On the Efficiency of a Multiresolution Block-Adaptive Strategy for Multiphysics Problems

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Introduction

Many physical systems are characterized by the presence of multiple spatial scales. Numerically simulating such types of systems poses a significant challenge. One of the most widely-adopted approaches to address this issue involves the use of a non-uniformly spaced grid with a hierarchy of grid resolutions. Such approaches can be broadly classified as adaptive mesh refinement (AMR) methods.

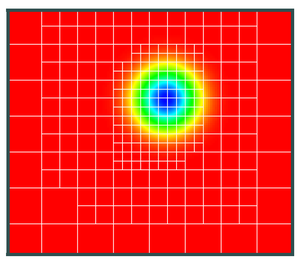

Figure 1: A density contour snapshot of an advecting vortex, with three active levels of AMR. AMR blocks with 8×8 cells each are outlined in white. Large regions with essentially no density gradient surrounding the vortex are refined at the finest level.

Rather than refine one computational cell individually, AMR methods typically create a hierarchy of grid levels by refining large collections of cells (blocks) around non-smooth features. Block-based refinement is typically preferred over cell-based refinement for reasons of computational efficiency.

One major drawback of block-structured AMR is the 'over-resolution' of many cells in the mesh which may be in a smooth region, but are grouped together with a non-smooth feature for refinement. While the cost is generally justified over existing alternatives, it remains an unwanted side-effect (see Figure (1)).

A method for dealing with multi-scale problems on *uniform* grids was introduced by Harten [2], which used a multiresolution (MR) representation of the data in order to decrease excessive computations in smooth regions. The idea was to reduce the number of costly flux evaluations while maintaining a prescribed level of accuracy. In the present work, this scheme is generalized to block-structured AMR discretizations.

Numerical Methods

Finite Volume Framework

In the present work we are interested in numerically solving conservation laws of the form

$$\begin{cases} u_t + f(u)_x = s(u) \\ u(x,0) = u_0(x), \end{cases} \tag{1}$$

where u represents a conserved quantity, $f(u)$ is the flux function, and $s(u)$ is a source term. In the finite volume formulation, the solution $u(x,t)$ is approximated by a volume average defined over a target cell $I_i = \left[x_i - \frac{h}{2}, x_i + \frac{h}{2}\right]$ as $u_i(t) = \frac{1}{h}\int_{x_i-\frac{h}{2}}^{x_i+\frac{h}{2}} u(\xi,t)d\xi$, where h is the cell width. The governing equations are cast into the following semi-discrete conservative scheme

$$\frac{du_i(t)}{dt} = -\frac{1}{h}\left(\hat{f}_{i+\frac{1}{2}} - \hat{f}_{i-\frac{1}{2}},\right), \tag{2}$$

where $\hat{f}_{i\pm\frac{1}{2}}$ are numerical fluxes.

Multiresolution Scheme

The role of the multiresolution representation is to obtain regularity information about the underlying function. This is done by decomposing data into successively more coarse levels of resolution, then at each level measuring the difference made between the fine grid data and its approximation based on data on the adjacent coarse grid. We define on each active AMR block a new hierarchy of nested grids $\mathcal{G}^l, 0 \le l \le L$,

$$\mathcal{G}^l = \left\{x_i^l\right\}_{i=0}^{N_l}, \quad x_i^l = i \cdot h^l, \;\; h^l = 2^l \cdot h^0, \;\; N_l = N_0/2^l. \tag{3}$$

Here $l = 0$ represents the (finest) level of resolution defined on the AMR block, and $l = L$ represents the most coarse.

