The Halfway Way:
A fresh look at the midpoint method

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The Executive Summary

- The midpoint method is second order and absolutely stable;
- It is B-stable;
- It preserves linear and quadratic conservation quantities;
- It produces reliable error estimates;
- Safe time-steps are calculated accurately, efficiently, and adaptively;
- Existing backward Euler codes upgrade with one line of new code;
- C, C++, Fortran, FreeFem, MATLAB, Octave, Python, R versions.

Multiple Identities

Methods are for ODE's, rules for numerical quadrature.

	f(mid)	$\frac{1}{2}$(f(left)+f(right))
explicit:	explicit midpoint method	Heun's method or Improved Euler method
implicit:	implicit Runge-Kutta 2 or **implicit midpoint method**	trapezoidal method

From now on, "midpoint method" refers to *implicit midpoint method.*

Two Ways to Look at It

The implicit midpoint method can be seen as:

$$y_{n+1} = y_n + \tau_n f(t_{n+1/2}, y_{n+1/2})$$

or as Backward and Forward Euler steps of size $\frac{\tau_n}{2}$:

$$y_{n+1/2} = y_n + \frac{\tau_n}{2} f(t_{n+1/2}, y_{n+1/2}) \quad \text{(BE: Backward Euler)}$$

$$y_{n+1} = y_{n+1/2} + \frac{\tau_n}{2} f(t_{n+1/2}, y_{n+1/2}) \quad \text{(FE: Forward Euler)}$$

The second step can be rewritten simply as:

$$y_{n+1} = 2y_{n+1/2} - y_n \quad \text{(FE: Forward Euler)}$$

so the implicit problem only needs to be solved once, in BE.

The resulting method can be designated as (BEFE):

$$y_{n+1/2} = y_n + \frac{\tau_n}{2} f(t_{n+1/2}, y_{n+1/2}) \quad \text{(BE: Backward Euler)}$$

$$y_{n+1} = 2y_{n+1/2} - y_n \quad \text{(FE: Forward Euler)}$$

and admits a generalization to Cauchy's one-leg θ method:

$$y_{n+\theta_n} = y_n + \theta_n \tau_n f(t_{n+\theta_n}, y_{n+\theta_n})$$

$$y_{n+1} = \frac{1}{\theta_n} y_{n+\theta_n} - (\frac{1}{\theta_n} - 1) y_n$$

Stability

The θ-method for $\frac{1}{2} \leq \theta_n \leq 1$, and the BEFE special case are unconditionally stable, A-stable, and B-stable.

We say a method is **B-stable** if, for all u, v elements of a Banach or Hilbert space, and $\forall f()$ for which $\langle f(u) - f(v), u - v \rangle \leq 0$, we have $||y_{n+1} - z_{n+1}|| \leq ||y_n - z_n||$, for any two sequences y and z of approximations computed with the method, and any index n.

B-stability implies A-stability.

Error Estimates for Adaptive Stepsize

For a smooth exact solution $y(x)$, the local truncation error for BEFE is

$$T_{n+1} \equiv y(t_{n+1}) - y_{n+1} = \frac{1}{24}\,\tau_n^3\, y'''(t_n + 1/2) + \mathcal{O}(\tau_n^5)$$

For a given local error tolerance `tol`, propose the next time step as

$$\tau_{n+1} = \kappa\,\tau_n \Big(\frac{\text{tol}}{||\,T_{n+1}||}\Big)^{\frac{1}{3}}$$

where the safety factor $\kappa \leq 1$.

How to go MAD: Midpoint Adaptive method:

t_0, y_0, $\texttt{tol}$, T, κ given.

t_1, y_1, τ_0 from one step second-order method in convergence range

$t^{\text{new}} = t_1$, $\tau^{\text{new}} = \tau_0$, $n = 1$

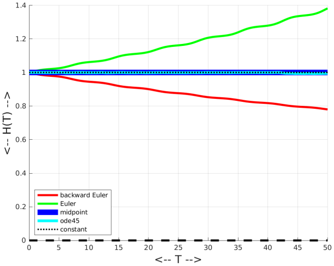
while $t_n \leq T$ **do**

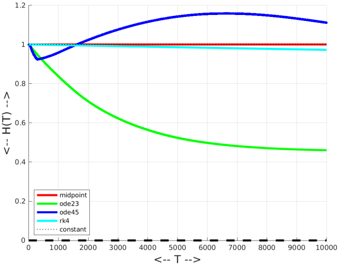
- $\tau_n \leftarrow \tau^{\text{new}}$;
- evaluate y_{n+1} with the midpoint rule;
- evaluate $\widehat{T}_{n+1}$;
- $\tau^{\text{new}} \leftarrow \kappa\, \tau_n \left|\texttt{tol}/\|\widehat{T}_{n+1}\|\right|^{\frac{1}{3}}$;
- **if** $\|\widehat{T}_{n+1}\| \leq \texttt{tol}$ **then**
  - $t_{n+1} \leftarrow t_n + \tau^{\text{new}}$,
  - $n \leftarrow n + 1$
- **end**

end

Test: Rigid Body Rotation

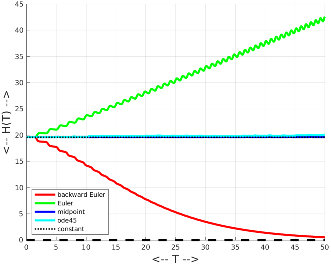
Conservation: $H(t) = u^2 + v^2 + w^2$

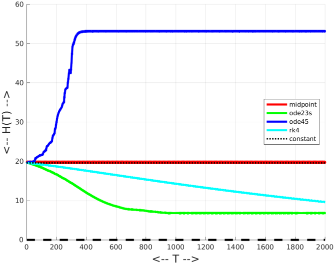




Test: Nonlinear Pendulum

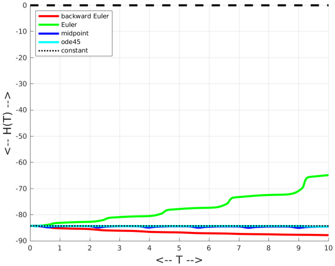
Conservation: $H(t) = \frac{mg}{l}(1-\cos(u)) + \frac{1}{2}mv^2$

