Who Knew What, and When?
The Timing of Discoveries in Early Greek Astronomy

Dennis Duke,
Department of Physics and School of Computational Science,
Florida State University
NDVII 9 July 2005

Trigonometry in ancient Indian texts is based on increments of 3¾° for the sine function, corresponding to increments of 7½° for the Greek chord function. In 1972 Neugebauer suggested that the Indian schemes are of early Greek origin.[1] In terms of chords, a table would look like this:[2]

Table of chords

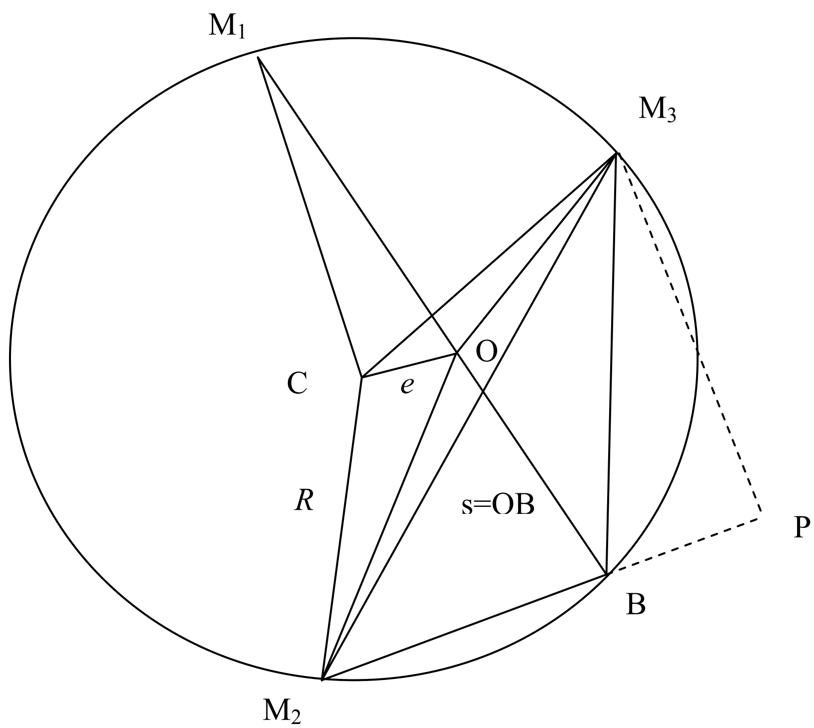
Angle(degrees)	Chord
0	0
7 ½	450
15	897
22 ½	1341
30	1780
37 ½	2210
45	2631
52 ½	3041
60	3438
67 ½	3820
75	4186
82 ½	4533
90	4862
97 ½	5169
105	5455
112 ½	5717
120	5954
127 ½	6166
135	6352
142 ½	6511
150	6641
157 ½	6743
165	6817
172 ½	6861
180	6875

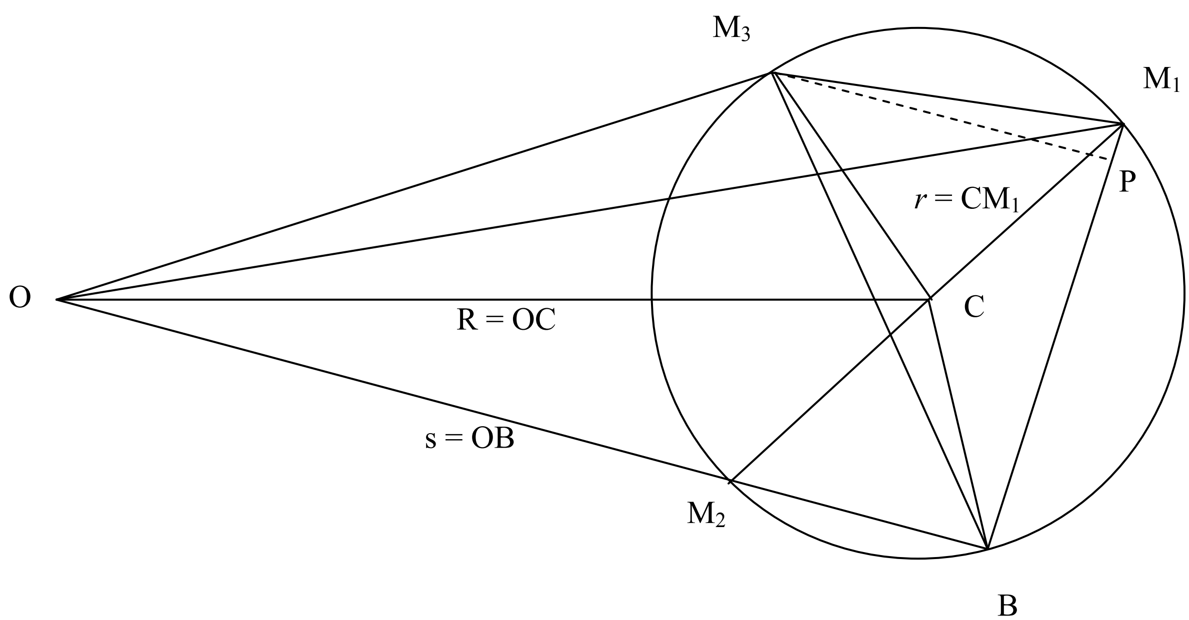
The rather peculiar numbers result from assuming a base circle of circumference $360° \times 60\,'^{/\circ} = 21{,}600\,'$, corresponding to a diameter of $D = 21{,}600/\pi \simeq 6875$.

