Who Knew What, and When? The Timing of Discoveries in Early Greek Astronomy

Dennis Duke
Department of Physics and School of Computational Science
Florida State University

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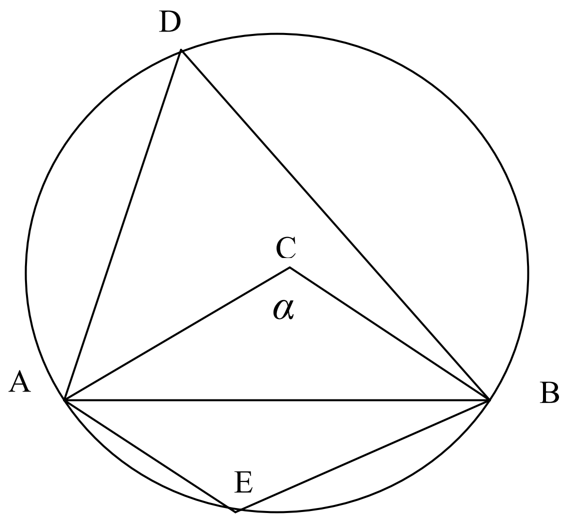
Two questions:

- was sophisticated plane trigonometry used by Hipparchus?
- does the equant precede Ptolemy and the *Almagest*?

Perhaps surprisingly, both questions are best answered by looking at 500 A.D. astronomy texts from India.

Neugebauer (*PASP*, 1972)



$$\mathrm{crd}\,\alpha = \frac{AB}{R'} = 2\sin\frac{\alpha}{2}$$

$$R' = \frac{360^\circ \cdot 60'/^\circ}{2\pi} = \frac{21,600}{2\pi} \simeq 3438$$

