments [i.e. the armillary] and, on the basis of the mean courses of the sun at the times of the observations, we further calculated the time and position corresponding to greater precision in the diametrical opposition, we demonstrate from these the ratio of eccentricity and the apogee.

Second, while describing the second and third oppositions for Saturn, which he says occurred during daylight, he writes "We computed the time and place of exact opposition from nearby observations…". Using a zodiacal armillary the position of the planet could be measured relative to a star and/or the Moon, a procedure Ptolemy claims to use for the fourth observation of each outer planet, and for Mercury and Venus. The armillary also gives the degree of the ecliptic culminating at the time of the measurement, and that can be used to estimate the time of the observation. The underlying solar model gives the longitude of the mean Sun at the same moment, and one then easily and accurately estimates the moment of opposition by interpolation in the resulting table of data. The fractional endings that Ptolemy reports for the longitudes of opposition: 00, 50, 34, 11, 54, 23, 13, 40, and 14 are consistent with the idea that the oppositions are computed using some such process of data reduction, and not from direct analog measurement with an armillary, which would certainly give only round fractions of a degree. Ptolemy's description of the measuring process does not, of course, necessarily imply that Ptolemy himself used it to make any measurements (see the discussion in the Comments section, below).

While a simple eccentric model can be solved directly, an iterative scheme is used to solve the bisected equant model (see Appendix A). The actual input data to Ptolemy's numerical analysis are, in the notation of the Appendix, the angles α, β, γ, and δ. Ptolemy's values compare to exactly computed values, using Ptolemy's times and longitudes, as follows:

	Ptolemy	Exact
α	81;44°	81;43,26°
β	95;28°	95;27,31°
γ	67;50°	67;50°
δ	93;44°	93;44°

In the notation of the Appendix, Ptolemy gives[8] after each iteration the values of 2*e*, $\varphi_1 = \varphi - \alpha$, $\varphi_2 = \varphi$, and $\varphi_3 = \varphi + \beta$ (and since α and β are known, only one of the angles, in our case *φ*, is an independent computed output). The results for Mars are:

Exact Computation (Ptolemy's Result)				
Mars	2*e*	φ_1	φ_2	φ_3
iteration 1	13;02,21 (13;07)	36;22,18 (36;31)°	45;21,41 (45;13)°	39;10,18 (39;19)°
iteration 2	11;51,17 (11;50)	42;38,40 (42;45)°	39;05,19 (38;59)°	45;26,40 (45;33)°
iteration 3	12;01,23 (12;00)	41;22,20 (41;33)°	40;21,39 (40;11)°	44;10,20 (44;21)°
iteration 4	11;59,48	41;34,03°	40;09,56°	44;22,03°
iteration 5	12;00,03	41;32,14°	40;11,45°	44;20,14°
iteration 6	12;00,00	41;32,31°	40;11,28°	44;20,31°

Ptolemy proceeds to check his results by using 2*e* = 12;00 and his values for φ_i to compute the distance from the apsidal line for each opposition. He gets 34;30°(34;30,00), 33;20°(33;20,01), and 52;56°(52;56,07) (the accurately computed values, using his values for 2*e* and *φ*, are given in parentheses). Combining these in pairs returns the starting values for γ and *δ*, accurate to the nearest arcmin: 34;30° + 33;20° = 67;50°, and 180° – (33;20° + 52;56°) = 93;44°. Since the longitude of the third opposition was given by Ptolemy as 242;34°, he concludes that the longitude of apogee is

$$A = 242;34^\circ + 52;56^\circ + 180^\circ = 115;30^\circ,$$

which indeed agrees with the result of exact computation.

In order to complete the determination of the model parameters, Ptolemy produces two additional observations. The first (number four in the table above) is used in combination with the third opposition to determine the epicycle radius *r*. The second (number five in the table above) is derived from ancient records and is used, according to Ptolemy, to determine an accurate value for Mars' mean motion in anomaly. Ptolemy reports that the epicycle radius is 39;30 while accurate computation using Ptolemy's input data gives 39;34,54. The discrepancy is due to rounding error.[9]

Ptolemy then reports that at the moment of the fifth observation the distance of Mars from the apogee of its epicycle was 109;42°, while accurate computation again using Ptolemy's input data gives 109;45,10°. This particular discrepancy is quite significant in its context, since Ptolemy is using the number to determine the mean motion in anomaly to six sexagesimal places of accuracy. The value he quotes in *Almagest* 9.1 is 0;27,42,40,19,20,58 $^{o/d}$. Using Ptolemy's value for the ancient longitude and Ptolemy's time interval, one finds that the final digits of the mean motion are 19,28,7. If you add or subtract just 1´ to Ptolemy's longitude, the finals digits are 20,54,35 and 18,1,39, respectively. Thus it is likely that Ptolemy was simply choosing the best value to 1´

precision, among the neighboring values, that combined with his value of the anomaly at time t_3, results in a value of ω_a closest to the value he quotes in *Almagest* 9.1.

Having now fully determined the equant model parameters, Ptolemy extrapolates over the 475 years from the fifth (and already most ancient) observation back to his chosen Nabonassar epoch to determine the model epoch values, which he quotes as 3;32° in longitude, 327;13° in anomaly, and 106;40° for longitude of apogee. Accurate computation gives 3;30,03° for longitude, 327;16,41° for anomaly, and 106;39,52° for longitude of apogee. In some sense, these small 'adjustments' being made by Ptolemy are required, since he would certainly want the sum of the epoch values in mean longitude and anomaly to equal the mean longitude of the Sun at the same moment, which he takes as 330;45°. Altogether, though, these relatively minor discrepancies lead to small differences between the input data and the model output for the same moments in time.