In 1973 Toomer suggested[3] that the numbers we see in the chord table explain the equally strange numbers that Ptolemy attributes to Hipparchus in *Almagest* 4.11:

$$R / e = 3144 / 327\tfrac{2}{3} \qquad\qquad R / r = 3122\tfrac{1}{2} / 247\tfrac{1}{2}.$$

These numbers characterize the geometry of the Moon's orbit, as deduced from analyses of two lunar eclipses trios, one using an eccentre model, the other an epicycle.





As you can see in the page below from Toomer's paper, apart from minor tweaking the numbers simply appear in the natural course of calculation.

14 *G. J. Toomer*

$$M_{III}P = \frac{M_{III}B \text{ Crd } 2\frac{\alpha_{II}}{2}}{2R'} = \frac{M_{III}B \text{ Crd } 153;25^\circ}{2R'} = s \cdot \frac{5379 \cdot 6688}{3247 \cdot 2 \cdot 3438}.$$

$$M_{II}P = M_{II}B + BP = M_{II}B + \frac{M_{III}B \text{ Crd } 26;35^\circ}{2R'}$$

$$= s\left(\frac{3110\frac{1}{2}}{6574} + \frac{5379 \cdot 1580}{3247 \cdot 2 \cdot 3438}\right) = s \cdot \frac{\frac{3110\frac{1}{2} \cdot 3438}{6574} + \frac{5379 \cdot 1580}{3247 \cdot 2}}{3438}.$$

$$M_{II}M_{III} = \sqrt{M_{III}P^2 + M_{II}P^2} = s \cdot \frac{6268}{3438}. \qquad (1)$$

$$R = \frac{M_{II}M_{III} \cdot R'}{\text{Crd } \alpha_{II}} = s \cdot \frac{6268 \cdot 3438}{3438 \cdot 6688} = s \cdot \frac{6268}{6688} = s \cdot \frac{3134^{14}}{3344}.$$

$$\text{Crd } \widehat{M_{III}CB} = \frac{M_{III}B \cdot R'}{R} = \frac{s \cdot 5379 \cdot 3438 \cdot 3344}{s \cdot 3247 \cdot 3134} = 6078.$$

$$\widehat{M_{III}CB} = 124;24^\circ.$$

$$\widehat{M_{I}CB} = \widehat{M_{III}CB} + \alpha_{III} = 124;24^\circ + 46;36^\circ = 171^\circ.$$

$$M_IB = \frac{R \text{ Crd } 171^\circ}{R'} = s \cdot \frac{3134 \cdot 6853}{3344 \cdot 3438}.$$

$$R^2 - e^2 = (R + e)(R - e) = s \cdot M_IO = s\,(M_IB - s)$$

$$= s^2\left(\frac{3134 \cdot 6853 - 3344 \cdot 3438}{3344 \cdot 3438}\right).$$

$$\frac{R^2}{e^2} = \frac{R^2}{R^2 - s\,(M_IB - s)}.$$

$$\frac{R}{e} = \frac{R}{\sqrt{R^2 - s\,(M_IB - s)}} = \frac{\frac{3134}{3344}}{\sqrt{\left(\frac{3134}{3344}\right)^2 - \frac{3134 \cdot 6853 - 3344 \cdot 3438}{3344 \cdot 3438}}}.$$

Multiplying through by 3344,

$$\frac{R}{e} = \frac{3134}{\sqrt{3134^2 + 3344^2 - \frac{3134 \cdot 3344 \cdot 6853}{3438}}} \qquad =3134/338$$

A few years later, however, Toomer discovered he had made an error while analyzing the epicycle trio, which he explained in a footnote in his 1984 edition of the *Almagest*, and expressed doubt about the whole matter.[4]