or

$$D' = \frac{21600}{\pi} \simeq 6875$$

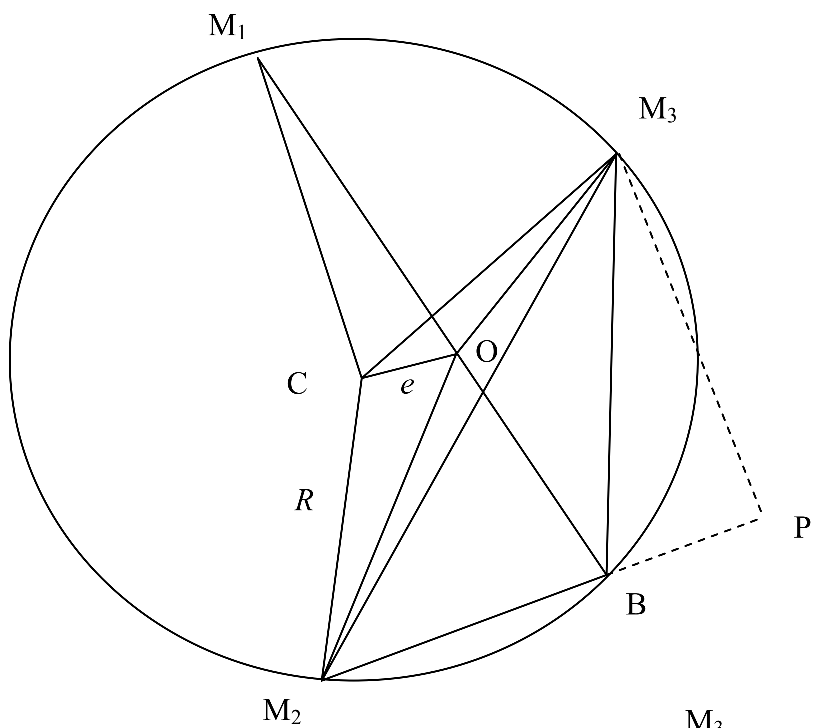
Table of chords

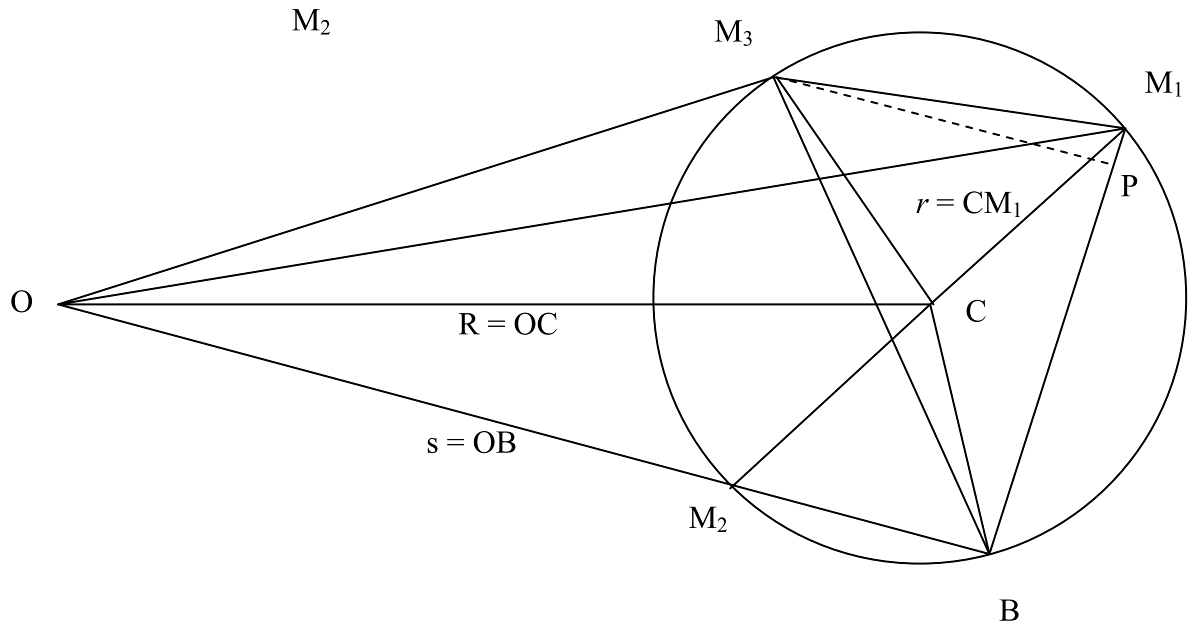
Angle(degrees)	Chord
0	0
7 ½	450
15	897
22 ½	1341
30	1780
37 ½	2210
45	2631
52 ½	3041
60	3438
67 ½	3820
75	4186
82 ½	4533
90	4862
97 ½	5169
105	5455
112 ½	5717
120	5954
127 ½	6166
135	6352
142 ½	6511
150	6641
157 ½	6743
165	6817
172 ½	6861
180	6875

Toomer (*Centaurus*, 1973)

$R/e = 3144 / 327\frac{2}{3}$ $R / r = 3122\frac{1}{2} / 247\frac{1}{2}$.

from *Almagest* 4.11 and Hipparchus.





$$M_{III}P = \frac{M_{III}B \text{ Crd } 2\frac{\alpha_{II}}{2}}{2R'} = \frac{M_{III}B \text{ Crd } 153;25°}{2R'} = s \cdot \frac{5379 \cdot 6688}{3247 \cdot 2 \cdot 3438}.$$

$$M_{II}P = M_{II}B + BP = M_{II}B + \frac{M_{III}B \text{ Crd } 26;35°}{2R'}$$

$$= s\left(\frac{3110\frac{1}{2}}{6574} + \frac{5379 \cdot 1580}{3247 \cdot 2 \cdot 3438}\right) = s \cdot \frac{\dfrac{3110\frac{1}{2} \cdot 3438}{6574} + \dfrac{5379 \cdot 1580}{3247 \cdot 2}}{3438}.$$

$$M_{II}M_{III} = \sqrt{M_{III}P^2 + M_{II}P^2} = s \cdot \frac{6268}{3438}. \quad (1)$$

$$R = \frac{M_{II}M_{III} \cdot R'}{\text{Crd } \alpha_{II}} = s \cdot \frac{6268 \cdot 3438}{3438 \cdot 6688} = s \cdot \frac{6268}{6688} = s \cdot \frac{3134}{3344}.$$ [14]

$$\text{Crd } M_{III}\hat{C}B = \frac{M_{III}B \cdot R'}{R} = \frac{s \cdot 5379 \cdot 3438 \cdot 3344}{s \cdot 3247 \cdot 3134} = 6078.$$

$$M_{III}\hat{C}B = 124;24°.$$

$$M_{I}\hat{C}B = M_{III}\hat{C}B + \alpha_{III} = 124;24° + 46;36° = 171°.$$

$$M_{I}B = \frac{R \text{ Crd } 171°}{R'} = s \cdot \frac{3134 \cdot 6853}{3344 \cdot 3438}.$$

$$R^2 - e^2 = (R+e)(R-e) = s \cdot M_{I}O = s\,(M_{I}B - s)$$

$$= s^2\left(\frac{3134 \cdot 6853 - 3344 \cdot 3438}{3344 \cdot 3438}\right).$$

$$\frac{R^2}{e^2} = \frac{R^2}{R^2 - s\,(M_{I}B - s)}.$$

$$\frac{R}{e} = \frac{R}{\sqrt{R^2 - s\,(M_{I}B - s)}} = \frac{\dfrac{3134}{3344}}{\sqrt{\left(\dfrac{3134}{3344}\right)^2 - \dfrac{3134 \cdot 6853 - 3344 \cdot 3438}{3344 \cdot 3438}}}.$$