Jupiter

The input data for Jupiter are:

	Ptolemy	Model
t_1	1769773.45830	
t_2	1770975.41830	
t_3	1771377.70830	
t_4	1772018.70830	
t_5	1633644.74997	
λ_1	233;11°	233;10,45°
λ_2	337;54°	337;53,05°
λ_3	14;23°	14;22,22°
λ_4	75;45°	75;42,12°
λ_5	97;33°	97;30,52°
$\lambda_1(S)$		53;11,12°
$\lambda_2(S)$		157;52,51°
$\lambda_3(S)$		194;23,32°

As in the case for Mars, the third column gives the longitudes predicted for Jupiter and the mean Sun using the final *Almagest* models. Ptolemy's values for α and β compare to exactly computed values, using Ptolemy's times and longitudes, as follows:

	Ptolemy	Exact
α	99;55°	99;54,35°
β	33;26°	33;26,21°
γ	104;43°	104;43°
δ	36;29°	36;29°

The results for Jupiter's iterations are:

Exact Computation (Ptolemy's Result)				
Jupiter	$2e$	φ_1	φ_2	φ_3
iteration 1	5;20,03 (5;23)°	79;00,24 (79;30)°	1;04,35 (0;35)°	32;21,24 (32;51)°
iteration 2	5;28,55 (5;30)°	77;34,40 (77;15)°	2;30,19 (2;50)°	30;55,40 (30;36)°
iteration 3	5;29,40°	77;22,51°	2;42,08°	30;43,51°
iteration 4	5;29,44°	77;21,28°	2;43,31°	30;42,28°
iteration 5	5;29,44°	77;21,19°	2;43,40°	30;42,19°

Ptolemy again checks his results by using $2e$ = 5;30 and his values for φ_i to compute the distance from the apsidal line for each opposition. He gets 72;11°(72;11,12), 3;06° (3;06,19), and 33;23°(33;22,55) (the accurately computed values are given in parentheses). When combined these again return precisely the intervals for γ and δ: 180° – (72;11° + 3;06°) = 104;43° and 3;06° + 33;23° = 36;29°. Since the longitude of the third opposition was given by Ptolemy as 14;23, he concludes that the longitude of apogee is

$$A = 14;23^\circ - 33;23^\circ + 180^\circ = 161;00^\circ .$$

Accurate computation, however, gives 160;53°, a discrepancy that must occur since Ptolemy's final values for the φ_i do not agree with the results of exact computation.

As he did for Mars, Ptolemy produces two additional observations to determine the epicycle radius *r* and an accurate value for Jupiter's mean motion in anomaly. Ptolemy reports that the epicycle radius is 11;30 while accurate computation using Ptolemy's input data gives 11;35,45. Ptolemy then reports that at the moment of the fifth observation the distance of Jupiter from the apogee of its epicycle was 77;02°, while accurate computation again using Ptolemy's input data gives 77;04,44°.

Having now fully determined the equant model parameters, Ptolemy extrapolates over the nearly 507 years from the fifth (and already most ancient) observation back to his chosen Nabonassar epoch to determine the model epoch values, which he quotes as 184;41° in longitude, 146;04° in anomaly, and 152;09° for longitude of apogee. Accurate computation gives 184;38,19° for longitude, 146;06,40° for anomaly, and 152;8,52° for longitude of apogee. Ptolemy was probably again adjusting values for consistency with his solar theory. Just as for Mars, these relatively minor discrepancies thus lead to small differences between the input data and the model output for the same moments in time.

Saturn

The input data for Saturn are:

	Ptolemy	Model
t_1	1767529.25000	
t_2	1769790.16667	
t_3	1770921.00000	
t_4	1771818.33333	
t_5	1637841.25000	
λ_1	181;13°	181;13,00°
λ_2	249;40°	249;39,05°
λ_3	284;14°	284;14,08°
λ_4	309;10°	309;05,13°
λ_5	159;30°	159;27,18°
$\lambda_1(S)$		1;12,57°
$\lambda_2(S)$		69;39,18°
$\lambda_3(S)$		104;14,39°

Once again, the third column gives the longitudes predicted for Saturn and the mean Sun using the final *Almagest* models. Ptolemy's values for α and β compare to exactly computed values, using Ptolemy's times and longitudes, as follows:

	Ptolemy	Exact
α	75;43°	75;42,53°
β	37;52°	37;52,11°
γ	68;27°	68;27°
δ	34;34°	34;34°

The results for Saturn's iterations are:

Exact Computation (Ptolemy's Result)				
Saturn	2*e*	φ_1	φ_2	φ_3
iteration 1	7;03,32 (7;08)	56;16,34 (57;43)°	19;26,25 (19;51)°	57;18,25 (55;52)°
iteration 2	6;48,59 (6;50)	57;42,45 (57;05)°	18;00,14 (18;38)°	55;52,14 (56;30)°
iteration 3	6;49,55	57;20,36°	18;22,23°	56;14,23°
iteration 4	6;49,51	57;25,15°	18;17,44°	56;09,44°
iteration 5	6;49,51	57;24,19°	18;18,40°	56;10,40°

Once again Ptolemy checks his results by using 2*e* = 6;50 and his values for φ_i to compute the distance from the apsidal line for each opposition. He gets 51;47°(51;46,45), 16;40°(16;39,20), and 51;14°(51;14,00) (the accurately computed values are given in parentheses). And once again he recovers precisely the values for *γ* and *δ*: 51;47° + 16;40° = 68;27° and 51;14° – 16;40° = 34;34°. Since the longitude of the third opposition was given by Ptolemy as 284;14°, he concludes that the longitude of apogee is

$$A = 284;14^\circ - 51;14^\circ = 233;00^\circ .$$

And in this case, accurate computation gives 233;18°, again reflecting the fact that Ptolemy's final values for the φ_i do not agree with the results of exact computation.

As he did for Mars and Jupiter, Ptolemy produces two additional observations to determine the epicycle radius r and an accurate value for Saturn's mean motion in anomaly. Ptolemy reports that the epicycle radius is 6;30 while accurate computation using Ptolemy's input data gives 6;31,41. Ptolemy then reports that at the moment of the fifth observation the distance of Saturn from the apogee of its epicycle was 183;17°, while accurate computation again using Ptolemy's input data gives 183;15,04°.

