The Number of Nodes in a Delaunay Mesh Refinement

http://people.sc.fsu.edu/~jburkardt/presentations/delaunay_refinement.pdf
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1 The Delaunay Mesh Refinement Procedure

In any computation using a grid, a common procedure is to *refine* the grid, typically by increasing the number of grid cells by a power of 2. This procedure may be repeated several times, in order to improve the results, or to investigate questions of accuracy.

In cases of grid computation where a regular mesh is not suitable, a Delaunay mesh can always be constructed. In such a mesh, each cell is a triangle. Delaunay meshes can be constructed on a sphere as well; our current interest is to understand a few properties of such a mesh under refinement.

Suppose, then, that we begin with $N = N(0)$ points on a sphere. Let us construct the Delaunay triangulation of the points, which we regard as a mesh M_N. Now we construct a refinement of the mesh by augmenting the original nodes with the set of all midpoints of sides of the Delaunay triangles. The total number of nodes in this set is $N(1)$.

This refinement can be repeated, using a set of $N(2)$ nodes, and so on. The question is, is there a formula for the total number of nodes $N(K)$ on the K-th step of refinement, given $N(0)$, the initial number of nodes, and K the number of refinements?

The answer is *yes*, and the formula is

$$N(K) = 4^K * (N(0) - 2) + 2 \tag{1}$$

or, more usefully,

$$N(K) - 2 = 4^K * (N(0) - 2) \tag{2}$$

This formula can be put into recursive form:

$$N(K+1) = 4 * (N(K) - 2) + 2 \tag{3}$$

or, more usefully,

$$N(K+1) - 2 = 4 * (N(K) - 2) \tag{4}$$

One object we have been studying is the soccer ball, made up of 12 pentagons and 20 hexagons. To begin a mesh, we put a node at the center of each of the polygons. Our initial mesh is then just the Delaunay triangulation of these 32 nodes. If we now carry out the mesh refinement process and count the total number of nodes at each step, we find the results in Table 1. Subtracting 2 from $N(K)$ makes the growth factor of 4 obvious.

Table 1: Number of nodes for Delaunay refinement

K	N(K)	N(K)-2
N(0)	32	30
N(1)	122	120
N(2)	482	480
N(3)	1922	1920
N(4)	7682	7680

PROOF: For any polygonal network on a sphere, define

- F = the number of faces,
- E = the number of edges, and
- V = the number of vertices (or nodes).

Then Euler's formula for the sphere is

$$F - E + V = 2. \tag{5}$$

Now consider the Delaunay triangulation of our points N as such a polygonal network. Then

- $V = N$, because each point is a vertex of the triangulation;
- $E = (3 * F)/2$, because each face is a triangle, contributing 3 edges, but each edge is counted twice because it is shared by two triangles.

Substituting these two facts into the Euler formula gives us the following relationship :

$$F - (3 * F)/2 + N = 2 \tag{6}$$

which simplifies to

$$F = 2 * N - 4 = 2 * (N - 2) \tag{7}$$

which gives us the number of triangles in the mesh. Now, knowing the number of faces, we can immediately deduce

$$E = 3 * (2 * (N - 2))/2 = 3 * (N - 2) \tag{8}$$

which is the number of edges.

Now every edge of the current Delaunay triangulation contributes exactly one new node to the refined mesh. So if mesh M_K has $N = N(K)$ nodes, then our refined mesh M_{K+1} will have

$$N(K+1) = N(K) + 3 * (N(K) - 2) = 4 * (N(K) - 2) + 2 \tag{9}$$

or

$$N(K+1) - 2 = 4 * (N(K) - 2) \tag{10}$$

and hence

$$N(K) - 2 = 4^K * (N(0) - 2) \tag{11}$$

or

$$N(K) = 4^K * (N(0) - 2) + 2 \tag{12}$$

2 The Voronoi Mesh Refinement Procedure

Similarly, we may consider a *Voronoi refinement*, in which, for a given set of nodes, we determine the Voronoi diagram, and then construct a new set of nodes comprising the original set, augmented with the Voronoi vertices. The number of Voronoi vertices NV is the same as the number of Delaunay triangles, so we have

$$N(K+1) = N(K) + 2 * (N(K) - 2) = 3 * (N(K) - 2) + 2 \tag{13}$$

or

$$N(K+1) - 2 = 3 * (N(K) - 2) \tag{14}$$

so

$$N(K) - 2 = 3^K * (N(0) - 2) \tag{15}$$

Applying the Voronoi refinement scheme to our soccer ball, this gives the results in Table 2. Subtracting 2 from each result makes the growth factor of 3 obvious.

Table 2: Number of nodes for Voronoi refinement

K	N(K)	N(K)-2
N(0)	32	30
N(1)	92	90
N(2)	272	270
N(3)	812	810
N(4)	2432	2430