Multiplying through by 3344,

$$\frac{R}{e} = \frac{3134}{\sqrt{3134^2 + 3344^2 - \dfrac{3134 \cdot 3344 \cdot 6853}{3438}}}$$ =3134/338

However, Toomer made an error in analyzing the epicycle case, and expressed doubt about the entire thesis. However, Toomer overlooked something:

The problem is to find the ratio of the radius of the deferent, R (= OC), to the radius of the epicycle r. This is achieved via the intermediate step of finding r in terms of s, s being the distance OB between O and the place where the straight line from O to one of the points M meets the circle again in B. The solution is as follows (R' is throughout the radius of the base circle, or 3438′; *Crd* is the chord expressed in terms of that base circle).

$$O\widehat{M_1}B = \tfrac{1}{2}\alpha_1 - \delta_1 = 84;25^\circ - 8;22^\circ = 76;3^\circ.$$

$$O\widehat{M_3}B = 180^\circ - \tfrac{1}{2}\alpha_2 - \delta_2 = 180^\circ - 69;49\tfrac{1}{2}^\circ - 9;19^\circ = 100;52\tfrac{1}{2}^\circ$$

$$M_1B = s\,\frac{\text{Crd } 2\delta_1}{\text{Crd } 2\,O\widehat{M_1}B} = s\,\frac{\text{Crd } 16;44^\circ}{\text{Crd } 152;6^\circ} = s\cdot\frac{1000}{6669\frac{1}{3}}.$$

$$M_3B = s\,\frac{\text{Crd } 2\delta_2}{\text{Crd } 2\,O\widehat{M_3}B} = s\,\frac{\text{Crd } 18;38^\circ}{\text{Crd } 201;45^\circ} = s\,\frac{\text{Crd } 18;38^\circ}{\text{Crd } 158;15^\circ} = s\cdot\frac{1112\frac{1}{2}}{6750\frac{1}{2}}.$$

$$M_3P = \frac{M_3B \text{ Crd } 2\frac{\alpha_3}{2}}{2R'} = \frac{M_3B \text{ Crd } 51;33^\circ}{2R'} = s\cdot\frac{1112\frac{1}{2}\cdot 2989}{6750\frac{1}{2}\cdot 2\cdot 3438} = s\cdot\frac{246\frac{1}{3}}{3438}.$$

$$M_1P = M_1B - \frac{M_3B \text{ Crd } 2\left(\frac{180^\circ - \alpha_3}{2}\right)}{2R'} = M_1B - \frac{M_3B \text{ Crd } 128;27^\circ}{2R'}$$

$$= s\left(\frac{1000}{6669\frac{1}{3}} - \frac{1112\frac{1}{2}\cdot 6189\frac{1}{2}}{6750\frac{1}{2}\cdot 2\cdot 3438}\right) = s\left(\frac{515\frac{1}{2} - 510\frac{1}{6}}{3438}\right) = s\cdot\frac{5\frac{1}{3}}{3438}.$$

$$M_1M_3 = \sqrt{M_3P^2 + M_1P^2} = s\cdot\frac{246\frac{1}{3}}{3438}. \qquad (1)$$

$$r = \frac{M_1M_3\cdot R'}{\text{Crd } \alpha_3} = s\cdot\frac{246\frac{1}{3}}{3438}\cdot\frac{3438}{2989} = s\cdot\frac{246\frac{1}{3}}{2989}.$$

=3162 x 231 3/4 / 2960 2/5

$$\text{Crd } \widehat{M_3C}B = \frac{M_3B\cdot R'}{r} = \frac{s\cdot 1112\frac{1}{2}\cdot 2989\cdot 3438}{s\cdot 6750\frac{1}{2}\cdot 246\frac{1}{3}} = 6875.$$

=247 1/2

$$\widehat{M_3C}B = 180^\circ. \qquad (2)$$

(accurate calculation with modern sine tables gives 179° here).

$$\widehat{M_2C}B = \widehat{M_3C}B - \alpha_2 = 180^\circ - 139;37^\circ = 40;23^\circ. \qquad (3)$$

$$BM_2 = \frac{r \text{ Crd } 40;23^\circ}{R'} = s\cdot\frac{246\frac{1}{3}\cdot 2372}{2989\cdot 3438}.$$

But where does 3162 come from?

