The Secant Method

We can obtain this from Newton's method by replacing the tangent slope $f'(x)$ by the chord or secant slope $\frac{f(x)-f(x-h)}{h}$. Using a Taylor expansion of f about x, we find

$$\frac{f(x)-f(x-h)}{h}=\frac{f(x)-hf'(x)+\frac{h^2}{2}f''(x+\vartheta h)-f(x)}{h}=f'(x)+\frac{h}{2}f''(x+\vartheta h)$$

so that the error is O(h) if f'' is bounded. The obvious choice for h is so that if $x=x_n$, the n-th iterate, $x-h=x_{n-1}$, the previous iterate. Putting this into Newton's method, we obtain

a formula that does need two starting values.
The formula can be written with a common denominator as

$$x_{n+1}=\frac{x_{n-1}f(x_n)-x_nf(x_{n-1})}{f(x_n)-f(x_{n-1})} \tag{2}$$

which is recognizable as the formula for *inverse interpolation*, that is, it linearly interpolates the values of f at x_n and x_{n-1} to approximate the value of x at which f vanishes. This form should not be used in practical evaluation, since it computes the ratio of two quantities that are liable to be differences of nearly equal numbers.
This formulation suggests a variant of the method, in which convergence is guaranteed provided starting values can be found for which the values of f are opposite in sign. This is defined by

$$x_{n+1}=\frac{x_mf(x_n)-x_nf(x_m)}{f(x_n)-f(x_m)}$$

where $m\le n$ is the greatest index for which $\operatorname{sgn} f_m \ne \operatorname{sgn} f_n$. Convergence is guaranteed because there is always a zero of f between x_n and x_m. The rate of convergence is better than linear, but not as good as that of the original secant method.

Example Evaluate. $\sqrt{15}$

We use the secant method to solve the equation $x^2-15=0$. The secant formula is

$$x_{n+1} = x_n - \frac{(x_n^2 - 15)(x_n - x_{n-1})}{x_n^2 - x^2_{n-1}} = x_n - \frac{x_n^2 - 15}{x_n + x_{n-1}}$$

Starting with x_0=3, x_1=4, we calculate to 9D successively
x_2=3.857142857,
x_4=3.872983871, x_5=3.872983347,x_6=3.872983347.

This looks nearly as good as Newton's method, but our starting values were fairly close.

Convergence Rate of Secant Method

Subtracting α from both sides of equation of the secant method, we find

$$x_{n+1} - \alpha = \frac{(x_{n-1} - \alpha) f(x_n) - (x_n - \alpha) f(x_{n-1})}{f(x_n) - f(x_{n-1})} =$$

$$\frac{(x_{n-1} - \alpha)(f(x_n) - f(\alpha)) - (x_n - \alpha)(f(x_{n-1}) - f(\alpha))}{f(x_n) - f(x_{n-1})} =$$

$$\frac{(x_{n-1} - \alpha)(x_n - \alpha) f\left[x_n, \alpha\right] - (x_n - \alpha)(x_{n-1} - \alpha) f\left[x_{n-1}, \alpha\right]}{(x_n - x_{n-1}) f\left[x_n, x_{n-1}\right]} =$$

$$(x_n - \alpha)(x_{n-1} - \alpha) f\left[\alpha, x_{n-1}, x_n\right] \big/ f\left[x_{n-1}, x_n\right] =$$

$$(x_n - \alpha)(x_{n-1} - \alpha) \frac{f''(\xi_n)}{f'(\eta_n)}$$

$$\begin{aligned} u_n &= c_{n-1} + q u_{n-1} \\ &= c_{n-1} + q(c_{n-2} + q u_{n-2}) \\ &= c_{n-1} + q c_{n-2} + q^2 u_{n-2} \\ &= \vdots\vdots \\ &= c_{n-1} + q c_{n-2} + \ldots + q^{n-1} c_0 + q^n u_0 \end{aligned}$$

with values of ξ_n, η_α in appropriate intervals. Set

$$e_n = |x_n - \alpha|, \quad \left|\frac{f''(\xi_n)}{f'(\eta_n)}\right| = \kappa_{n-1} \qquad \text{for} \quad n = 0, 1, \ldots;$$

then

$$e_{n+1} = \kappa_{n-1} e_n e_{n-1}.$$

Increasing each subscript by 1, taking logs and setting $v_n = \log e_n, c_n = \log \kappa_n$, we obtain the difference equation

$$v_{n+2} - v_{n+1} - v_n = c_n .$$

We use the *displacement operator*, *E*, defined by $Ev_n = v_{n+1}$ to write this as $(E^2 - E - 1)v_n = c_n$, or $((E-q)(E-p)v_n = c_n$, where:

$$p = \frac{1+\sqrt{5}}{2}$$
$$q = \frac{1-\sqrt{5}}{2} .$$

Now set

$$(E-p)v_n = u_n, \qquad (3)$$

whence we obtain

$$\begin{aligned} u_n &= c_{n-1} + qu_{n-1} \\ &= c_{n-1} + q(c_{n-2} + qu_{n-2}) \\ &= c_{n-1} + qc_{n-2} + q^2 u_{n-2} \\ &= \vdots\vdots \\ &= c_{n-1} + qc_{n-2} + \ldots + q^{n-1}c_0 + q^n u_0 \end{aligned}$$

Putting this back into (   ), we find

$$v_{n+1} - pv_n = c_{n-1} - qc_{n-2} + \ldots + q^{n-1}c_0 + q^n(v_1 - pv_{0).}$$

Provided that the $c_i, i = 0,1,\ldots,c_{n-1}$ are all of the same sign, the first *n* terms of the right hand side form an alternating series, since -1<*q*<0, and so $v_{n+1} - pv_n \to L$, say, where $|L| < \infty$, which gives

$$\frac{e_{n+1}}{e_n^p} \to e^L.$$

This gives the order of the secant method as $p = \frac{1+\sqrt{5}}{2} \cong 1.618$; As we had suspected, the rate of convergence is somewhere between linear and second order. In order to be sure that $c_1, c_2, \ldots, c_n$ are of the same sign, we need

$\kappa_1, \kappa_2, \ldots, \kappa_n$ to be all greater than 1 or all less than 1. Since $\kappa_n \to \left|\frac{f''(\alpha)}{f'(\alpha)}\right|$, we can achieve this by starting with an x_0 sufficiently near to α.

Finally, there is a classical variant of the secant method, which is defined by

$$x_{n+1} = \frac{x_0 f(x_n) - x_n f(x_0)}{f(x_n) - f(x_0)}$$

Here the first and last values of the iterate are used. Following through the convergence analysis above, we find

$$x_{n+1} - \alpha = \frac{1}{2}(x_0 - \alpha)(x_n - \alpha)\frac{f''(\xi)}{f'(\eta)}$$

which shows that the convergence is only linear.