And so things rested for nearly 20 years[5], until in the summer of 2003, when, just after returning home from NDVI, I looked into the situation. First of all, Toomer had analyzed the eccentre trio correctly, so his conclusion that Hipparchus was using a chord table based on a reference circle of diameter 6875 was correct for that trio. But perhaps because I was, somewhat lazily, doing my work almost entirely in a computer, I realized that Toomer had overlooked something. Just as, when we write a numerical computer program today we must specify values for each variable, so too must have Hipparchus, since he was working without the benefit of symbolic algebra. And Toomer, in his analysis, had in fact left a particular value, the one labeled *s* in the figures above, unspecified,[6] no doubt reasoning that all it does is set the overall distance scale of the figure and must cancel out of any derived ratio of distances, such as *R/e* and *R/r*. So I realized fairly quickly that, with that extra degree of freedom, I could simply ask what value of *s* would be required to give the crucial number 247½ , and it is easy to figure out that the required number must be very close to *s* = 3162.

10 G. J. Toomer

The problem is to find the ratio of the radius of the deferent, R (= OC), to the radius of the epicycle r. This is achieved via the intermediate step of finding r in terms of s, s being the distance OB between O and the place where the straight line from O to one of the points M meets the circle again in B. The solution is as follows (R' is throughout the radius of the base circle, or 3438′; *Crd* is the chord expressed in terms of that base circle).

$$\mathrm{O\widehat{M_1}B} = \tfrac{1}{2}\alpha_1 - \delta_1 = 84;25^\circ - 8;22^\circ = 76;3^\circ.$$

$$\mathrm{O\widehat{M_3}B} = 180^\circ - \tfrac{1}{2}\alpha_2 - \delta_2 = 180^\circ - 69;49\tfrac{1}{2}^\circ - 9;19^\circ = 100;52\tfrac{1}{2}^\circ$$

$$\mathrm{M_1B} = s\,\frac{\mathrm{Crd}\ 2\delta_1}{\mathrm{Crd}\ 2\,\mathrm{O\widehat{M_1}B}} = s\,\frac{\mathrm{Crd}\ 16;44^\circ}{\mathrm{Crd}\ 152;6^\circ} = s\cdot\frac{1000}{6669\frac{1}{3}}.$$

$$\mathrm{M_3B} = s\,\frac{\mathrm{Crd}\ 2\delta_2}{\mathrm{Crd}\ 2\,\mathrm{O\widehat{M_3}B}} = s\,\frac{\mathrm{Crd}\ 18;38^\circ}{\mathrm{Crd}\ 201;45^\circ} = s\,\frac{\mathrm{Crd}\ 18;38^\circ}{\mathrm{Crd}\ 158;15^\circ} = s\cdot\frac{1112\frac{1}{2}}{6750\frac{1}{2}}.$$

$$\mathrm{M_3P} = \frac{\mathrm{M_3B\ Crd}\ 2\frac{\alpha_3}{2}}{2R'} = \frac{\mathrm{M_3B\ Crd}\ 51;33^\circ}{2R'} = s\cdot\frac{1112\frac{1}{2}\cdot 2989}{6750\frac{1}{2}\cdot 2\cdot 3438} = s\cdot\frac{246\frac{1}{3}}{3438}.$$

$$\mathrm{M_1P} = \mathrm{M_1B} - \frac{\mathrm{M_3B\ Crd}\ 2\left(\frac{180^\circ - \alpha_3}{2}\right)}{2R'} = \mathrm{M_1B} - \frac{\mathrm{M_3B\ Crd}\ 128;27^\circ}{2R'}$$

$$= s\left(\frac{1000}{6669\frac{1}{3}} - \frac{1112\frac{1}{2}\cdot 6189\frac{1}{2}}{6750\frac{1}{2}\cdot 2\cdot 3438}\right) = s\left(\frac{515\frac{1}{2} - 510\frac{1}{6}}{3438}\right) = s\cdot\frac{5\frac{1}{3}}{3438}.$$

$$\mathrm{M_1M_3} = \sqrt{\mathrm{M_3P^2 + M_1P^2}} = s\cdot\frac{246\frac{1}{3}}{3438}. \qquad (1)$$

$$r = \frac{\mathrm{M_1M_3}\cdot R'}{\mathrm{Crd}\ \alpha_3} = s\cdot\frac{246\frac{1}{3}}{3438}\cdot\frac{3438}{2989} = s\cdot\frac{246\frac{1}{3}}{2989}.$$