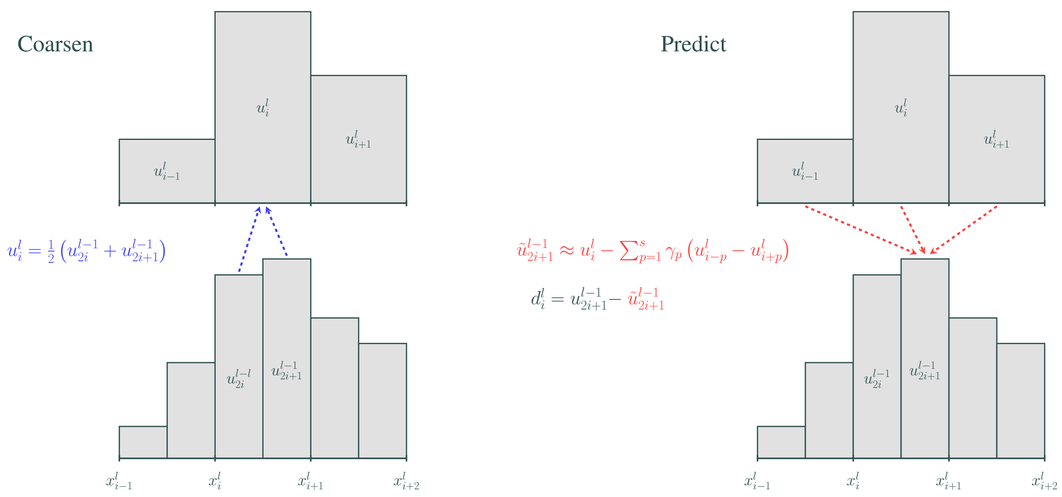

Figure 2: Left: fine-scale cell averages are coarsened. Right: quadratic average-interpolating polynomial prediction from coarse-scale l to fine-scale $l-1$, given cell averages $\{u_j\}_{j=i-1}^{i+1}$.

The forward transform proceeds by predicting data between each adjacent level of resolution based on average-interpolating polynomials of degree $2s$, given by

$$\tilde{u}_{2i+1}^{l-1} \approx u_i^l - \sum_{p=1}^{s} \gamma_p \left(u_{i-p}^l - u_{i+p}^l\right). \tag{4}$$

The detail coefficients are then computed as

$$d_i^l = u_{2i+1}^{l-1} - \tilde{u}_{2i+1}^{l-1}. \tag{5}$$

These steps are illustrated in Figure (2). **Once the detail coefficients have been obtained, the MR scheme proceeds by setting a threshold ϵ and truncating coefficients which have an absolute value below the threshold.** Lastly, the inverse transform then starts from grid $l = L$ and at each interface either computes fluxes using the fine-grid scheme, or interpolates them using the MR basis. The fluxes are interpolated by

$$\tilde{f}_{2i+1}^{l-1} \approx \sum_{p=1}^{s+1} \beta_p \left(\hat{f}_{i-p+1}^l + \hat{f}_{i+p}^l\right), \tag{6}$$

where the interpolants are of degree $2s+1$. The process repeats until all fluxes are either computed or interpolated on the fine grid $l = 0$.

Results

Convergence Study

We solve the one-dimensional Euler equations on the periodic domain $[0, 2\pi]$ with

$$w = (\rho, \rho u, E), \quad f(w) = \left(\rho u, \rho u^2 + p, u(E+p)\right), \tag{7}$$

where ρ is the mass density, u is the velocity, E is the total energy, and p is the pressure. The initial conditions are $\rho(x,0) = 1 + 0.2\sin(x)$, $u(x,0) = 1$, $p(x,0) = 1$, and the ratio of specific heats is $\gamma = 1.4$. The solution is carried forward for several timesteps only on a uniform grid. **It is found that the MR scheme does not significantly alter the rate of convergence of the fine-grid scheme.**