In Indian astronomy

$$\pi = \frac{62832}{20000} = 3.1416$$

and

$$\pi \simeq \sqrt{10} \simeq 3.1622...$$

This suggests that Hipparchus was using a circle of circumference 10,000 (*i.e.* a Greek myriad), and hence a diameter of

$$s = \frac{10,000}{\pi} = \frac{10,000}{\sqrt{10}} = 1,000\sqrt{10} \simeq 3162 \quad 1)$$

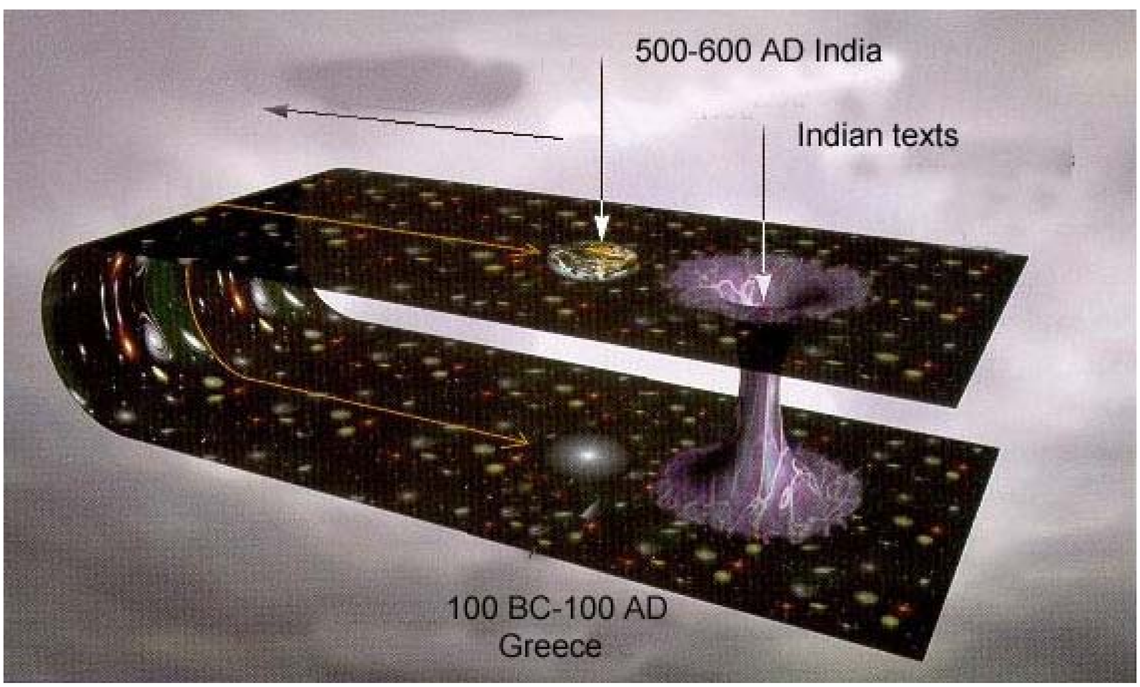
So Toomer was correct after all.

In fact, there are many ideas common to Greek and Indian astronomy, but Indian astronomy is generally less advanced than the *Almagest*:

- *The equation of time*.
- *Obliquity of the ecliptic*.
- *Parallax*.
- *Trigonometry scales*.
- *Retrograde motion*.
- *Model of Mercury*.
- *Determination of orbit elements*.
- *Values of orbit elements*.
- *Star catalog*.
- *Zodiacal signs*.
- *The second lunar anomaly*.

