Theorem (Convergence Rate for Newton-Raphson Iteration) Assume that Newton-Raphson iteration produces a sequence $\{p_k\}_{k=0}^{\infty}$ that converges to the root p of the function $f(x)$.

If p is a simple root, then convergence is quadratic and $\left|E_{k+1}\right| \approx \frac{|f''(p)|}{2|f'(p)|}(|E_k|)^2$ for k sufficiently large.

If p is a multiple root of order m, then convergence is linear and $\left|E_{k+1}\right| \approx \frac{m-1}{m}\left|E_k\right|$ for k sufficiently large.

Proof.

Expand $f(x)$ in a Taylor polynomial of degree n = 1, about $x = p_k$ to get

$$f(x) = f(p_k) + f'(p_k)(x-p_k) + \frac{1}{2!}f''(c_k)(x-p_k)^2.$$

Since p is a zero of $f(x)$, set $x = p$ in the above equation and obtain

$$0 = f(p_k) + f'(p_k)(p-p_k) + \frac{1}{2!}f''(c_k)(p-p_k)^2.$$

Which can be rewritten as

$$f(p_k) + f'(p_k)(p-p_k) = -\frac{f''(c_k)}{2!}(p-p_k)^2.$$

Now assume that $f'(x) \neq 0$ for all x near the root p, and observe that $f'(p_k) \neq 0$, so that we can divide by it and obtain:

$$\frac{f(p_k)}{f'(p_k)} + \frac{f'(p_k)}{f'(p_k)}(p-p_k) = -\frac{f''(c_k)}{2f'(p_k)}(p-p_k)^2.$$

Rearrange the terms and simplify to get

$$(p - p_k) + \frac{f(p_k)}{f'(p_k)} = -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2.$$

The above equation can be rewritten as:

$$p - \left(p_k - \frac{f(p_k)}{f'(p_k)}\right) = -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2.$$

Now use the Newton-Raphson iteration formula $p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}$ and substitute it into the above equation to obtain:

$$p - p_{k+1} = -\frac{f''(c_k)}{2f'(p_k)}(p - p_k)^2.$$

Assuming $f'(p_k) \approx f'(p)$ and $f''(c_k) \approx f''(p)$ when k is sufficiently large yields

$$p - p_{k+1} \approx -\frac{f''(p)}{2f'(p)}(p - p_k)^2,$$

$$E_{k+1} \approx -\frac{f''(p)}{2f'(p)}(E_k)^2.$$

Now take absolute values and obtain the desired conclusion

$$\left|E_{k+1}\right| \approx -\frac{|f''(p)|}{|2f'(p)|}(|E_k|)^2.$$

Q. E. D.

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