Having now fully determined the equant model parameters, Ptolemy extrapolates over the 518 years from the fifth (and already most ancient) observation back to his chosen Nabonassar epoch to determine the model epoch values, which he quotes as 296;43° in mean longitude, 34;02° in anomaly, and 284;10° for longitude of apogee. Accurate computation gives 296;45,21° for mean longitude, 33;59,38° for anomaly, and 284;08,58° for longitude of apogee, and as before probably adjusted for consistency. Just as before, these relatively minor discrepancies thus lead to small differences between the input data and the model output for the same moments in time.

Comments

The attentive reader might have noticed a few instances where Ptolemy's account is not consistent with the results of exact calculation. First, for Mars Ptolemy's values for 2*e* and φ_i after three error-prone iterations agree virtually exactly with the values reached after accurate computation to full convergence. Such a circumstance is not impossible, but seems unlikely.

Second, for Jupiter and Saturn Ptolemy's values after two equally error-prone iterations are closer to the accurately computed and fully converged values than to the values Ptolemy would have gotten by computing accurately and stopping at two iterations. Nevertheless, the agreement is not as good as we noticed for Mars.

Finally, Ptolemy's values for 2*e* and *A* are nicely round values for all three planets. The likelihood of that happening from purely empirical data cannot be very large. One might suspect simply convenient rounding, but that cannot be the case for the apogee values, which are accurately computed as round numbers with no rounding involved.

So how might we reconcile our reconstruction of the correct calculations with Ptolemy's own quite specific account? One option would be to accept his claims that he determined the times and longitudes of the oppositions from empirical observations, but then assume that he simply followed the iterative algorithm far enough to see where it was headed, and finally steered it to round numbers for 2*e* and *A*.

However, reality cannot be this simple. In this scenario the correlation between Ptolemy's input data and his final rounded numbers would be effectively washed out, so that those numbers could never reproduce the original data. Yet Ptolemy's verification for all three planets indeed does combine his round value for 2*e* and his computed values for the φ_i, each of which is in error by anywhere from 10´ to 40´ relative to what he would have gotten by computing accurately for two or three iterations, and he invariably recovers, accurate to an arcmin, his input values for γ and δ. Such accurate correlation between input data and poorly computed, rounded output parameters is impossible, yet it occurs for all three planets.[10]

An alternative, and almost as direct, explanation is that Ptolemy did exactly what we did: start with purported observations and perform many more iterations than he told us about. Then he would have *discovered* the correct final values, and he could go back and steer his calculations of the first few iterations to arrive at what he knew to be the correct values for 2*e* and φ_i. This cannot be the case, either, but to understand why requires some detailed discussion.

We first must consider whether the cases of Jupiter and Saturn can be construed as evidence in favor of the idea that Ptolemy's data is, in fact, derived from empirical observation. The answer to that question is no, and the proof is implicit in Ptolemy's own calculations.

Let us consider Saturn first. After completing two iterations Ptolemy announces that his final values are $2e$ = 6;50, φ_1 = 57;05°, φ_2 = 18;38°, and φ_3 = 56;30°. He then computes the value of ψ_i implied by each pair ($2e$, φ_i). The telling point is that for the second pair his purported verification actually fails. He quotes ψ_2 = 16;40°, while accurate computation gives 16;39,20°, which of course rounds accurately to 16;39°. Such a trifling mistake would, in most circumstances, be of no interest, but in the present context Ptolemy is consistently verifying to 1´ accuracy, and so we have to be careful.

Combining the ψ_i's now gives γ = 68;26° and δ = 34;35°, instead of 68;27° and 34;34°. Using the quartet ($\alpha, \beta, \gamma, \delta$) = (75;43°, 37;52°, 68;26°, 34;35°) and iterating to convergence produces $2e$ = 6;49,50, φ_1 = 56;58,22°, φ_2 = 18;44,37° and φ_3 = 56;36, 47°, all values now quite close to Ptolemy's final values. Computing ψ_3 now gives 51;20,19°, and combining with λ_3 gives A = 232;53,41°, again much closer to Ptolemy's 233;00°. Thus it is quite likely that somewhere along the line Ptolemy made a small slip in estimating the circumstances of the second opposition.

The case of Jupiter is similar. During his verification Ptolemy finds φ_1 = 72;11° and φ_2 = 3;06°, which combine to $\gamma = 180° - (\varphi_1 + \varphi_2)$ = 104;43°. However, exact calculation gives γ = 180° – (72;11,12° + 3;06,19°) = 104;42,29°, which rounds to 104;42°. Using the quartet ($\alpha, \beta, \gamma, \delta$) = (95;55°, 33;26°, 104;42°, 36;29°) and iterating to convergence produces $2e$ = 5;29,33, φ_1 = 77;14,15° φ_2 = 2;50,44° and φ_3 = 30;35, 15°, all virtually identical to Ptolemy's final values. Computing ψ_3 now gives 33;21,53°, and combining with λ_3 gives A = 161;01,07°, again almost identical to Ptolemy's 161;00°. Thus it is again quite likely that somewhere along the line Ptolemy made a small slip, this time even smaller than for Saturn, and in this case in estimating the circumstances of the first opposition.

Note that the correcting of the small errors in Ptolemy's verifications of Jupiter and Saturn is not the result of some wide search for better numbers, but results entirely from numbers that Ptolemy provides most directly. Hence, while it might be imprudent to claim that the cases of Jupiter and Saturn give airtight proofs that Ptolemy was using other than empirical data for his inputs, it may certainly be claimed that the cases of Jupiter and Saturn provide no evidence whatsoever in support of any claim that Ptolemy *was* using empirical observations as the source of his input data.

Thus the issue is no longer whether Ptolemy started with a set of empirical observations, analyzed them far enough to see where things were headed, and then steered his computations to pleasingly round numbers. The issue is rather that Ptolemy started with a set of input data that indeed converges to precisely the round numbers that Ptolemy assumes as his final values. Such a numerical accident is very unlikely for even one planet, and of course completely out of the question for all three. The three cases of computing apogee, where combining two very non-round numbers always produces a round one, is the most compelling argument against Ptolemy's claim that he really relies on empirical input data.