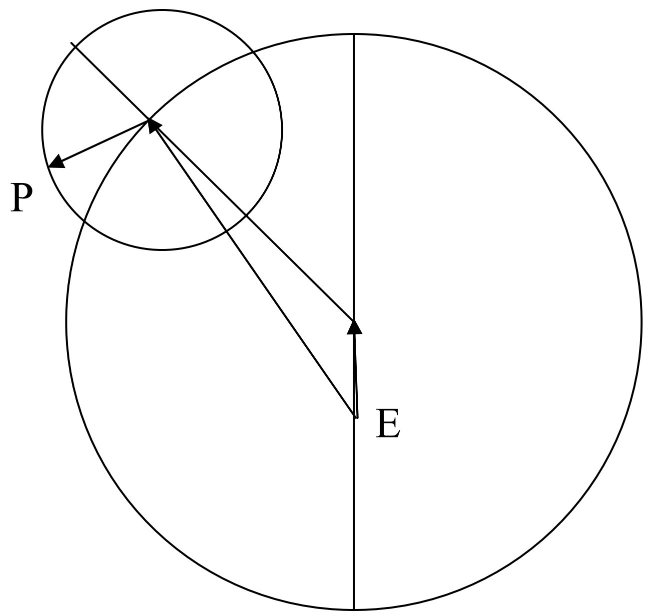
This has led to what might be called the Pingree – van der Waerden Hypothesis:

> *The texts of ancient Indian astronomy give us a sort of wormhole through space-time back into an otherwise inaccessible era of Greco-Roman developments in astronomy.*

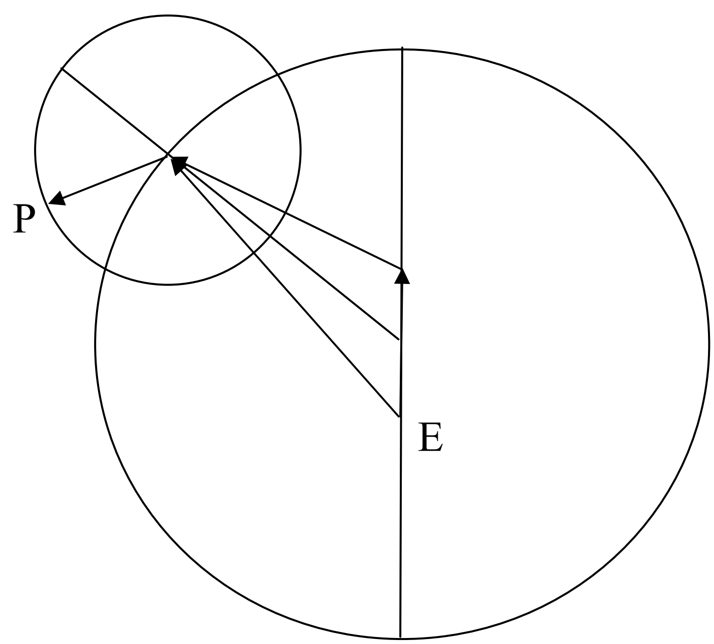


Thus the essentially universally accepted view that the astronomy we find in the Indian texts is pre-Ptolemaic. Summarizing the prevailing opinion, Neugebauer wrote in 1956:

> "Ptolemy's modification of the lunar theory is of importance for the problem of transmission of Greek astronomy to India. The essentially Greek origin of the Surya-Siddhanta and related works cannot be doubted – terminology, use of units and computational methods, epicyclic models as well as local tradition – all indicate Greek origin. But it was realized at an early date in the investigation of Hindu astronomy that the Indian theories show no influence of the Ptolemaic refinements of the lunar theory. **This is confirmed by the planetary theory, which also lacks a characteristic Ptolemaic construction, namely, the "*punctum aequans,*"** to use a medieval terminology".



eccentric plus epicycle model



equant plus epicycle model

Indian Planetary Theories

eccentric orbits (*manda*)

$$\sin q(\alpha) = -e \sin \alpha$$

epicycles (*sighra*)

$$\tan p(\gamma) = \frac{r \sin \gamma}{1 + r \cos \gamma}$$

(1)	$\alpha = \overline{\lambda} - \lambda_A$	$\nu_1 = \overline{\lambda} + \frac{1}{2} q(\alpha)$
(2)	$\gamma = \overline{\lambda}_S - \nu_1$	$\nu_2 = \nu_1 + \frac{1}{2} p(\gamma)$
(3)	$\alpha = \nu_2 - \lambda_A$	$\nu_3 = \overline{\lambda} + q(\alpha)$
(4)	$\gamma = \overline{\lambda}_S - \nu_3$	$\lambda = \nu_3 + p(\gamma)$

Aryabhata's text says:

half the *mandaphala* obtained from the apsis is minus and plus to the mean planet. Half from the *sigraphala* is minus and plus to the manda planets. From the apsis they become *sphutamadhya* [true-mean]. From the *sigraphala* they become *sphuta* [true].