=3162 x 231 3/4 /
2960 2/5

$$\mathrm{Crd\ \widehat{M_3C}B} = \frac{\mathrm{M_3B}\cdot R'}{r} = \frac{s\cdot 1112\frac{1}{2}\cdot 2989\cdot 3438}{s\cdot 6750\frac{1}{2}\cdot 246\frac{1}{3}} = 6875.$$

$$\mathrm{\widehat{M_3C}B} = 180^\circ. \qquad (2)$$

(accurate calculation with modern sine tables gives 179° here).

$$\mathrm{\widehat{M_2C}B} = \mathrm{\widehat{M_3C}B} - \alpha_2 = 180^\circ - 139;37^\circ = 40;23^\circ. \qquad (3)$$

$$\mathrm{BM_2} = \frac{r\ \mathrm{Crd}\ 40;23^\circ}{R'} = s\cdot\frac{246\frac{1}{3}\cdot 2372}{2989\cdot 3438}.$$

But at the time I couldn't see any particular reason why Hipparchus would have chosen that number, seemingly out of the blue, and so the spreadsheet sat idle in my computer while I went on the other things, one of which was, in Spring 2004, a closer look into ancient Indian astronomy. Now you won't look far into that field without noticing that even the earliest texts routinely use two different values for π, one of which is $\frac{62832}{20000}$, and is often attributed to Archimedes,[7] and one of which is the somewhat cruder $\pi \simeq \sqrt{10} \simeq 3.1622...$ This then clearly suggests that Hipparchus was using for the scale of his diagram a reference circle of circumference not 21,600, but instead 10,000, and the diameter of his circle would then be

$$s = \frac{10,000}{\pi} = \frac{10,000}{\sqrt{10}} = 1,000\sqrt{10} \simeq 3162,$$

precisely the value he appears to have used for s to get the numbers we find in the epicycle trio in *Almagest* 4.11. And of course, as numbers go, the myriad 10,000 is about as Greek as you can get.[8]

So Toomer was correct after all and it is established that Hipparchus[9]

(a) used the trio method for analyzing the lunar orbit
(b) was fairly proficient in plane trigonometry
(c) used at least two different base circle conventions, with circumferences of 21,600 and 10,000, both of which appeared in Indian texts some 700 years later (but in no known Greek text).

Indeed, there are many features of early Greek astronomy that appear in these ancient Indian texts, and like the trigonometry examples just discussed, they tend to be rather crude compared to the relatively sophisticated features that we find in the *Almagest*. Examples are:

- *The equation of time*. Throughout the first millennium the Indians used an abbreviated version which includes only the effect of the zodiacal anomaly of the Sun, and neglects the effect of the obliquity of the ecliptic.
- *Obliquity of the ecliptic*. When used in spherical trigonometry, the Indians use either 24° or 23;40°, both associated with Hipparchus, but never the Eratosthenes/*Almagest* value 23;51,20°.
- *Accurate discussions of parallax*. The Indians were aware of parallax and used it for computing eclipses, but always used various approximations.
- *Trigonometry scales*. The Indians used a variety of values for the diameter of the circle, and mostly the values $D = 6875$ and $D = 3162$ in the earliest texts. These are values used by Hipparchus but apparently abandoned by the time of Ptolemy.
- *Retrograde motion*. When mentioned at all, the Indians quoted specific values of the sighra anomaly that correspond to first and second station. There is no mention of the variation in the size of retrograde arcs with zodiacal position.
- *Model of Mercury*. Unlike Ptolemy, who used a complicated crank mechanism to generate a pair of perigees for Mercury, the Indians used the same model for Mercury and Venus, which is also often the same or closely related to the model used for the outer planets.
- *Determination of orbit elements*. While the bulk of the *Almagest* is devoted to explaining how to determine orbit elements from empirical data, it is not all obvious that any comparable derivation is even possible in the context of the Indian approximation schemes.
- *Values of orbit elements*. The values used in the Indian schemes for e, r, and A are generally different from the values found in the *Almagest*. Except for Mercury, the resulting Indian model predictions for true longitudes are generally inferior to those in the *Almagest*.
- *Star catalog*. The Indian coordinates for star positions are generally inaccurate, and bear no relation to those found in the *Almagest* star catalog.
- *Zodiacal signs*. The Indian texts routinely divide circles such as epicycles into 30° segments and refer to them in terms of the zodiacal signs. The only other known use of this practice is in Hipparchus' similar description of circles of constant latitude in the *Commentary to Aratus*.
- *The second lunar anomaly*. The Indians did not discuss evection until the beginning of the second millennium, and then in a form different from that used by Ptolemy.