All of this strongly suggests that the times and longitudes of opposition that Ptolemy reports in the *Almagest* are not, in fact, the result of data reduction based on observation. Perhaps the simplest scenario consistent with our analysis is that Ptolemy did indeed begin with some estimates of time and longitude based on observation, and that he did indeed proceed to analyze these with the iterative algorithm far enough to see where things were headed. He then chose round values for 2e and A, and finally used these and the equant model to recompute what his input data needed to be to produce the final round values he had chosen, and it is this tweaked data that we find in the *Almagest*.

An equally tenable alternative scenario is that Ptolemy dispensed with observation altogether and simply used the equant model with *a priori* known, fairly round parameters to *compute* the circumstances of each opposition. This would not be hard to do, since the approximate dates of mean opposition were certainly known *a priori.* One would simply compute longitudes of the planet and the mean Sun a few, perhaps 10, days before and after the expected opposition, and use these to estimate the date of opposition by interpolation. The estimate would certainly be close, but suppose it was slightly early. One would then compute longitudes of the planet and the mean Sun a short time later, but after opposition, and use this new pair to re-estimate opposition. Such a process obviously converges quickly and accurately and requires minimal effort.

It is equally simple to imagine how the round values for 2e and A arose in the first place.[11] If one has a sequence of empirical observations, it is straightforward to look at the difference in longitude of consecutive oppositions and observe that the longitude differences indeed vary as the pair moves around the ecliptic, suggesting that the longitude of minimum pair-wise difference is the direction of apogee for the planet. Having determined apogee, it is straightforward to find the eccentricity and the radius of the epicycle. Or, with somewhat more work, one could just do a trio analysis on a dozen or so trios of oppositions and estimate the most likely values of the parameters, perhaps by taking the median. Under any such scenario, we would certainly expect nicely round values for the parameters, just as Ptolemy quotes.

In the end it is clear that Ptolemy's manipulation of data for the outer planets is completely consistent with his treatment of the inner planets, the Sun and the Moon. In each case he produces purported 'observations' which he claims he made, and which he claims are the empirical basis for his model parameters. And in each case those claims have been found to be not the case. Especially in the case of the outer planets, the original source of the model parameters must be considered as unknown.

Appendix A

1. Using three oppositions to find the eccentricity and apsidal direction for the bisected equant model

Ptolemy's method for solving this problem is based on classical geometric analysis, and generally works in terms of angles in the range 0° to 90, so that the algebraic signs of terms are never negative. This requires that each configuration of input angles be analyzed by inspection. The solution presented below is mathematically, but not step-by-step, equivalent, and allows the angles to have any values and any combination of algebraic signs. It is thus far more convenient for computer calculation.[12]

First consider an eccentric deferent model. We have a deferent circle of unit radius centered at point F. Let point E be the position of the Earth, so that the apogee A lies on the extension of the line EF. The length of EF is $2e$ (in anticipation of the equant). Let λ_1, λ_2 and λ_3 be three longitudes of the planet at opposition, and let points Q_1, Q_2 and Q_3 be the center of the planet's epicycle at the times t_1, t_2 and t_3 of the oppositions, so that the lines FQ_i, i = 1,3 are all (unit) radii of the deferent circle. Let angle $AFQ_2 = \varphi$, angle $AEQ_2 = \psi$, angle $FQ_2E = \zeta = \varphi - \psi$, angle $Q_1FQ_2 = \alpha$, angle $Q_2FQ_3 = \beta$, angle $Q_1EQ_2 = \gamma$, and angle $Q_2EQ_3 = \delta$.

Given the mean speed ω of the planet's epicycle center on the deferent and the times of the oppositions, the angles α and β are determined by $\alpha = \omega(t_2 - t_1)$ and $\beta = \omega(t_3 - t_2)$.

The angles γ and δ are determined by $\gamma = \lambda_2 - \lambda_1$ and $\delta = \lambda_3 - \lambda_2$. We have $\varphi = \zeta + \psi$, so the problem is to determine e, ψ, and ζ in terms of α, β, γ, and δ.

Applying the law of sines to triangle EFQ_2 gives $2e\sin\psi = \sin\zeta$. Similarly triangle EFQ_1 gives $2e\sin(\psi-\gamma) = \sin(\zeta-\alpha+\gamma)$ and triangle EFQ_3 gives $2e\sin(\psi+\delta) = \sin(\zeta+\beta-\delta)$. Expanding these latter two equations gives

$$2e\sin\psi\cos\gamma - 2e\cos\psi\sin\gamma = \sin\zeta\cos(\alpha-\gamma) - \cos\zeta\sin(\alpha-\gamma)$$

and

$$2e\sin\psi\cos\delta + 2e\cos\psi\sin\delta = \sin\zeta\cos(\beta-\delta) + \cos\zeta\sin(\beta-\delta),$$

and letting $x = 2e\cos\psi$, $y = \sin\zeta = 2e\sin\psi$, and $z = \cos\zeta$, we get