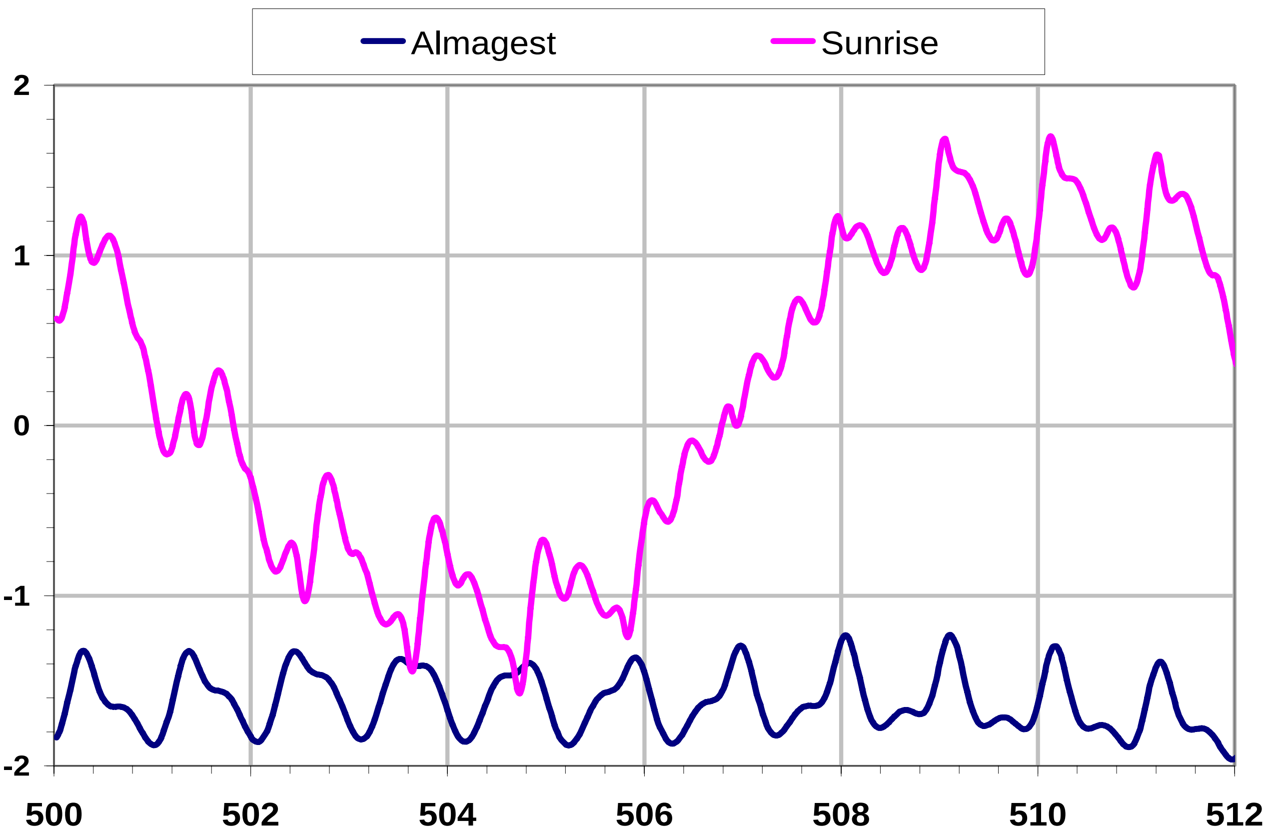
Pingree wrote in 1971:

> "The orbits of the planets are concentric with the center of the earth. The single inequalities recognized in the cases of the two luminaries are explained by *manda*-epicycle (corresponding functionally to the Ptolemaic eccentricity of the Sun and lunar epicycle, respectively), the two inequalities recognized in the case of the five star-planets by a *manda*-epicycle (corresponding to the Ptolemaic eccentricity) and a *sighra*-epicycle (corresponding to the Ptolemaic epicycle). *The further refinements of the Ptolemaic models are unknown to the Indian astronomers.*"

and again in 1980:

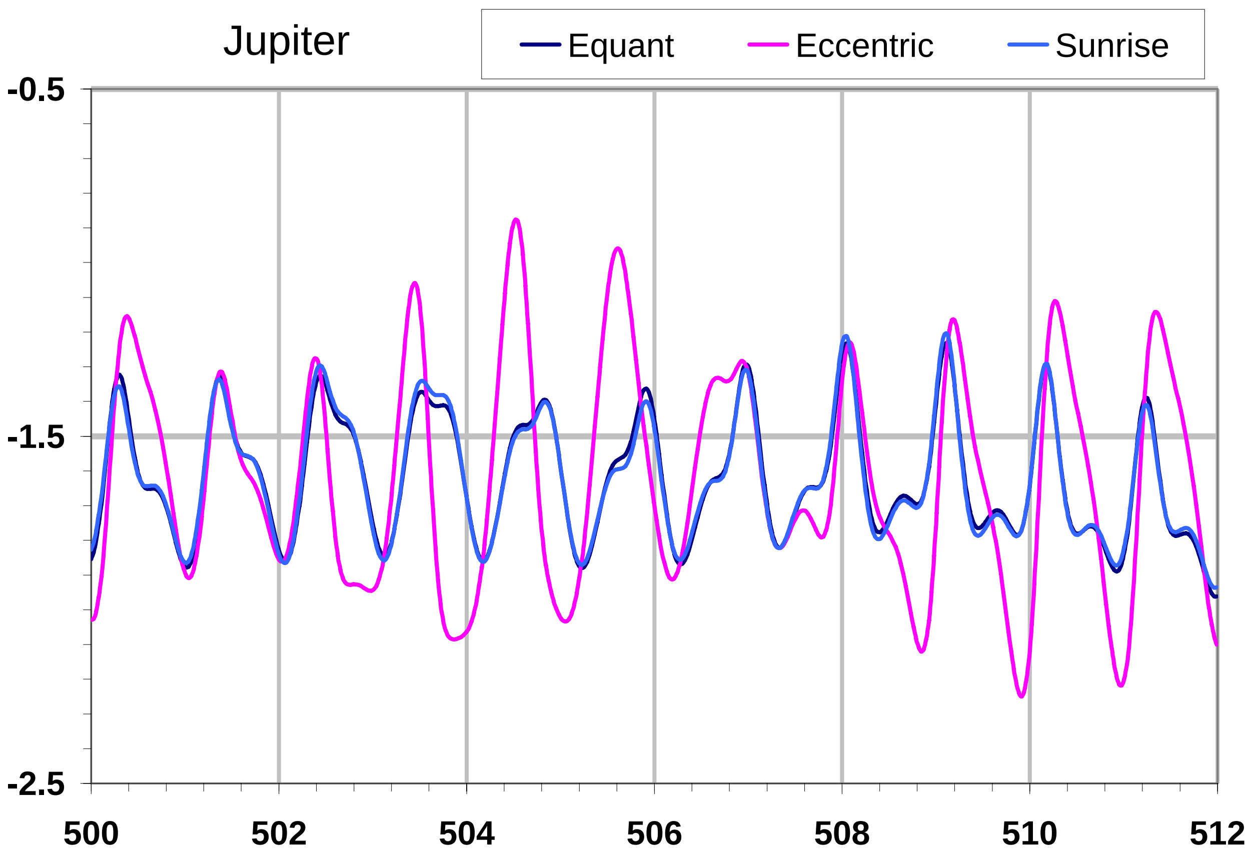
> "The Indians' geometrical models for computing the corrections to the planets' mean longitudes, while derived from Greek sources, *are crude in comparison to Ptolemy's*…..The dimensions of the epicycles of the five planets selected by Aryabhata I are unrelated to Ptolemy's eccentricities and epicycles, as is clear from Table 4 in that same article; and Aryabhata I *has nothing corresponding to the Ptolemaic equant*."