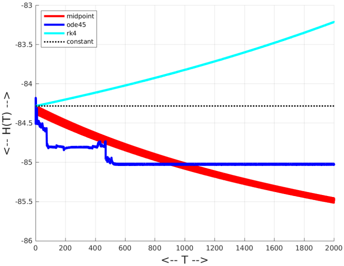




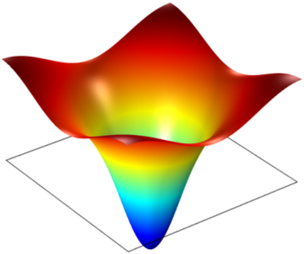
Test: Predator Prey

Conservation: $H(t) = \delta u - \gamma \log(u) + \beta v - \alpha \log(v)$
(not quadratic!)





Implicit Solvers Must Handle Nonlinear Equations



Any implicit ODE method must reliably solve a sequence of systems of nonlinear equations. Given the small stepsizes of a typical ODE method, the previous ODE solution is often a good first approximation to the solution at a new (but very close) time.

A simple method that usually works is to apply a fixed point iteration.

The developers of MINPACK provided the function `hybrd()` for solving general systems of nonlinear equations, and versions of this code are available in MATLAB, Python, and R under the name `fsolve()`.

Fixed Point Iteration

Ada Lovelace: *"The calculation will eat its own tail."*

```
tm = t0 + 0.5 * dt
ym = y0 + 0.5 * dt * f(t0,y0)

for j = 1 : it_max
  ym = y0 + 0.5 * dt * f(tm,ym)
end

t1 = t0 + dt
y1 = 2 * ym - y0
```

Using fsolve()

If you're close enough, you can't miss!

```
th = to + 0.5 * dt;
yh = yo + 0.5 * dt * ( f ( to, yo ) )';
yh = fsolve ( @(yh)residual(f,to,yo,th,yh), yh );
     ...
function value = residual ( f, to, yo, th, yh )
  value = yh - yo - ( th - to ) * ( f ( th, yh ) )';
  return
end
```

Codes available

Enter your ODE, and crank out the result!

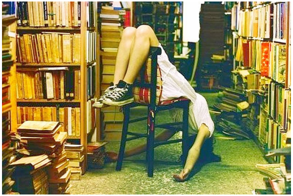
Language	fixed point	fsolve	adaptive
C	midpoint_fixed.c	midpoint.c	
C++	midpoint_fixed.cpp	midpoint.cpp	
Fortran77	midpoint_fixed.f	midpoint.f	
Fortran90	midpoint_fixed.f90	midpoint.f90	
FreeFem		midpoint.edp	
MATLAB	midpoint_fixed.m	midpoint.m	midpoint_adaptive.m
Octave	midpoint_fixed.m	midpoint.m	midpoint_adaptive.m
Python	midpoint_fixed.py	midpoint.py	midpoint_adaptive.py
R	midpoint_fixed.R	midpoint.R	

Professional Codes available

Or you may prefer your ODE to be handled by a professional!

Language	library	code
C	Gnu Scientific Library	gsl_odeiv2_step_rk2imp()
C++	Gnu Scientific Library	gsl_odeiv2_step_rk2imp()
Julia	ODE	Midpoint

References



Catalin Trenchea, John Burkardt,
Refactorization of the midpoint rule,
Applied Mathematics Letters,Volume 107, September 2020,

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Refactorization of the midpoint rule,
Technical Report TR-MATH 20-02,
https://www.mathematics.pitt.edu/sites/default/files/midpoint3_technicalreport.pdf,

John Burkardt, Wenlong Pei, Catalin Trenchea,
A stress test for the midpoint time-stepping method,
International Journal of Numerical Analysis and Modeling, Volume 19, Number 2-3, pages 299-314, 2022.

Links

Links to source code in any language:

https://people.sc.fsu.edu/∼jburkardt/...

Language	subdirectory	Sample directory
C	c_src/	midpoint.html
C++	cpp_src/	midpoint.html
Fortran77	f77_src/	midpoint.html
Fortran90	f_src/	midpoint.html
FreeFem++	freefem_src/	midpoint.html
MATLAB	m_src/	midpoint.html
Octave	octave_src/	midpoint.html
Python	py_src/	midpoint.html
R	r_src/	midpoint.html

For most languages, there are actually several implementations: fixed point/fsolve/adaptive.

The End

- The midpoint method is powerful, accurate, and stable.
- The method is A-stable, B-stable, linearly and nonlinearly stable.
- It is a symplectic method for general Hamiltonian systems.
- The correct estimator for local truncation error only involves the differentiation defect, but not the interpolation defect.
- Implementations are provided in a variety of computing languages.