$$y\cos\gamma - x\sin\gamma = y\cos(\alpha-\gamma) - z\sin(\alpha-\gamma)$$

and

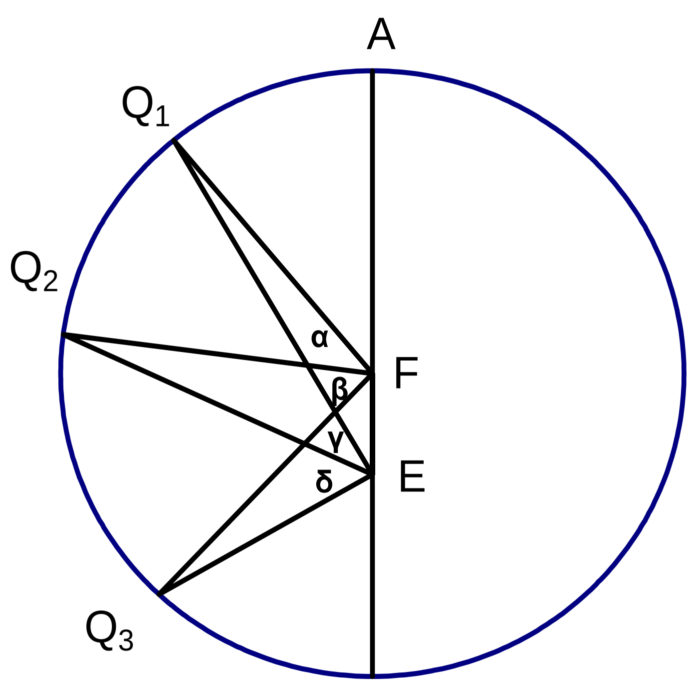
$$y\cos\delta + x\sin\delta = y\cos(\beta-\delta) + z\sin(\beta-\delta).$$

From these we get

$$\tan\psi = \frac{y}{x} = \frac{\sin\delta\sin(\alpha-\gamma) - \sin\gamma\sin(\beta-\delta)}{\sin(\alpha+\beta-\gamma-\delta) - \cos\gamma\sin(\beta-\delta) - \cos\delta\sin(\alpha-\gamma)}$$

$$\tan\zeta = \frac{y}{z} = \frac{\sin\delta\sin(\alpha-\gamma) - \sin\gamma\sin(\beta-\delta)}{\sin\gamma\cos(\beta-\delta) + \sin\delta\cos(\alpha-\gamma) - \sin(\gamma+\delta)}$$

The signs in the equations for $\tan\zeta$ and $\tan\psi$ are such that if the denominator in the equation for $\tan\zeta$ is positive then ζ and ψ will be in the correct quadrant (e.g. when using the atan2(y,x) function in some programming languages). If that denominator is not positive, then reverse the signs in the numerators and denominators of both equations.

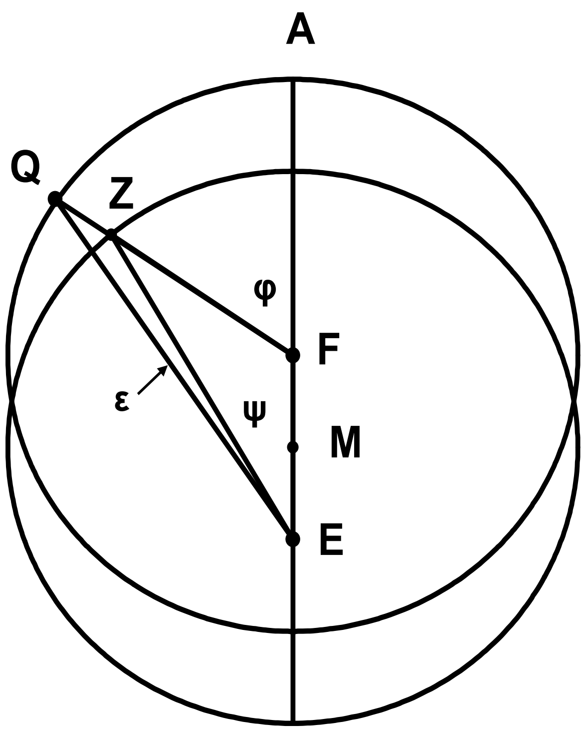


Finally,

$$\varphi = \zeta + \psi \qquad \text{and} \qquad 2e = \frac{\sin\zeta}{\sin\psi}$$

and the longitude of the apogee at time t_2 is $\lambda_A = \lambda_2 - \psi$.

In order to correct for the fact that the equant point F is not really at the center of the deferent, an iterative algorithm is used.



Consider a new point M a distance *e* from E in the direction of point A, hence bisecting EF, and let M be the center of a (second) deferent circle of unit radius. The planet's epicycle will be centered at point Z on this new circle. The extension of the line FZ intersects the original circle at point Q. Since motion is still considered uniform about the point F, we still have angle AFZ = φ and now angle AMZ = χ. Angle AEZ is still ψ, so we have angle FZM = $\varphi - \chi$ and angle MZE = $\chi - \psi$. Let angle ZEQ = ε and let angle MEQ = ψ'. Then letting Z be the three planetary positions in turn, we proceed as follows, with the various subscripts denoting the three planetary positions: let $\psi_1 = \psi - \gamma$,

$\psi_2 = \psi$, $\psi_3 = \psi - \delta$, and let $\varphi_1 = \varphi - \alpha$, $\varphi_2 = \varphi$, and $\varphi_3 = \varphi - \beta$. Then

$\chi_i = \phi_i - \sin^{-1}(e \sin \phi_i)$, i = 1,3, and we estimate ε_i from

$$\tan \varepsilon_i = \frac{e \sin \psi_i - \sin(\chi - \psi_i)}{e \cos \psi_i + \cos(\chi - \psi_i)},$$

which we use to find $\psi_2' = \psi + \varepsilon_2$. We then 'correct' the input values of γ and δ according to

$$\text{new } \gamma = \text{ original } \gamma + (\varepsilon_2 - \varepsilon_1)$$

and

$$\text{new } \delta = \text{ original } \delta + (\varepsilon_3 - \varepsilon_2).$$

The iteration continues by restarting the algorithm described in the first part of this section with the new values of γ and δ (note that the original values never change), and continues until the values of ε_1, ε_2, and ε_3, and hence e and λ_A, stabilize.

Appendix B

In some cases the trio algorithm discussed in Appendix A can be extremely sensitive to small variations in the values of the input variables α, β, γ, and δ. Ptolemy is certainly generally aware of the numerical sensitivity in these trio analyses, and in fact uses it in *Almagest* 4.11 to explain how relatively small errors in the computed intervals of time and longitude led Hipparchus astray while analyzing two lunar trios.

