2.1 Factorizations

We want to solve $Ax = b$. If $A$ has a triangular form this very easy. Backsubstitution will solve the equation, for example

$$
\begin{bmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
7 \\
8 \\
9
\end{bmatrix}
$$

we find the solution for $x$ starting with the last row:

$0x_1 + 0x_2 + 6x_3 = 9 \quad \rightarrow \quad x_3 = \frac{9}{6}$

$0x_1 + 4x_2 + 5x_3 = 8 \quad \rightarrow \quad 0x_1 + 4x_2 + \frac{9}{6} = 8 \quad \rightarrow \quad x_2 = \frac{1}{4}$

$1x_1 + 2x_2 + 3x_3 = 7 \quad \rightarrow \quad 1x_1 + 2\left(\frac{1}{4}\right) + \frac{9}{6} = 7 \quad \rightarrow \quad x_2 = \frac{1}{4}$

Solving the above equation is easy if the matrix is in triangular form, therefore the problem reduces to find the triangular matrix.

2.1.1 QR-factorization

We want to put the matrix $A$ into the form

$$
A = QR
$$

where $A$ is a $m \times n$ matrix and $Q$ is a $m \times n$ orthonormal matrix, and $R$ is a triangular $n \times n$ matrix. Having $Q$ and $R$ we can substitute,

$$
Ax = b \\
QRx = b \\
Q^{-1}QRx = Q^{-1}R \\
Rx = Q^{-1}R
$$

This boils down to a little rough algorithm ??: Of course, we still need to know how step 1 in our algorithm is done.

2.1.2 Gram-Schmidt factorization

One of the first factorization of a matrix into triangular matrix is the Gram-Schmidt factorization (Algorithm ??).
Algorithm 1 QR factorization
Compute factorization \( A = QR \)
Compute \( y = Qb \)
Solve \( Rx = y \) for \( x \)

Algorithm 2 Gram-Schmidt factorization
\[
\text{for } j = 1 \text{ to } n \text{ do }
\]
\[
v_j = a_j
\]
\[
\text{for } j = 1 \text{ to } j - 1 \text{ do }
\]
\[
q_i = a_j
\]
\[v_{ij} = q_i a_j
\]
\[
v_j = v_j - r_{ij} q_i
\]
\[
\text{end for}
\]
\[
r_{ij} = ||v_i||_2
\]
\[
q_i = v_j / r_{ij}
\]
\[
\text{end for}
\]

2.1.3 Householder factorization, Householder reflection

2.2 Uses of these factorizations