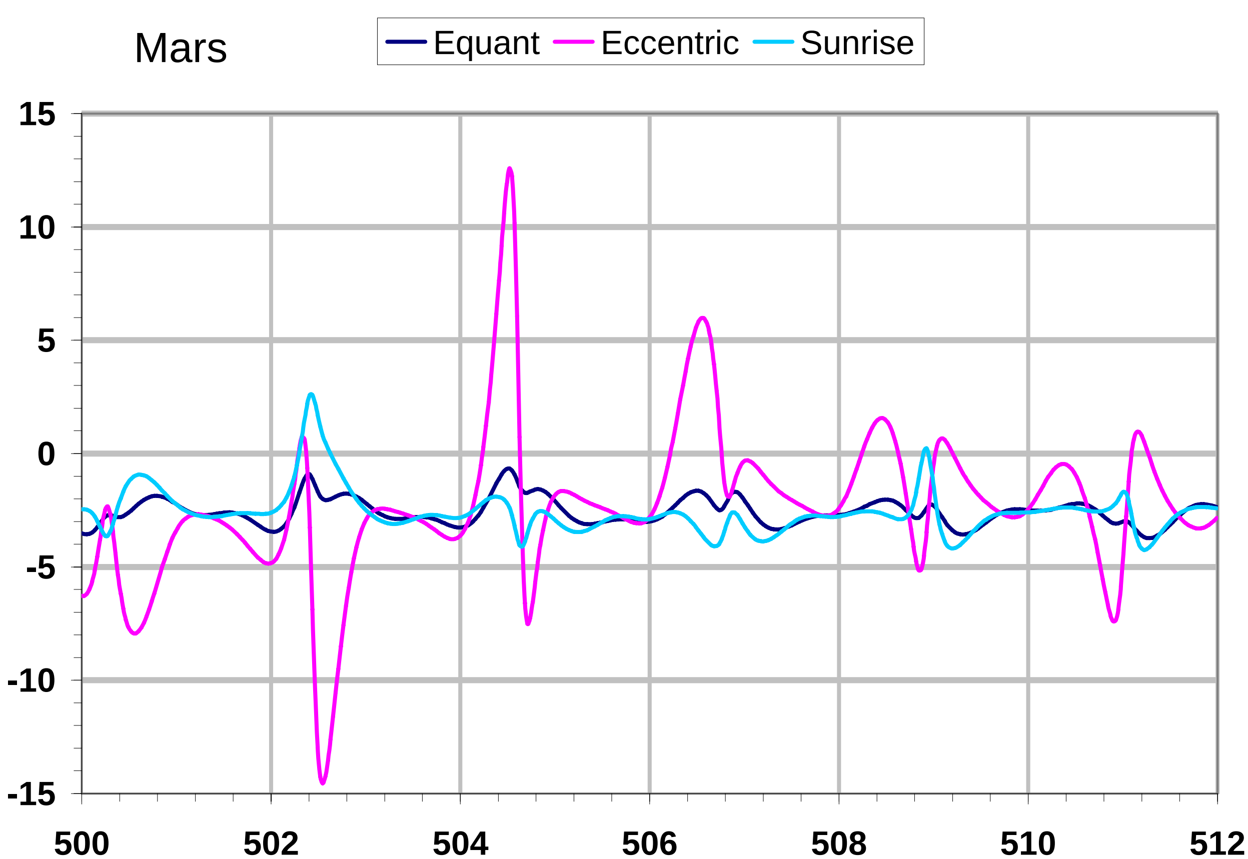
Jupiter



Most of the difference is due to poor orbit elements in the Sunrise model.

What happens if we use *identical* orbit elements in both models?


Jupiter
Equant
Eccentric
Sunrise
-0.5
-1.5
-2.5
500
502
504
506
508
510
512


Mars
Equant
Eccentric
Sunrise
15
10
5
0
-5
-10
-15
500
502
504
506
508
510
512

Therefore, it is clear that the *Almagest* equant and the Indian models share the same mathematical basis.

Let's do an audience survey by a show of hands:

Who knows, apart from anything I have told you, that in 1961 a paper was published claiming that the Indian theories were based on the equant?

In fact, all of this has been known since:

B. L. van der Waerden, “Ausgleichspunkt, ‘methode der perser’, und indische planetenrechnung”, *Archive for history of exact sciences*, 1 (1961), 107-121.

- power series identical through $O(e,r,e^2,r^2,er)$:

$$-2e\sin\alpha + 2e^2\sin\alpha\cos\alpha + r\sin\gamma - r^2\sin\gamma\cos\gamma +$$
$$er(2\sin\alpha\cos\gamma - \cos\alpha\sin\gamma) + O(er^2, e^2 r)$$

- factor ½ in steps (1), (2) follows directly from *bisected* equant

- Indian algorithm decouples the two anomalies (accomplished in the *Almagest* by a clever interpolation scheme).

Conclusions:

- yes, Hipparchus was very good at trigonometry

- did Ptolemy invent the equant? It doesn't seem likely, since while Indian planetary models are based on the equant, almost everything else is less developed than what we find in the *Almagest*.

- it appears more likely that Greek astronomy in the period 120 B.C – 120 A.D. was rather advanced, but we have very little idea who knew what, or when.

If you want to know more:

these slides and a written version of this talk are at www.csit.fsu.edu/~dduke/articles

“Hipparchus’ Eclipse Trios and Early Trigonometry”, *Centaurus,* 47 (2005) 163-177.

“The Equant in India: the Mathematical Basis of Ancient Indian Planetary Models”, *Archive for History of Exact Sciences*, (2005), forthcoming.