The tables below show the changes in the second sexagesimal place of the output variables φ_3, ψ_3, $2e$, and A (thus in units of arcseconds for the angles) that result from a change from Ptolemy's values of one unit in the first sexagesimal place (thus 1´) of the input angles α, β, γ, and δ:

Mars	ψ_3	φ_3	$2e$	A
α	0,08	0,39	0,47	-0,08
β	3,22	2,57	0,01	3,20
γ	0,10	0,46	-0,56	0,10
δ	3,57	3,28	-0,01	-3,57
Jupiter	ψ_3	φ_3	$2e$	A
α	7,31	6,52	-0,10	7,21
β	13,18	12,21	-1,34	-12,32
γ	7,40	7,00	0,12	-7,42
δ	10,35	10,30	1,30	11,30
Saturn	ψ_3	φ_3	$2e$	A
α	6,50	7,50	0,40	-8,45
β	15,05	15,45	0,35	15,00
γ	7,15	8,15	-0,40	7,15
δ	17,14	17,50	-0,40	-16,10

Thus, for example for Jupiter, a change in γ of 1´ from 104;43° to 104;42° (as in fact happens at one point in the discussion in the main text) changes the longitude of apogee by about 7⅔´. For Saturn, if one decreases γ by 1´ and increases δ by 1´(as also happens in the discussion above), then the longitude of apogee changes by about 23½´. This extreme level of sensitivity for Jupiter and Saturn, compared to Mars, is the result of two factors. First, the size of the eccentricity for Mars is about twice as large as for Jupiter or Saturn, and it is relatively easier to determine the direction of a long line segment than of a short line segment. Second, for Mars the oppositions are strategically placed near octants, while for Jupiter and Saturn the second opposition is too close to the direction of the apsidal line.[13]

This numerical sensitivity has consequences for Ptolemy's entire program of using just five data points to determine the parameters of each outer planet. This can be illustrated by the following sequence of computational exercises.

First we use the *Almagest* models for Mars and the mean Sun to compute very accurately the time and longitudes of the oppositions that Ptolemy used in years 130, 135, and 139. We also compute the position of Mars at two other times: 3^d later and 410^y234^d earlier than the moment of the third opposition, corresponding roughly to Ptolemy's time intervals for his fourth and fifth observations. When this data is analyzed according to Ptolemy's protocol, one gets the values $2e$ = 11;59,39, r = 39;30,00, and ω_a =

$0;27,41,40,19,25,50^{o/d}$, and the epoch values for mean longitude, anomaly and apogee are 3;31,40°, 327;13,20°, and 106;38,33°, resp. When these values are used to recompute the longitudes of Mars at the five times used, the comparison is as follows:

time	input	recomputed
1768888.541666	80;58,54°	80;57, 8°
1770418.326979	148;47,31°	148;46,41°
1771974.420828	242;31,59°	242;32,12°
1771977.420828	241;34, 0°	241;34,13°
1622093.420828	212;39,26°	212;39,26°

The agreement in longitude for the fifth observation is exact because Ptolemy's protocol uses the extrapolation from the fifth observation to find his epoch values. The small disagreements for the first four observations result from Ptolemy's neglect of the slow movement of the apsidal line at that stage of his calculations. Note that even with this degree of care, the errors in the first and second longitudes are 1-2´, and so would affect Ptolemy's calculations, which are done to a precision of at least 1´.

As a second exercise, we can use as input the times and longitudes reported by Ptolemy for each of the five observations, and proceed as above to execute his protocol without calculation errors. One gets the values $2e$ = 11;59,34, r = 39;34,53, and ω_a = $0;27,41,40,09,28,16^{o/d}$, and the epoch values for mean longitude, anomaly and apogee are

3;24,06°, 327;20,54°, and 106;36,52°, resp. When these values are used to recompute the longitudes of Mars at the five times used, the comparison is as follows:

time	input	recomputed
1768888.54167	81;00°	80;39,50°
1770418.37500	148;50°	148;28,33°
1771974.41667	242;34°	242;04,14°
1771977.35903	241;36°	241;06,08°
1622092.75000	212;15°	212;15,00°

Once again the agreement with the fifth observation is guaranteed, but the errors in the recomputed values of the first four observations are now much more substantial, typically 30´ or so.

As a final exercise we can use data from real oppositions of Mars with the real mean Sun.[14] Because of Ptolemy's pervasive systematic error one finds longitudes about 1° larger than before, but the times are still roughly the same. One gets the values 2*e* = 11;39,10 , *r* = 39;26,15, and ω_a = 0;27,41,40,16,44,18$^{o/d}$, and the epoch values for mean longitude, anomaly and apogee are 3;33,15°, 327;11,45°, and 111;13,16°, resp. When these values are used to recompute the longitudes of Mars at the five times used, the comparison is as follows:

time	input	recomputed
1768888.11938	81;43,27°	82;19, 8°
1770418.62428	150;15,41°	150;53,42°
1771973.90440	242;31,59°	241;42,31°
1771976.90440	241;34, 0°	240;44,25°
1622092.90440	212;42, 0°	212;42, 0°

As above the agreement with the fifth observation is guaranteed, but now the errors in the recomputed values of the first four observations are even larger, typically about 40´.

From these examples we see that if you use input data that are known to be consistent with the theoretical model in question (in this case the *Almagest* equant model), then Ptolemy's analysis protocol is nearly self-consistent (and could be made completely so by adding a moving apsidal line into the analysis). But if one uses any other kind of data, then one will in general *not* find self-consistent results.