This has led to an essentially universally accepted view that the astronomy we find in the Indian texts is pre-Ptolemaic. In modern terms, ancient Indian astronomy gives us a sort of wormhole through space-time and documents an otherwise lost history of perhaps 300 years in early Greek mathematical astronomy.[10] Summarizing this point of view, Neugebauer wrote in 1956:

> "Ptolemy's modification of the lunar theory is of importance for the problem of transmission of Greek astronomy to India. The essentially Greek origin of the Surya-Siddhanta and related works cannot be doubted – terminology, use of units and computational methods, epicyclic models as well as local tradition – all indicate Greek origin. But it was realized at an early date in the investigation of Hindu astronomy that the Indian theories show no influence of the Ptolemaic refinements of the lunar theory. This is confirmed by the planetary theory, which also lacks a characteristic Ptolemaic construction, namely, the "*punctum aequans*," to use a medieval terminology"[11]

This brings us to the planetary theories of the Indian texts. These theories have in common with the familiar Greek theories a treatment of two irregularities in the orbits. First, the planet's speed changes as it circles the zodiac, and second, the detailed motion of each planet is clearly correlated with the location of the Sun – when the inner planets Venus and Mercury are near the Sun and when the outer planets Mars, Jupiter and Saturn are more or less opposite the Sun as seen from Earth, they appear to stop moving, go in reverse, and stop moving again before resuming forward motion. Now Ptolemy tells us that astronomers at least as early as Hipparchus were aware of both of these anomalies, and that no astronomer at the time of Hipparchus was able to construct a theory that explained both anomalies "by means of eccentric circles or by circles concentric with the ecliptic, and carrying epicycles, or even by combining both." And as it happens, Ptolemy's brief description of the theories that "were faulty and at the same time lacked proofs; some of them did not achieve their object at all, the others only to a limited extent" sounds a lot like the Indian theories. The Indian texts tell us that the zodiacal, or manda, correction is computed using

$$\sin q(\alpha) = -e \sin \alpha \,,$$

and the solar, or sighra, correction is computed using

$$\tan p(\gamma) = \frac{r \sin \gamma}{1 + r \cos \gamma} \,,$$

where e and r control the size of the corrections, and α and γ are the uniformly changing angles around the deferent and the major epicycle. These corrections are essentially identical with those found in Greek geometrical models. The really curious part of the Indian models is the prescription for combining these corrections. If $\overline{\lambda}_A$ is the longitude of the apogee, and $\overline{\lambda}$ is the planet's mean longitude and $\overline{\lambda}_S$ the Sun's, then for an outer planet the steps in the algorithm are:

(1) with manda argument $\alpha = \overline{\lambda} - \lambda_A$ compute $\nu_1 = \overline{\lambda} + \frac{1}{2} q(\alpha)$.
(2) with sighra argument $\gamma = \overline{\lambda}_S - \nu_1$ compute $\nu_2 = \nu_1 + \frac{1}{2} p(\gamma)$.

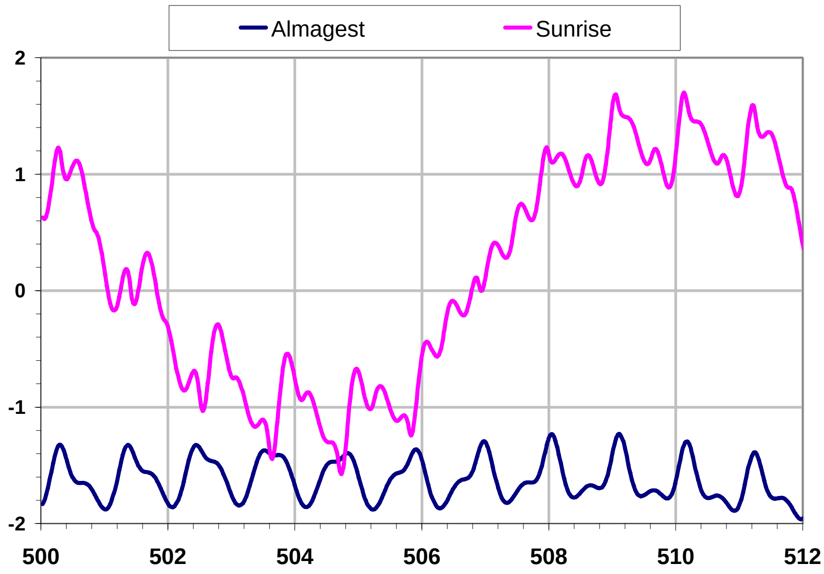
(3) with manda argument $\alpha = \nu_2 - \lambda_A$ compute $\nu_3 = \bar{\lambda} + q(\alpha)$.
(4) with sighra argument $\gamma = \bar{\lambda}_S - \nu_3$ compute the true longitude $\lambda = \nu_3 + p(\gamma)$.