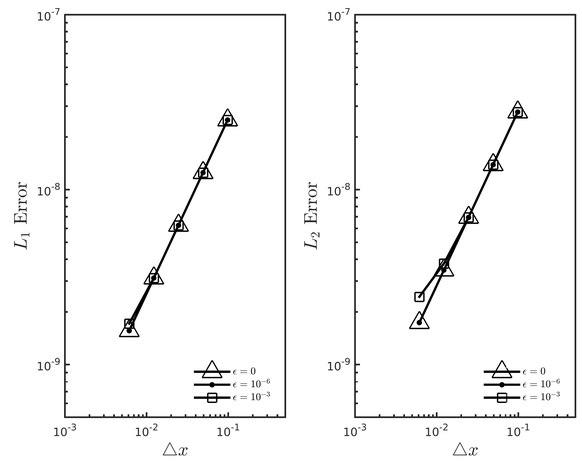

Figure 3: Errors in the L_1 and L_2 norm for the sine wave problem using a first-order Godunov method and first-order multiresolution interpolation.

Two Blast Waves

We illustrate the efficiency of the scheme on the problem of two blast waves [3]. The multiresolution scheme is applied to the mass and momentum fluxes, with just one MR level, $L = 1$. The threshold parameter ϵ is set to be of the same order as the local truncation error of the fine-grid scheme on each AMR block. **We find that throughout the simulation, approximately 13.4 percent of flux calculations are replaced by an approximation from the MR basis.** After the waves collide, we find only small regions where the MR scheme is active. This is expected, as the entire domain has been affected by the shocks at these times.

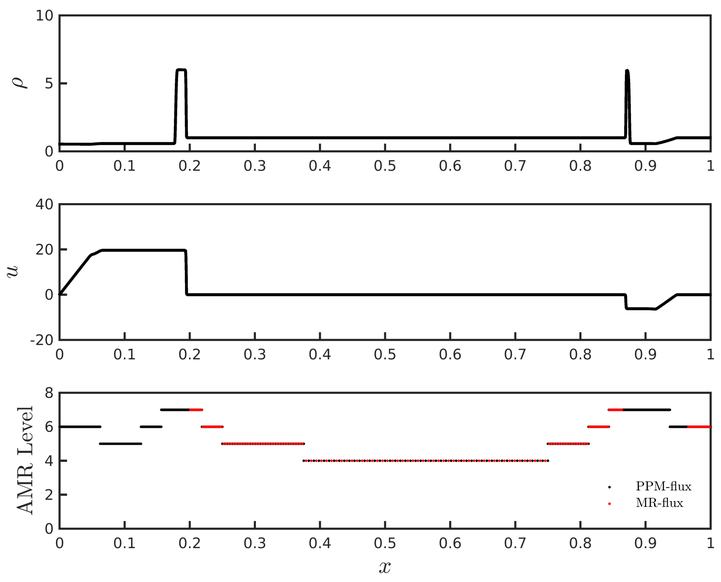

Figure 4: Results are shown from the problem of two interacting blast waves after $t = 0.004$ seconds. Top: two density waves moving towards each other. Middle: velocity curve at the same time. Bottom: AMR levels corresponding to each cell interface in the domain are plotted, showing interfaces where the flux is computed using the fine-grid scheme (Piecewise Parabolic Method) in black, and interfaces approximated by the MR basis in red.

Conclusions

The multiresolution scheme is used to reduce the number of flux calculations on block-structured AMR grids. It is found that the scheme significantly reduces computational effort without adversely affecting the numerical solution. In future work we plan to **expand the test suite to include multidimensional problems**, **include source terms in the MR scheme for handling reactive flows** [1], and **conduct parallel scaling and detailed benchmark studies.**

References

[1] B.L. Bihari and D. Schwendeman. Multiresolution schemes for the reactive euler equations. *Journal of Computational Physics*, 154:197–230, 1999.

[2] A. Harten. Adaptive multiresolution schemes for shock computations. *Journal of Computational Physics*, 115(2):319–338, 1994.

[3] P. Woodward and P. Colella. The numerical simulation of two-dimensional fluid flow with strong shocks. *Journal of Computational Physics*, 54:115–173, 1984.