In order to understand why this happening, it is useful to do one last computational exercise. Let us find all real oppositions of Mars with the real mean sun over some substantial time frame, e.g. the interval 0 AD – 150 AD. For each such opposition we also generate two additional real observations, the first 3 days later, the second 410^y234^d earlier. We then form successive trios of alternate oppositions, e.g. nos. 1, 3, 5, nos. 3, 5, 7, etc., just as Ptolemy did with his single trio. Each of these trios is analyzed using Ptolemy's protocol. The results are shown in Figures 1 and 2. One notices a striking dependence of the eccentricity and epicycle radius upon longitude. This variation in effective parameter values is caused by the mismatch between the patterns of speed and distance variation of the planet in its orbit while following equant motion and Kepler motion, a crucial point first understood by Kepler himself.[15] There is certainly no *single* best set of parameters for the equant model, within the variance displayed of about 4% in $2e$ and 2.5% in r. Thus Ptolemy's data analysis protocol is unavoidably mixing varying effective parameters in a way completely beyond his control.

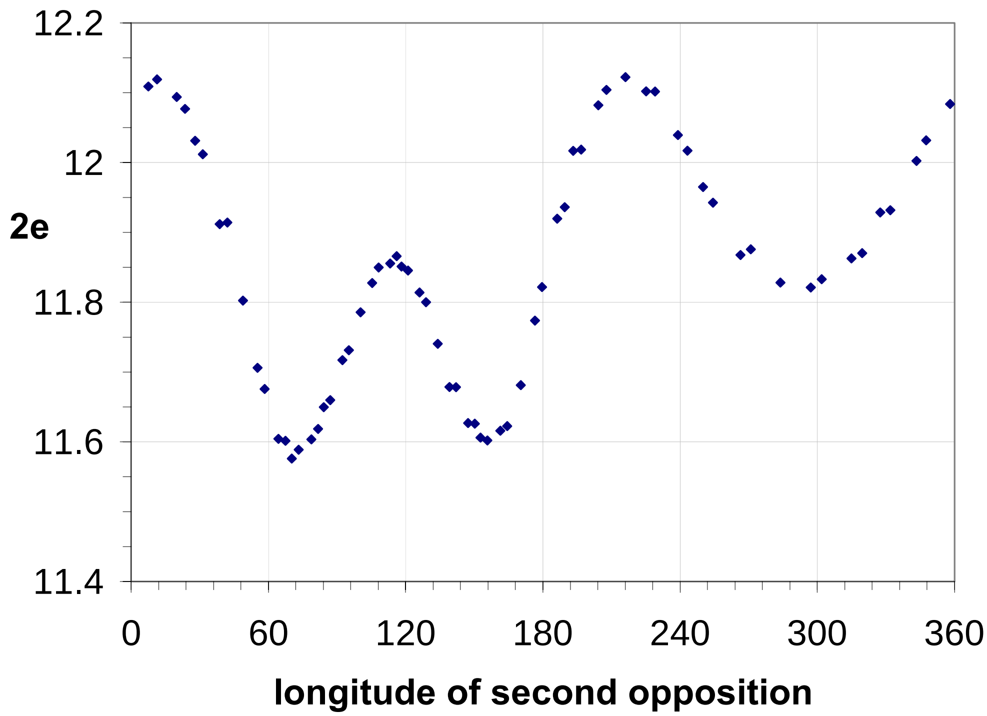

12.2
12
11.8
11.6
11.4
2e
0
60
120
180
240
300
360
longitude of second opposition

Figure 1

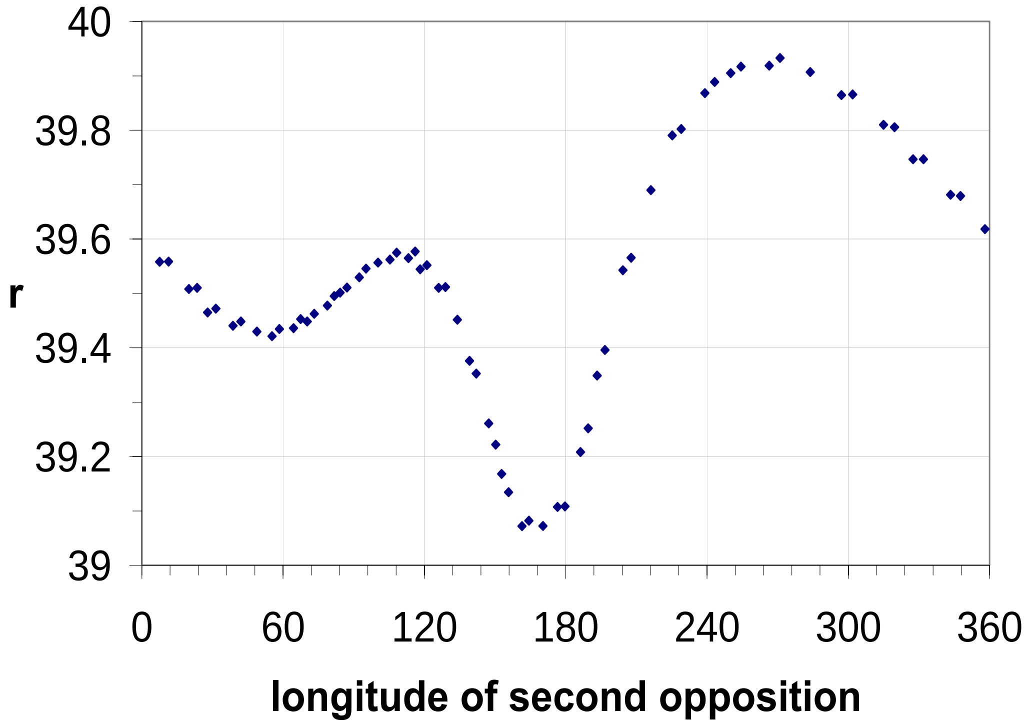

40
39.8
39.6
r
39.4
39.2
39
0
60
120
180
240
300
360
longitude of second opposition