Unlike the usual Greek theories, it is hard to imagine a geometrical diagram that accurately illustrates this algorithm (and the Indian texts indeed include no diagrams), for we are sometimes told to adjust the manda and sighra arguments according to what was found in the previous step, and the size of the adjustment is sometimes ½ of the correction and sometimes the whole correction.

With only one exception (which I will come back to), historians have always assumed that the basis of the Indian theory is what we would call in Greek terms simply an eccentric deferent carrying an epicycle. For example, Pingree wrote in 1971:

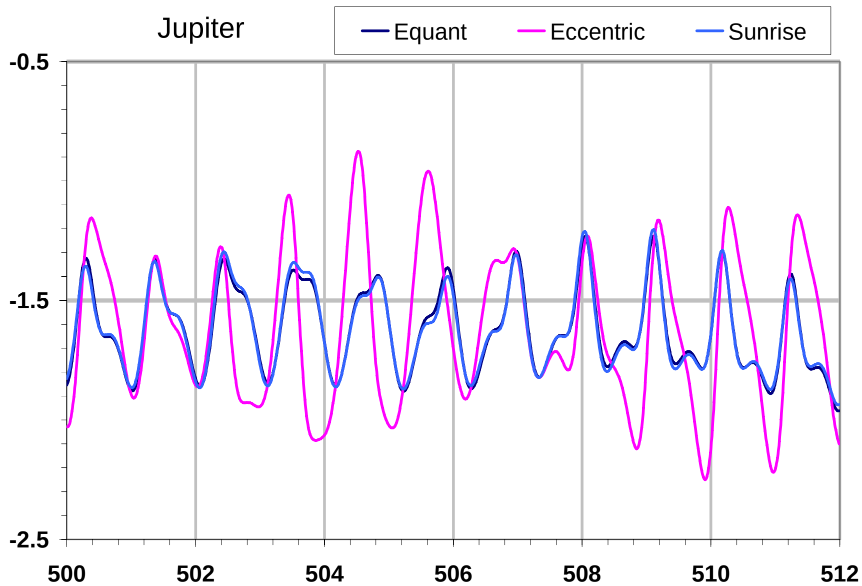
> "The orbits of the planets are concentric with the center of the earth. The single inequalities recognized in the cases of the two luminaries are explained by *manda*-epicycle (corresponding functionally to the Ptolemaic eccentricity of the Sun and lunar epicycle, respectively), the two inequalities recognized in the case of the five star-planets by a *manda*-epicycle (corresponding to the Ptolemaic eccentricity) and a *sighra*-epicycle (corresponding to the Ptolemaic epicycle). The further refinements of the Ptolemaic models are unknown to the Indian astronomers."[12]
>
> and again in 1980:
>
> "The Indians' geometrical models for computing the corrections to the planets' mean longitudes, while derived from Greek sources, *are crude in comparison to Ptolemy's*…..The dimensions of the epicycles of the five planets selected by Aryabhata I are unrelated to Ptolemy's eccentricities and epicycles, as is clear from Table 4 in that same article; and Aryabhata I *has nothing corresponding to the Ptolemaic equant*."[13]

Indeed, if you compare the predictions of the *Almagest* model for, say Jupiter, with the predictions of an Indian model, you do see that in general the *Almagest* model is a superior predictor of Jupiter's positions.[14]