Figure 2

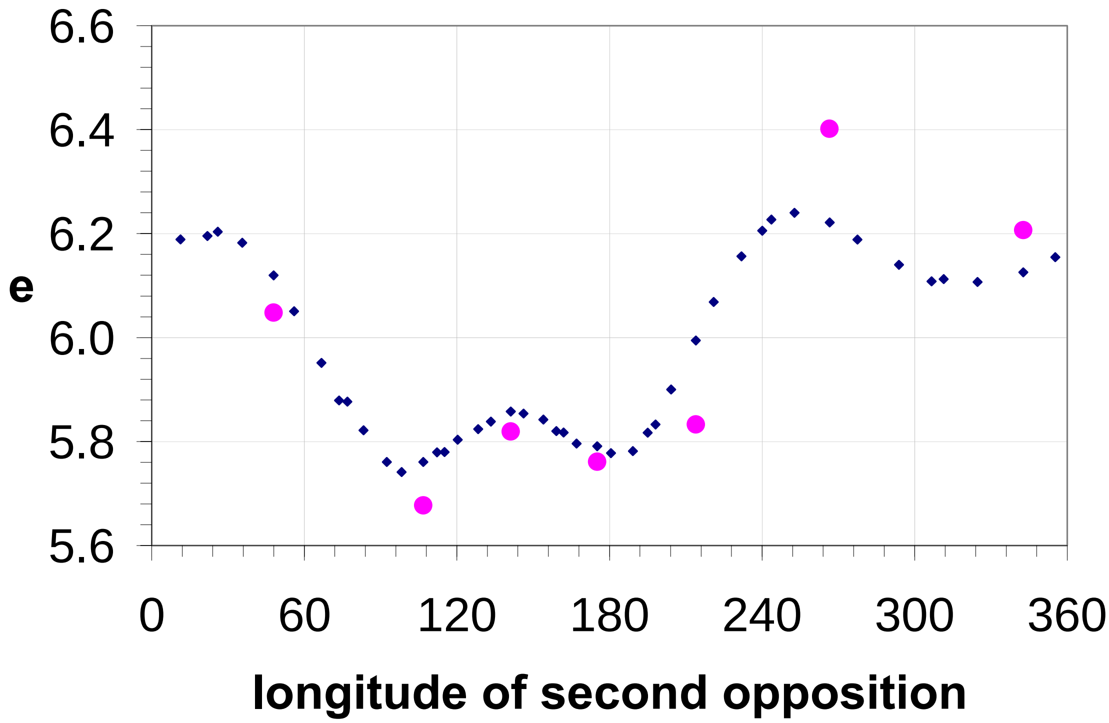

6.6
6.4
6.2
e
6.0
5.8
5.6
0
60
120
180
240
300
360
longitude of second opposition

Figure 3.

References

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[1] *Ptolemy's Almagest*, transl. by G. J. Toomer (London, 1984).

[2] J. P. Britton, *Models and Precision: the quality of Ptolemy's observations and parameters*, (Princeton, 1992) and derived from his unpublished 1966 Yale PhD thesis; C. Wilson, "The Inner Planets and the Keplerian revolution", *Centaurus*, 17 (1972) 205-248; R. R. Newton, *The crime of Claudius Ptolemy*, (Baltimore, 1977); N. M. Swerdlow, "Ptolemy's Theory of the Inferior Planets", *Journal for the history of astronomy*, 20 (1989) 29-60.

[3] Indeed, except for the Sun there seems no particular reason to assume that Ptolemy even knew how the various model parameters were originally determined.

[4] Technically, Ptolemy determines the longitude of the 4th observation by sighting each planet both with respect to a star and to the Moon. As always, he gets exactly the same longitude from both determinations.

[5] Newton, *op. cit.* (Ref. 2) found $2e = 12.00036$ by direct numerical solution of the equant model, thus avoiding any iteration steps.

[6] Hugh Thurston, "Ptolemy's Backwardness", *DIO* 4.2 (1994) 58-60.

[7] This translation was provided by Alexander Jones.

[8] Actually, Ptolemy always gives positive angles referred to either apogee or perigee, whichever results in an angle less than 90°. In order to facilitate comparison I have consistently converted my results for φ_1, φ_2, and φ_3 to Ptolemy's conventions.

[9] It is curious that Ptolemy would choose a fourth observation only three days after an opposition, since the radius of Mars' epicycle, which advances in anomaly at about 1/2° per day, will still be pointing nearly straight at the observer. Detailed analysis shows that

if Ptolemy's observed longitude for the fourth observation changes by 1/4°, his calculated value for the radius would change from 39;30 to about 39;00.

[10] See Appendix B for a detailed discussion of the sensitivity of Ptolemy's calculations to changes in his input data.

[11] James Evans, *The History and Practice of Ancient Astronomy*, (New York, 1998), p. 362-368.

[12] All the derivations in this appendix follow closely K. Stumpff, *Himmelsmechanik*, Bd. 1, Berlin, 1956. The original geometrical derivations are clearly explained in O. Pedersen, *A Survey of the Almagest*, Odense, 1974; O. Neugebauer, *A history of ancient mathematical astronomy*, (3 vols., Berlin, 1975); H. Thurston, *Early Astronomy*, New York, 1994.

[13] The effect of numerical errors that we are discussing here is several steps removed, and considerably simpler to discuss, than the effect of errors in the sequence of time and planetary longitude pairs that must be made to determine the time of opposition. Since the longitude of opposition is dependent upon the determined time, either directly through a solar model or less directly through interpolation of measurements, the errors in time and longitude of opposition will be correlated, and ideally speaking, only the measured time is a true independent variable. If the correlation is less than 100%, which will generally be the case, then the analysis is even more complicated.

[14] These are computed using programs of Pierre Bretagnon and Jean-Louis Simon, *Planetary Programs and Tables from –4000 to +2800*, (Richmond, 1986). The mean Sun is taken from Jean Meeus, *Astronomical Algorithms,* (Richmond, 1998) p. 212.

[15] Figure 3 shows the result of analyzing a similar set of oppositions from Kepler's time, along with the results (large dots) of analyzing the ten oppositions measured by Tycho Brahe and used by Kepler in *Astronomia Nova*. The effect we are looking at appears to be just barely noticeable, remembering that Kepler would not have had the benefit of the trail of small dots in Fig. 3 to guide his eye. In any event, there is no record of Kepler doing such an analysis.