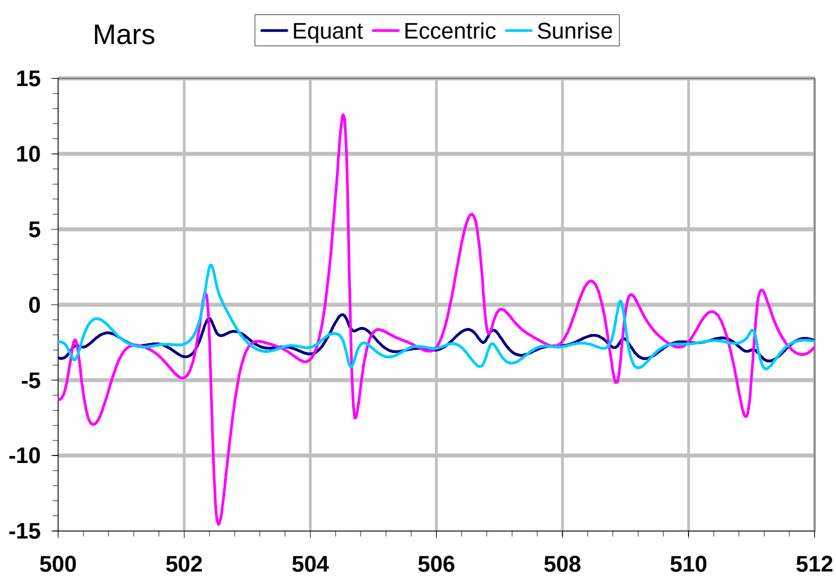


Now the parameters used for each model are somewhat different, but then there is no reason to expect that the optimum parameter values for unrelated models should be the same. However, we need to be a little careful here, because although we can get some idea of how the model parameters for Jupiter were determined by reading the *Almagest*, the Indian texts tell us absolutely nothing about how those parameters were determined. This, of course, raises the question of how the models would compare if they were using the *same* parameter values.

This is very easy to check, of course, and it is sort of surprising that nobody, as far as I can tell, ever bothered to do it.[15] Even without a computer one could, using identical model parameters, easily compute a series of planetary positions by hand in an hour or so, and with a computer you can compute many thousand in a single second. Comparing the predictions of the Indian theory with a Greek eccentre plus epicycle and with the *Almagest* equant, one finds



The figure shows that the Indian theory and the *Almagest* equant are, for the parameters of Jupiter, effectively the *very same theory*, and that contrary to an apparently widely held belief, the Indian theory is certainly not derived from an eccentric plus epicycle theory. The same is true for Saturn, but for Mars the agreement is somewhat degraded.



The reason is that while the equant is clearly the underlying mathematical basis of the Indian theory, the latter is in fact a rather clever algorithm for approximating the equant while decoupling the two anomalies,[16] a problem solved in the *Almagest* by means of an even more clever interpolation scheme.

I mentioned above that there was one exception to the conventional wisdom.

[ask for audience show of hands: who knows, apart from anything I have told you, that in 1961 a paper was published claiming that the Indian theories were based on the equant]

In 1961 van der Waerden showed, on the basis of the first few terms in a power series expansion in e and r, that the equant and Indian theories are closely related, and in particular that the factor of 1/2 used in the initial steps of the Indian models is directly related to the bisection of the equant.[17] van der Waerden published no numerical comparisons, and so it was perhaps not entirely clear whether or not his conclusion was solid. Curiously, though, his result appears to have been systematically ignored in the principal Indian[18] and western literature.[19]

So where does this leave us?

We have seen that the trigonometry in the Indian texts is much closer to Hipparchus than to Ptolemy. Furthermore, we have a long list of features in Indian astronomy that are clearly much less developed than the astronomy in the *Almagest*. Yet we find that the most sophisticated element in the *Almagest* – the equant construction for coupling the zodiacal and solar anomalies – is in fact *precisely* the mathematical basis of the Indian texts, which most people think "are crude in comparison to Ptolemy's" and have "nothing corresponding to the Ptolemaic equant."

It seems to me that the most conservative conclusion is that the idea that Ptolemy invented the equant warrants a careful reconsideration, and that to establish Ptolemy's priority one must establish a different relationship between the Indian models and ancient Greek astronomy than the one universally accepted until now.

But in reality it is most likely the case that mathematical astronomy was far more advanced in the years between Hipparchus and Ptolemy than we have previously thought.[20] Let us remember that in order to finally establish the planetary theories in the *Almagest* Ptolemy leads us through an elaborate development of solar theory, lunar theory, and precession. Whatever we may think of Ptolemy's account of the development of this empirical and theoretical framework, at every stage using the results of instrumental measurement to determine the values of the parameters that fix the theories, it seems safe to conclude that *some* astronomer, and probably a number of them, must have, at some time during the period 120 B.C. – 120 A.D., accomplished all that Ptolemy describes in the *Almagest*.[21]

And, apparently, we have very little idea who did it, or when.

REFERENCES

[1] O. Neugebauer, "On Some Aspects of Early Greek Astronomy", *Proceedings of the American Philosophical Society*, 116 (1972), pp. 243-251.

[2] Of course no such table ever appears in the ancient Indian texts. Instead of the chord function in the range 0°-180° they use the sine function in the range 0°-90°, they never list the actual angles, and instead of the sine values they give the difference between successive sine values (so the first numbers are 225, 224, 222, 219, etc).

[3] G. J. Toomer, "The Chord Table of Hipparchus and the Early History of Greek Trigonometry", *Centaurus* 18 (1973), pp. 6-28.

[4] G. J. Toomer, *Ptolemy's Almagest*, London (1984), p. 215, fn 75.

[5] an exception is the attempt to explain the ratios by D. Rawlins, *DIO* 1.2-3 (1991).

[6] In the *Almagest* Ptolemy analyzes several such trios, both for the Moon and for the similar case of oppositions of the outer planets, and he always sets $s = 120$.

[7] This claim goes back at least to the late 1800's. In "Measurement of the Circle" Archimedes inscribes and circumscribes a circle with 96-sided polygons to derive the bound $3\frac{10}{71} < \pi < 3\frac{1}{7}$. The granularity of the 96-sided polygon is precisely the granularity of the Indian sine tables. Continuing Archimedes' algorithm of successive halving by two more steps and normalizing to a denominator of 20,000 gives $\frac{62831\frac{1}{7}}{20000} < \pi < \frac{62833\frac{2}{3}}{20000}$. The midpoint is $\frac{62832\frac{2}{5}}{20000}$, which rounds to the value found in early Indian texts.

[8] Full details may be found in Dennis W. Duke, "Hipparchus' Eclipse Trios and Early Trigonometry", *Centaurus,* 47 (2005) 163-177.

[9] It is also worth remarking what is *not* established by this analysis. It is not established that Hipparchus used a *table* of trig values, although that would certainly be the easiest thing to do. Likewise, we have not established the granularity of his grid of values, which could be finer than the 7½° shown in the table. Finally, we cannot establish the precise values he used for lunar mean motions, as long as they were reasonably close to the correct values.

[10] It is also the case that many features that we recognize as Babylonian in origin appear in the Indian texts, but they also appear to have been transmitted through some Greek intermediary, and not directly from Babylon.

[11] O. Neugebauer, "The Transmission of planetary theories in ancient and medieval astronomy", *Scripta mathematica*, 22 (1956) , 165-192.

[12] D. Pingree, "On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle", *Journal for the history of astronomy*, ii (1971), 80-85

[13] D. Pingree, "Reply to van der Waerden", *Journal for the history of astronomy*, xi (1980), 58-62.

[14] In this and the succeeding plots, the y-axis gives the difference in degrees of the planet's position computed from modern models and the position computed from an ancient model. Hence the curve is, for all practical purposes, the error in the prediction of the ancient model.

[15] an exception is H. Thurston, “Greek and Indian Planetary Longitudes”, *Archive for History of Exact Sciences*, 44 (1992) 191-195.

[16] For the details see Dennis W. Duke, “The equant in India: the basis of ancient Indian planetary models”, *Archive for history of exact sciences*, (2005), forthcoming.

[17] B. L. van der Waerden, “Ausgleichspunkt, ‘methode der perser’, und indische planetenrechnung”, *Archive for history of exact sciences*, 1 (1961), 107-121.

[18] see, e.g. B. Chatterjee, *The Khandakhadyaka of Brahmagupta* (1972), appendix 7; S. N. Sen, “Epicyclic Eccentric Planetary Theories in Ancient and Medieval Indian astronomy”, *Indian Journal of History of Science*, 9 (1974), 107-121; S. N. Sen, “Survey of studies in European languages”, *Indian Journal of History of Science*, 20 (1985), 49-121; D. A. Somayaji, “The yuga system and the computation of mean and true planetary longitudes”, *Indian Journal of History of Science*, 20 (1985), 145-187.

[19] see, e.g., D. Pingree, “On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle”, *Journal for the history of astronomy*, ii (1971), 80-85; D. Pingree, “The Recovery of Early Greek astronomy from India”, *Journal for the history of astronomy*, vii (1976), 109-123; D. Pingree, “History of Mathematical astronomy in India”, *Dictionary of Scientific Biography*, 15 (1978), 533-633.

[20] this idea is developed in about as much detail as the scant evidence allows in L. Russo, *The Forgotten Revolution* (2004).

[21] Several proposals have been made to explain the empirical origin of the equant. See James Evans, “On the function and probable origin of Ptolemy’s equant”, *American journal of physics,* 52(1984), 1080-9; Noel Swerdlow, “The empirical foundations of Ptolemy’s planetary theory”, *Journal for the history of astronomy*, 35 (2004), 249-71; Alexander Jones, “A route to the ancient discovery of non-uniform planetary motion”, *Journal for the history of astronomy*, 35 (2004), 375-386; Dennis W. Duke, “Comment on the Origin of the Equant papers by Evans, Swerdlow, and Jones”, *Journal for the history of astronomy*, 36 (2005). None of the routes suggested are inconsistent with a pre-Ptolemaic origin, and equally, none require it.